K-Homology

December, 2006

Let A be a (separable) $G - C^*$ algebra.

A *covariant representation* of A is a triple (H, ψ, π) such that:

H is a separable Hilbert space.

$\psi : A \to \mathcal{L}(H)$ is a $*$-homomorphism.

$\pi : G \to \mathcal{U}(H)$ is a unitary representation of G.

$$\psi(ga) = \pi(g)\psi(a)\pi(g^{-1}) \qquad\qquad \forall (g, a) \in G \times A.$$

Let X be a proper G-space with compact quotient space $G\backslash X$.

An equivariant (odd) K - cycle for X is a 4 - tuple (H, ψ, π, T) such that:

- (H, ψ, π) is a covariant representation of the $G - C^*$ algebra $C_0(X)$.

- $T \in \mathcal{L}(H)$

- $T = T^*$

- $\pi(g)T - T\pi(g) = 0 \qquad \forall g \in G$

- $\psi(\alpha)T - T\psi(\alpha) \in \mathcal{K}(H) \qquad \forall \alpha \in C_0(X)$

- $\psi(\alpha)(I - T^2) \in \mathcal{K}(H) \qquad \forall \alpha \in C_0(X)$

$$\mathcal{E}_1^G(X) = \{(H, \psi, \pi, T)\}$$

Example

$G = \mathbb{Z} \qquad X = \mathbb{R}$

$\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$

$(n, t) \mapsto n + t \qquad \mathbb{Z}\backslash\mathbb{R} = S^1$

$H = L^2(\mathbb{R}) \qquad \psi(\alpha)u = \alpha u$

$$\alpha u(t) = \alpha(t)u(t)$$

$$\alpha \in C_0(\mathbb{R}) \quad u \in L^2(\mathbb{R}) \quad t \in \mathbb{R}$$

$(\pi(n)u)(t) = u(t - n) \qquad n \in \mathbb{Z} \quad u \in L^2(\mathbb{R})$

$-i\frac{d}{dx}$

$(L^2(\mathbb{R}), \psi, \pi, -i\frac{d}{dx})$

$-i\frac{d}{dx}$ is not a bounded operator on $L^2(\mathbb{R})$

Shall "normalize" $-i\frac{d}{dx}$ to obtain a bounded operator T

Since $-i\frac{d}{dx}$ is self-adjoint there is functional calculus,

and T can be taken to be the function $\frac{x}{\sqrt{1+x^2}}$

applied to $-i\frac{d}{dx}$

$$T = \left(\frac{x}{\sqrt{1+x^2}}\right)\left(-i\frac{d}{dx}\right)$$

Equivalently, T can be constructed using Fourier transform

Let $\mathcal{M}_x$ be the operator "multiplication by x"

$\mathcal{M}_x\Big(f(x)\Big) = xf(x)$

Fourier transform converts $-i\frac{d}{dx}$ to $\mathcal{M}_x$

i.e. there is commutativity in the diagram

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \\ {\scriptstyle -i\frac{d}{dx}}\downarrow & & \downarrow{\scriptstyle \mathcal{M}_x} \\ L^2(\mathbb{R}) & \xrightarrow[\mathcal{F}]{} & L^2(\mathbb{R}) \end{array}$$

$\mathcal{F}$ = Fourier transform

Let $\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}$ be the operator "multiplication by $\dfrac{x}{\sqrt{1+x^2}}$"

$$\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}\Big(f(x)\Big) = \frac{x}{\sqrt{1+x^2}} f(x)$$

$\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}$ is a bounded operator

$$\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

T is the unique bounded operator

$$T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

such that there is commutativity in the

diagram

$$\begin{tikzcd}
L^2(\mathbb{R}) \arrow[r, "\mathcal{F}"] \arrow[d, "T"'] & L^2(\mathbb{R}) \arrow[d, "\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}"] \\
L^2(\mathbb{R}) \arrow[r, "\mathcal{F}"'] & L^2(\mathbb{R})
\end{tikzcd}$$

$\mathcal{F}$ = Fourier transform

$(L^2(\mathbb{R}), \psi, \pi, T) \in \mathcal{E}_1^{\mathbb{Z}}(\mathbb{R})$

X a proper G-space with compact quotient space $G\backslash X$.

$\mathcal{E}_1^G(X) = \{(H, \psi, \pi, T)\}$

$K_1^G(X) := \mathcal{E}_1^G(X)/\sim$

$\sim$ = "homotopy"

"homotopy" will be made precise later.

$K_1^G(X) = \mathcal{E}_1^G(X)/\sim$ is an abelian group.

$$(H, \psi, \pi, T) + (H', \psi', \pi', T') =$$

$$(H \oplus H', \psi \oplus \psi', \pi \oplus \pi', T \oplus T')$$

$-(H, \psi, \pi, T) = (H, \psi, \pi, -T)$

Let X be a proper G-space with compact quotient space $G\backslash X$.

An equivariant (even) K - cycle for X is a 4 - tuple (H, ψ, π, T) such that:

- (H, ψ, π) is a covariant representation of the $G - C^*$ algebra $C_0(X)$.

- $T \in \mathcal{L}(H)$

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X a proper G-space with compact quotient space $G\backslash X$.

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$$(H \oplus H', \psi \oplus \psi', \pi \oplus \pi', T \oplus T')$$

$-(H, \psi, \pi, T) = (H, \psi, \pi, T^*)$

X, Y proper G-spaces with compact quotient spaces $G\backslash X$, $G\backslash Y$.

$f : X \longrightarrow Y$ continuous G-equivariant map.

$\tilde{f} : C_0(X) \longleftarrow C_0(Y)$

$\tilde{f}(\alpha) = \alpha \circ f \qquad\qquad \alpha \in C_0(Y)$

$f_* : K_j^G(X) \longrightarrow K_j^G(Y)$

homomorphism of abelian groups $j = 0, 1$

$(H, \psi, \pi, T) \longmapsto (H, \psi \circ \tilde{f}, \pi, T)$

$(H, \psi, \pi, T) \in \mathcal{E}_j^G(X)$

Definition. A G-space is a topological space X with a given continuous action of G.

$$G \times X \longrightarrow X$$

Definition. A G-space X is proper if:

- X is paracompact and Hausdorff.

- The quotient space $G\backslash X$ (with the quotient topology) is paracompact and Hausdorff.

- For each $p \in X$ there exists a triple (U, H, ρ) such that :

  1. U is an open neighborhood of p in X with $gu \in U$ for all $(g, u) \in G \times U$.

  2. H is a compact subgroup of G.

  3. $\rho : U \longrightarrow G/H$ is a G-map from U to G/H.

X a proper G-space with compact quotient space $G\backslash X$

There is a map of abelian groups

$$K_j^G(X) \longrightarrow K_j C_r^* G$$

$$(H, \psi, \pi, T) \longmapsto \mathrm{Index}(T)$$

j = 0, 1

This map

$$K_j^G(X) \longrightarrow K_j C_r^* G$$
$$(H, \psi, \pi, T) \longmapsto \text{Index}(T)$$

is natural, i.e. If X, Y are proper G-spaces with compact quotient spaces $G\backslash X$, $G\backslash Y$ and if

$f : X \longrightarrow Y$ is a continuous G-equivariant map.

Then there is commutativity in the diagram

$$\begin{tikzcd}
K_j^G(X) \arrow[rr, "f_*"] \arrow[dr] & & K_j^G(Y) \arrow[dl] \\
& K_j C_r^* G &
\end{tikzcd}$$

$\underline{E}G$ universal example for proper actions of G

Definition. $\Delta \subseteq \underline{E}G$ is G-compact if

1. $gx \in \Delta$ for all $(g, x) \in G \times \Delta$.

2. The quotient space $G \backslash \Delta$ is compact.

Set $K_j^G(\underline{E}G) = \lim_{\Delta \subseteq \underline{E}G | \Delta \text{ is } G\text{-compact}} K_j^G(\Delta)$.

The direct limit is taken over all G-compact subsets Δ of $\underline{E}G$.

$K_j^G(\underline{E}G)$ is the equivariant K-homology of $\underline{E}G$ with G-compact supports.

$\mu : K_j^G(\underline{E}G) \longrightarrow K_j C_r^* G$

$(H, \psi, \pi, T) \longmapsto \text{Index}(T)$

Conjecture (P.B. and A.Connes , 1980) Let G be a locally compact Hausdorff second countable topological group. then

$$\mu : K_j^G(\underline{E}G) \longrightarrow K_j C_r^* G$$

is an isomorphism.

j=0, 1

THE CONJECTURE WITH COEFFICIENTS

Definition. A $G - C^*$ algebra is a C^* algebra A with a given continuous action of G.

$$G \times A \longrightarrow A$$

G acts on A by C^* algebra automorphisms.

The continuity condition is: For each $a \in A$,

$$G \longrightarrow A, g \mapsto ga$$

is a continuous map from G to A.

Let A be a $G - C^*$ algebra. Form the reduced crossed-product C^* algebra $C_r^*(G, A)$.

$K_j C_r^*(G, A) =$?

$K_j^G(\underline{E}G, A)$ denotes the equivariant K-homology of $\underline{E}G$ with G-compact supports and coefficients A.

Conjecture(P.B. and A.Connes, 1980) Let G be a locally compact Hausdorff second countable topological group, and let A be any $G-C^*$ algebra, then

$$\mu : K_j^G(\underline{E}G, A) \longrightarrow K_j C_r^*(G, A)$$

is an isomorphism.

j = 0, 1.

EXPANDER GRAPHS

A.LUBOTZKY and P.SARNAK

Let Γ be a finitely presented discrete group which contains an expander in its Cayley graph.

Such a Γ is a counter-example to the conjecture with coefficients.

Does such a Γ exist?

M. Gromov outlined a proof that such a Γ exists. A number of mathematicians are now filling in the details.