C^*-graph algebra maps that lift to Leavitt path algebras

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Homotopy inverse of ϕ need not be $*$-homomorphism.

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Full subcategories:

$$\mathrm{Alg}^*_{\mathbb{C}} \supset \mathrm{Leavitt}^* = \{L(E) : E \text{ finite, no sinks}\}$$
$$kk \supset \langle \mathrm{Leavitt} \rangle_{kk} = \{j(L(E)) : E \text{ finite, no sinks}\}$$

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i) $\hom_{\mathrm{Alg}^*_{\mathbb{C}}}(L(E), L(F))^P \to [[C^*(E), C^*(F)]]_{M_2} \setminus \{0\}$, *is onto and maps the subset of unital $*$-homomorphisms onto the subset of homotopy classes of unital C^*-algebra homomorphisms.*

ii) *Let* $\phi \in \hom_{\mathrm{Alg}^*_{\mathbb{C}}}(L(E), L(F))^P$. *Then* $\hat{\phi} : C^*(E) \to C^*(F)$ *is an M_2-continuous homotopy equivalence if and only if ϕ is a polynomial M_2-homotopy equivalence.*

SPI GRAPHS, CONTINUED

A $*$-homomorphism $\phi : A \to B$ of unital $*$-algebras has *property (P)* if $\phi(1)B$ contains an isometry.

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ii) *Let $\phi \in \hom_{\mathrm{Alg}^*_{\mathbb{C}}}(L(E),L(F))^P$. Then $\hat{\phi} : C^*(E) \to C^*(F)$ is an M_2-continuous homotopy equivalence if and only if ϕ is a polynomial M_2-homotopy equivalence. If furthermore ϕ is unital, then $\hat{\phi}$ is a continuous homotopy equivalence if and only if ϕ is a polynomial homotopy equivalence.*

Proof of Theorem B

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\downarrow & & \downarrow \\
kk(L(E), L(F)) & \xrightarrow{\mathrm{comp}} & KK(C^*(E), C^*(F)) \\
\downarrow & \swarrow & \\
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$$((\pi \circ \xi_1) \circ \eta_1) \circ (\pi \circ \xi_2 \circ \eta_2)$$

$$\eta_1 \circ \pi \in K_{-1}(\mathbb{C})^{F^0} = 0.$$

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$$\begin{array}{ccc}
\hom_{\mathrm{Alg}^*_{\mathbb{C}}}(L(E),L(F))^P & & \\
\downarrow & \overset{\textcircled{1}}{\searrow} & \\
[L(E),L(F)]_{M_2}\setminus\{0\} & \overset{\textcircled{3}}{\searrow} & \\
\downarrow & & \\
[[C^*(E),C^*(F)]]_{M_2}\setminus\{0\} & \xrightarrow{\textcircled{2}} & \hom(\mathfrak{BF}(E),\mathfrak{BF}(F))
\end{array}$$

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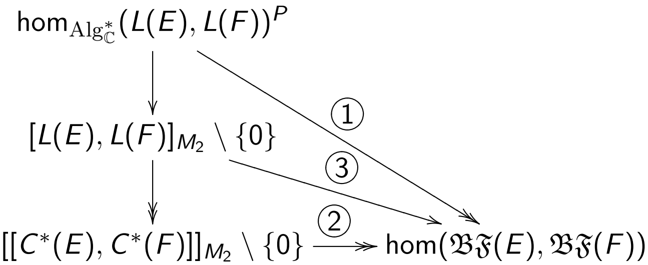

(1) is onto by [2].

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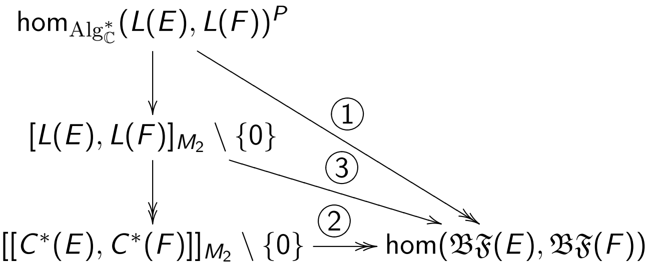

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$$\xi - k(\hat{\phi}) \in \mathrm{Ker}(2).$$

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$$\xi = [k(\hat{\phi})] + \partial[u] = [\widehat{\phi^u}]$$

$\phi^u(e) = ue$. □

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Thank you!