A, B separable C^* algebras

$\mathcal{E}^1(A,B) = \{(\mathcal{H}, \psi, T)\}$

$\mathcal{H}$ is a countably generated Hilbert B-module

$\psi : A \to \mathcal{L}(\mathcal{H})$ is a $*$-homomorphism

$T \in \mathcal{L}(\mathcal{H})$

$$\begin{cases} T = T^* & \\ \psi(a)(I - T^2) \in \mathcal{K}(\mathcal{H}) & \forall\, a \in A \\ \psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H}) & \forall\, a \in A \end{cases}$$

$(\mathcal{H}_0, \psi_0, T_0)$ and $(\mathcal{H}_1, \psi_1, T_1) \in \mathcal{E}^1(A,B)$

are <u>isomorphic</u> if $\exists$ an isomorphism of Hilbert B-modules $\Phi : \mathcal{H}_0 \to H_1$ with

$$\Phi\psi_0(a) = \psi_1(a)\Phi \qquad \forall a \in A$$

$$\Phi T_0 = T_1 \Phi$$

A, B, D separable C^* algebras

$\varphi : B \to D$ $\quad$ $*$-homomorphism

$\varphi_* : \mathcal{E}^1(A, B) \to \mathcal{E}^1(A, D)$

$$\varphi_*(\mathcal{H}, \psi, T) = (\mathcal{H} \underset{B}{\otimes} D, \psi \underset{B}{\otimes} I, T \underset{B}{\otimes} I)$$

$I =$ identity operator of D

$$I(\alpha) = \alpha \quad , \qquad \forall\, \alpha \in D$$

$$C([0,1], B) \underset{\rho_1}{\overset{\rho_0}{\rightrightarrows}} B \qquad \begin{array}{l} \rho_0(f) = f(0) \\ \rho_1(f) = f(1) \end{array}$$

$(\mathcal{H}_0, \psi_0, T_0)$ and $(\mathcal{H}_1, \psi_1, T_1)$ in $\mathcal{E}^1(A, B)$ are <u>homotopic</u> if $\exists\ (\mathcal{H}, \psi, T) \in \mathcal{E}^1(A\ ,\ C([0,1], B))$ with $(\rho_j)_*(\mathcal{H}, \psi, T) \cong (\mathcal{H}_j, \psi_j, T_j)$

"homotopy" has been made precise.

$$KK^1(A,B) = \mathcal{E}^1(A,B)/(\text{homotopy})$$

$$KK^0(A,B) = \mathcal{E}^0(A,B)/(\text{homotopy})$$

Topic 11: Equivariant K homology revisited

A $G - C^*$ algebra

Definition A G-Hilbert$A$-module is a Hilbert A-module $\mathcal{H}$ with a given continuous action

$$G \times \mathcal{H} \longrightarrow \mathcal{H} \qquad (g, v) \longmapsto gv$$

such that

$$g(u+v) = gu + gv \qquad u, v \in \mathcal{H} \qquad g \in G$$

$$g(ua) = (gu)(ga) \qquad u \in \mathcal{H} \;\; a \in A \qquad g \in G$$

$$\langle gu, gv \rangle = g\langle u, v \rangle \qquad u, v \in \mathcal{H} \qquad g \in G$$

"continuous" means that for each $u \in \mathcal{H}$, $g \mapsto gu$ is a continuous map $G \longrightarrow \mathcal{H}$.

Remark. A $G-C^*$ algebra

$\mathcal{H}$ G-Hilbert A-module

For each $g \in G$, denote by L_g the map

$$L_g : \mathcal{H} \longrightarrow \mathcal{H} \qquad g \in G$$

$$L_g(v) = gv \qquad v \in \mathcal{H}$$

Note that L_g might not be in $\mathcal{L}(\mathcal{H})$. But if $T \in \mathcal{L}(\mathcal{H})$, then $L_g T L_g^{-1} \in \mathcal{L}(\mathcal{H})$. Thus $\mathcal{L}(\mathcal{H})$ is a $G-C^*$ algebra with

$$gT = L_g T L_g^{-1}$$

Example. A $G-C^*$ algebra

n positive integer

A^n is a G-Hilbert A − module with

$$g(a_1, a_2, \ldots, a_n) = (ga_1, ga_2, \ldots, ga_n)$$

$A, B \quad G\text{-}C^*$ algebras $\quad A, B$ separable

$\mathcal{E}_G^0(A, B) = \{(H, \psi, T)\}$

H is a G-Hilbert B-module (countably generated)

$\psi : A \to \mathcal{L}(B)$ is a $*$-homomorphism with

$$\psi(ga) = g\psi(a) \qquad \forall\, (g, a) \in G \times A$$

$T \in \mathcal{L}(H)$

$$\left\{\begin{array}{ll} gT - T \in \mathcal{K}(H) & \forall\, g \in G \\ \psi(a)T - T\psi(a) \in \mathcal{K}(H) & \forall\, a \in A \\ \psi(a)(I - T^*T) \in \mathcal{K}(H) & \forall\, a \in A \\ \psi(a)(I - TT^*) \in \mathcal{K}(H) & \forall\, a \in A \end{array}\right\}$$

$$KK_G^0(A,B) = \mathcal{E}_G^0(A,B)/\text{homotopy}$$

$KK_G^0(A,B)$ is an abelian group

$$(H,\psi,T) + (H',\psi',T') = (H \oplus H', \psi \oplus \psi', T \oplus T')$$
$$-(H,\psi,T) = (H,\psi,T^*)$$

$A, B \quad G\text{-}C^*$ algebras $\quad A, B$ separable

$\mathcal{E}^1_G(A, B) = \{(H, \psi, T)\}$

H is a G-Hilbert B-module (countably generated)

$\psi : A \to \mathcal{L}(B)$ is a $*$-homomorphism with

$$\psi(ga) = g\psi(a) \qquad \forall\, (g, a) \in G \times A$$

$T \in \mathcal{L}(H)$

$$\left\{\begin{array}{ll} T = T^* & \\ gT - T \in \mathcal{K}(H) & \forall\, g \in G \\ \psi(a)T - T\psi(a) \in \mathcal{K}(H) & \forall\, a \in A \\ \psi(a)(I - T^2) \in \mathcal{K}(H) & \forall\, a \in A \end{array}\right\}$$

$$KK_G^1(A,B) = \mathcal{E}_G^1(A,B)/\text{homotopy}$$

$KK_G^1(A,B)$ is an abelian group

$$(H,\psi,T) + (H',\psi',T') = (H \oplus H', \psi \oplus \psi', T \oplus T')$$

$$-(H,\psi,T) = (H,\psi,-T)$$

Kasparov Product

A, B, D (separable) G-C^* algebras

$$KK_G^i(A,B) \otimes_{\mathbb{Z}} KK_G^j(B,D) \longrightarrow KK_G^{i+j}(A,D)$$

$i, j = 0, 1$