Topic 2: C^* algebras

Definition. A Banach algebra is an algebra A over $\mathbb{C}$ with a given norm $\| \ \|$

$$\| \ \| : A \longrightarrow \{t \in \mathbb{R} \mid t \geqq 0\}$$

such that A is a complete normed algebra:

$$\|\lambda a\| = |\lambda| \, \|a\| \qquad \lambda \in \mathbb{C} \qquad a \in A$$

$$\|a + b\| \leq \|a\| + \|b\| \qquad a, b \in A$$

$$\|ab\| \leq \|a\| \, \|b\| \qquad a, b \in A$$

$$\|a\| = 0 \quad \Longleftrightarrow \quad a = 0$$

Every Cauchy sequence is convergent in A (with respect to the metric $\|a - b\|$).

C^* algebras

A C^* algebra $\qquad\qquad *: A \longrightarrow A$

$A = (A, \| \ \|, *) \qquad\qquad a \longmapsto a^*$

$(A, \| \ \|)$ is a Banach algebra

$(a^*)^* = a$

$(a+b)^* = a^* + b^*$

$(ab)^* = b^* a^*$

$(\lambda a)^* = \overline{\lambda} a^* \qquad a, b \in A \qquad \lambda \in \mathbb{C}$

$\|aa^*\| = \|a\|^2 = \|a^*\|^2$

A $*$-homorphism is an algebra homorphism

$\varphi : A \longrightarrow B$ such that $\varphi(a^*) = (\varphi(a))^* \quad \forall a \in A.$

<u>Lemma:</u>

If $\varphi : A \longrightarrow B$ is a $*$-homorphism
then $\|\varphi(a)\| \leqq \|a\| \quad \forall a \in A$

EXAMPLES OF C^* ALGEBRAS

Example (1)

X topological space, Hausdorff, locally compact

$$X^+ = \text{one-point compactification of } X$$

$$= X \cup \{p_\infty\}$$

$$C_0(X) = \left\{ \alpha : X^+ \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} \alpha \text{ continuous} \\ \\ \alpha(p_\infty) = 0 \end{array} \right\}$$

$$\|\alpha\| = \sup_{p \in X} |\alpha(p)|$$

$$\alpha^*(p) = \overline{\alpha(p)}$$

$$(\alpha + \beta)(p) = \alpha(p) + \beta(p) \qquad p \in X$$

$$(\alpha\beta)(p) = \alpha(p)\beta(p)$$

$$(\lambda\alpha)(p) = \lambda\alpha(p) \qquad \lambda \in \mathbb{C}$$

If X is compact Hausdorff, then

$$C_0(X) = C(X) = \{\alpha : X \longrightarrow \mathbb{C} | \alpha \text{ continuous}\}$$

Example (2) H Hilbert space

(separable = H admits a countable (or finite) orthonormal basis.)

$$\mathcal{L}(H) = \{\text{bounded operators } T : H \longrightarrow H\}$$

$$\|T\| = \sup_{\substack{u \in H \\ \|u\|=1}} \|T\,u\| \qquad \text{operator norm} \quad \|u\| = \langle u,u\rangle^{1/2}$$

$$T^* = \text{ adjoint of } T \qquad \langle T\,u, v\rangle = \langle u, T^* v\rangle \quad u,v \in H$$

$$(T+S)u = T\,u + S\,u$$

$$(TS)u = T(S\,u)$$

$$(\lambda T)u = \lambda(T\,u) \qquad \lambda \in \mathbb{C}$$

Example 3 H Hilbert space

$$\mathcal{K}(H) = \{T \in \mathcal{L}(H) \mid T \text{ is a compact operator}\}$$

$$= \overline{\{T \in \mathcal{L}(H) \mid \dim_{\mathbb{C}} T(H) < \infty\}}$$

(closure taken in operator norm)

$\mathcal{K}(H)$ is a sub C^* algebra of $\mathcal{L}(H)$

$\mathcal{K}(H)$ is an ideal in $\mathcal{L}(H)$

G topological group

locally compact

Hausdorf

second countable

<u>Example 4</u>: C_r^*G the reduced C^* algebra of G

Fix a left-invariant Haar measure dg for G

"left-invariant" = whenever $f : G \to \mathbb{C}$ is

continuous and compactly supported

$\int_G f(\gamma g)dg = \int_G f(g)dg \qquad \forall\, \gamma \in G$

L^2G Hilbert space

$$L^2G = \left\{ u : G \to \mathbb{C} \mid \int_G |u(g)|^2 dg < \infty \right\}$$

$$\langle u, v \rangle = \int_G \overline{u(g)} v(g)\, dg$$

$u, v \in L^2G$

$\mathcal{L}(L^2G) = C^*$ algebra of all

bounded operators

$T : L^2G \to L^2G$

$$C_cG = \left\{ f : G \to \mathbb{C} \,\middle|\, \begin{array}{l} f \text{ is continuous} \\ \\ f \text{ has compact support} \end{array} \right\}$$

C_cG is an algebra

$(\lambda f)g = \lambda(fg) \qquad \lambda \in \mathbb{C} \qquad g \in G$

$(f + h)g = fg + hg$

multiplication in C_cG is convolution

$$(f * h)g_0 = \int_G f(g)h(g^{-1}g_0)dg \qquad g_0 \in G$$

$$0 \longrightarrow C_c G \longrightarrow \mathcal{L}(L^2 G)$$

Injection of algebras

$$f \mapsto T_f$$

$$T_f(u) = f * u \qquad\qquad u \in L^2 G$$

$$(f * u)g_0 = \int_G f(g)u(g^{-1}g_0)dg \qquad\qquad g_0 \in G$$

$$C_r^* G \subset \mathcal{L}(L^2 G)$$

$$C_r^* G = \overline{C_c G} = \text{closure of } C_c G$$

in the operator norm

$C_r^* G$ is a sub C^* algebra of $\mathcal{L}(L^2 G)$

Definition. A sub-algebra $\boldsymbol{A}$ of $\mathcal{L}(\boldsymbol{H})$ is a C* algebra of operators iff

(1) $\boldsymbol{A}$ is closed with respect to the operator norm.

(2) If $T \in \boldsymbol{A}$ then the adjoint operator $T^* \in \boldsymbol{A}$

Theorem. (I. Gelfand and V. Naimark)

Any C* algebra is isomorphic, as a C* algebra, to a C* algebra of operators.

Theorem. (I. Gelfand) Let A be a commutative C* algebra. Then A is (cannonically) isomorphic to $C_0(X)$ where X is the space of maximal ideals of A.

Thus a non-commutative C* algebra can be viewed as a "non-commutative locally compact Hausdorff topological space."