TWISTED BUNDLES AND TWISTED K-THEORY

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0. INTRODUCTION

Many papers have been devoted recently to twisted K-theory as originally defined in [15] and [29]. See for instance the references [2], [23] and the very accessible paper [30]. We offer here a more direct approach based on the notion of "twisted vector bundles". This is not an entirely new idea, since we find it in [9], [7], [6], [8] and [4] for instance, under different names and from various viewpoints. However, a careful look at this notion shows that we may interpret such bundles as modules over suitable algebra bundles. More precisely, the category of twisted vector bundles is equivalent to the category of vector bundles which are modules over algebra bundles with fibre $\mathrm{End}(V)$, where V is a finite dimensional vector space. This notion was first explored in [15] in order to define twisted K-theory. In the same vein, twisted Hilbert bundles may be used to define extended twisted K-groups, following [14] and [29].

More generally, we also analyse the notion of "twisted principal bundles" with structural group G. Under favourable circumstances, we show that the associated category is equivalent to the category of locally trivial fibration, with an action of a bundle of groups with fibre G, which is simply transitive on each fibre. Such bundles are classically called "torsors" in the literature. When the bundle of groups is trivial, we recover the usual notion of principal G-bundle.

As is well known, twisted K-theory is a graded group, indexed essentially by the third cohomology[1] of the base space X, namely $H^3(X; \mathbb{Z})$. The twisted vector bundles we define in this paper are also indexed by elements of the same group up to isomorphism. Roughly speaking, twisted K-theory appears as the Grothendieck group of the category of twisted vector bundles. This provides a geometric description of this theory, very close in spirit to Steenrod's definition of coordinate bundles [31]. The more subtle notion of <u>graded</u> twisted K-theory, indexed by $H^1(X; \mathbb{Z}/2) \times H^3(X; \mathbb{Z})$, may also be analyzed in this framework.

The usual operations on vector bundles (exterior powers, Adams operations...) are easily extended to twisted vector bundles, in a way

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[1]More precisely, it is indexed by 3-cocycles. Two cohomologous cocycles give twisted K-groups which are isomorphic (non canonically). This technical point is discussed in Appendix 8.3.

parallel to the operations defined in [2]. We have also added a section on cup-products, in order to show that the various ways to define them coincide up to isomorphism. This is essentially relevant in the last section of the paper, where we define an analog of the Chern character.

In this section, we define connections on twisted vector bundles in the finite and infinite dimensional cases, very much in the spirit of [26, pg. 78], [6], [22, Chapitre 1], in a quite elementary way. It is also described in [11] and [4] with a different method. From this analog of Chern-Weil theory, we deduce a "Chern character" from twisted K-theory to twisted cohomology. This character is defined in a much more elaborate way in [3], [27], [32] and [8], in the general framework of the "Connes-Karoubi Chern character" [12], [22], except in [3]. In the paper of Atiyah and Segal [3], classical topology tools are used to show that the twisted Chern character is essentially unique. Therefore, it coincides with the character defined by our elementary approach in this paper.

Finally, in a detailed appendix divided into three subsections, we study carefully the relation between Čech cohomology with coefficients in S^1 and de Rham cohomology. We also discuss more deeply multiplicative structures and the functorial aspects of twisted K-theory and of the Chern character.

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1. Twisted principal bundles

Let G be a topological group and let $\mathcal{U} =(U_i), i \in I$, be an open covering of a topological space X. The Čech cohomology set $H^1(\mathcal{U}; G)$ is well known (see [31], [18] for instance). One starts with "non abelian" 1-cocycles g, i.e. a set of continuous maps (also called "transition

functions")

$$g_{ji} : U_i \cap U_j \longrightarrow G,$$

such that $g_{kj} \cdot g_{ji} = g_{ki}$ over $U_i \cap U_j \cap U_k$. Two cocycles g and h are equivalent if there are continuous maps

$$u_i : U_i \longrightarrow G,$$

such that

$$u_j \cdot g_{ji} = h_{ji} \cdot u_i. \qquad (1)$$

The set of equivalence classes is denoted by $H^1(\mathcal{U}; G)$. A covering $\mathcal{V} = (V_s), s \in S$, is a refinement of $\mathcal{U}$ if there is a map $\tau : S \longrightarrow I$ such that $V_s \subset U_{\tau(s)}$. We then have a "restriction map"

$$R_\tau : H^1(\mathcal{U}; G) \longrightarrow H^1(\mathcal{V}; G),$$

assigning to the $g's$ the functions $k = \tau^*(g)$ defined by

$$k_{s,r} = g_{\tau(s),\tau(r)}.$$

It is shown in [18, pg. 48] for instance that the map R_τ is in fact independent of the choice of τ. We then define

$$H^1(X; G) = \underset{\mathcal{U}}{\text{Colim}}\, H^1(\mathcal{U}; G),$$

where $\mathcal{U}$ runs over the "set" of coverings of X.

Now let Z be a subgroup of the centre of G and let $\lambda = (\lambda_{kji})$ be a completely normalized 2-cocycle of $\mathcal{U}$ with values in Z. This means that $\lambda = 1$ if two of the three indices k, j, i are equal and that

$$\lambda_{\sigma(k)\sigma(j)\sigma(i)} = (\lambda_{kji})^{\varepsilon(\sigma)},$$

where σ is a permutation of the indices (k, j, i), with signature $\varepsilon(\sigma)$.

Remark 1.1. One can prove (see [21] for instance) that a Čech cocycle in any dimension is cohomologous to a completely normalized one. Moreover, if every open subset of X is paracompact, any cohomology class may be represented by a completely normalized Čech cocycle.

A λ-twisted 1-cocycle (simply called twisted cocycle if λ is implicit) is then given by transition functions $g = (g_{ji})$ as above, such that

$$g_{ii} = 1, g_{ji} = (g_{ij})^{-1}$$

and

$$g_{kj} \cdot g_{ji} = g_{ki} \cdot \lambda_{kji}$$

over $U_i \cap U_j \cap U_k$. If we compute the product $g_{lk} \cdot g_{kj} \cdot g_{ji}$ in two different ways using associativity, we indeed find that λ should be a 2-cocycle. On the other hand, one can easily show that the function $g_{ij} \cdot g_{jk} \cdot g_{ki}$ is invariant under a circular permutation of the indices and is changed to its inverse if we permute i and k. Since we have $\lambda_{kjk} = 1$, the cocycle λ should be completely normalized.

Two twisted cocycles g and h are equivalent if there are continuous maps $u_i : U_i \longrightarrow G$, such that we have a condition analogous to the above

$$u_j \cdot g_{ji} = h_{ji} \cdot u_i \qquad (1)$$

We define the twisted (non abelian) cohomology $H^1_\lambda(\mathcal{U}; G)$ as the set of equivalence classes.

Proposition 1.2. *Let μ be a 2-cocycle cohomologous to λ, i.e. such that we have the relation*

$$\mu_{kji} = \lambda_{kji} \cdot \eta_{ji} \cdot \eta_{ki}^{-1} \cdot \eta_{kj},$$

for some $\eta = (\eta_{ji})$ with $\eta_{ji} = (\eta_{ij})^{-1}$ and $\eta_{ii} = 1$. Then the map

$$\Theta : H^1_\lambda(\mathcal{U}; G) \longrightarrow H^1_\mu(\mathcal{U}; G),$$

sending (g) to the twisted cocycle (g') given by $g'_{ji} = g_{ji} \cdot \eta_{ji}$, is an isomorphism.

Proof. If we compute $g'_{kj} \cdot g'_{ji}$ we indeed find

$$g'_{kj} \cdot g'_{ji} = g'_{ki} \cdot \lambda_{kji} \cdot \eta_{kj} \cdot \eta_{ji} \cdot (\eta_{ki})^{-1} = g'_{ki} \cdot \mu_{kji}.$$

This shows that the map Θ is well defined. The inverse map is of course given by the correspondence $(g'_{ji}) \longmapsto (g'_{ji} \cdot \eta_{ji}^{-1})$. □

From the previous considerations one may define the following category. The objects are λ-twisted bundles on a covering $\mathcal{U}$, the morphisms between (g_{ji}) and (h_{ji}) being continuous maps (u_i), with the compatibility condition (1). In this category the covering $\mathcal{U}$ is fixed together with the 2-cocycle λ.

However, this category is too rigid for our purposes, since we want to consider covering refinements. The covering $\mathcal{V} = (V_s), s \in S$, is a refinement of $\mathcal{U} = (U_i), i \in I$ if there is a map $\tau : S \longrightarrow I$ such that $V_s \subset U_{\tau(s)}$. This map τ induces a morphism

$$\Theta_\tau : H^1_\lambda(\mathcal{U}; G) \longrightarrow H^1_\mu(\mathcal{V}; G)$$

which is not necessarily an isomorphism. Starting with a twisted cocycle (g_{ji}), its image by Θ_τ is the cocycle (h_{sr}) given by the formula

$$h_{sr} = g_{\tau(s)\tau(r)}.$$

The 2-cocycle associated to h is

$$\mu_{tsr} = g_{\tau(t)\tau(s)} \cdot g_{\tau(s)\tau(r)} \cdot g_{\tau(r)\tau(t)} = \lambda_{\tau(t)\tau(s)\tau(r)}.$$

Proposition 1.3. *Let τ and τ' be two maps from S to I such that $V_s \subset U_{\tau(s)}$ and $V_s \subset U_{\tau'(s)}$ and let x be an element of the set $H^1_\lambda(\mathcal{U}; G)$. Then $\Theta_\tau(x)$ and $\Theta_{\tau'}(x)$ are related through an isomorphism*

$$H^1_\mu(\mathcal{V}; G) \cong H^1_{\mu'}(\mathcal{V}; G),$$

made explicit in the proof below. This isomorphism does not depend on x and depends only on τ, τ' and the 2-cocycle λ.

Proof. Let h' be the following transition functions

$$h'_{sr} = g_{\tau'(s)\tau'(r)}.$$

We may write

$$h'_{sr} = g_{\tau'(s)\tau(s)} \cdot h_{sr} \cdot g_{\tau(r)\tau'(r)} \cdot \sigma_{sr},$$

where

$$\sigma_{sr} = \lambda_{\tau'(s)\tau(r)\tau(s)} \cdot \lambda_{\tau'(s)\tau'(r)\tau(r)}.$$

Since we have $h_{rs} = (h_{sr})^{-1}, h_{rr} = 1$ and the same properties for h', it follows that $\sigma_{rs} = (\sigma_{sr})^{-1}$ and $\sigma_{rr} = 1$. Therefore,

1) the twisted 1-cocycles (h_{sr}) and $(\overline{h}_{sr})$, where

$$\overline{h}_{sr} = g_{\tau'(s)\tau(s)} \cdot h_{sr} \cdot g_{\tau(r)\tau'(r)},$$

are isomorphic in the category of twisted bundles over $\mathcal{V}$ with the same twist.

2) the twisted bundles defined by the 1-cocycles $(\overline{h}_{sr})$ and (h'_{sr}) are also isomorphic through the isomorphism

$$H^1_\mu(\mathcal{V}; G) \cong H^1_{\mu'}(\mathcal{V}; G)$$

defined in the previous proposition.

We note that μ' is the following 2-cocycle with values in Z

$$\mu'_{tsr} = \overline{h}_{ts} \cdot \overline{h}_{sr}. \overline{h}_{rt} = h_{ts} \cdot h_{sr} \cdot h_{rt} \cdot \sigma_{ts} \cdot \sigma_{sr} \cdot \sigma_{rt} = \mu_{tsr} \cdot \sigma_{ts} \cdot \sigma_{sr} \cdot \sigma_{rt},$$

which is of course cohomologous to μ.

It remains to show that the isomorphism

$$H^1_\mu(\mathcal{V}; G) \cong H^1_{\mu'}(\mathcal{V}; G)$$

depends only of τ and τ' and not of the specific element x. The previous identity shows indeed that the 2-cocycles μ and μ' are cohomologous through the completely normalized 1-cochain σ which is a function of λ only. $\square$

Remark 1.4. Although we don't need it in the proof, this computation showing that μ and μ' are cohomologous is based on the existence of a twisted 1-cocycle (g_{ji}) associated to a 2-completely normalised cocycle λ. Unfortunately, this is not true in general. However, when X has the homotopy type of a CW-complex, we may also argue as follows in greater generality. First me may assume that X is pathwise connected, so that we can choose a base point on X. Now let PX be the path space of X and let

$$\pi : PX \to X$$

be the canonical map associating to a path starting at the base point its end point. In order to check that $\sigma_{ts} \cdot \sigma_{sr} \cdot \sigma_{rt} = \mu'_{tsr} \cdot (\mu_{tsr})^{-1}$, we consider the covering of PX defined by the pull-back $\pi^*(\mathcal{U})$ of the covering $\mathcal{U}$ of X. Since the nerve of $\pi^*(\mathcal{U})$ is contractible, there is a completely normalized 1-cochain $\overline{g}$, with values in the subgroup Z of G, such that λ_{tsr} is the associated twist, i.e. its coboundary. This

enables us to perform the previous computations on PX (with $\overline{g}$ as our 1-twisted cocycle) and hence on X, since the pull-back of functions from X to PX by the map π is injective.

2. Relation with torsors

There is another interpretation of twisted principal bundles in some favourable circumstances and which is more familiar. For this, we observe that G acts on itself by inner automorphisms and that the kernel of the map

$$G \to Aut(G)$$

is the centre of G. We now assume that the induced map $G/Z \to Aut(G)$ is a homeomorphism on its image for the quotient topology and that the map

$$G \to G/Z$$

is a locally trivial fibration. In the applications we have in mind, G is a Lie group or a Banach Lie group and is well known that these conditions are fulfilled if Z is a closed subgroup of the centre.

On the other hand, we notice that if P is a twisted principal bundle associated to a covering $\mathcal{U}$ with transition functions g_{ji}, we may define a bundle of groups $\mathrm{AUT}(P)$ as follows. Its transition functions are defined over $U_i \cap U_j$ by

$$g \longmapsto g_{ji} \cdot g \cdot (g_{ji})^{-1} = g_{ji} \cdot g \cdot g_{ij}.$$

Proposition 2.1. *Let $\widetilde{G}$ be a bundle of groups with fibre G and with structural group G/Z, acting by inner automorphisms on G. Then, if the covering $\mathcal{U} = (U_i)$ is fine enough, there is a twisted principal bundle P such that $\widetilde{G}$ is isomorphic to the bundle of groups $\mathrm{AUT}(P)$ defined above.*

Proof. The bundle of groups $\widetilde{G}$ is given by transition functions

$$\gamma_{ji} : U_i \cap U_j \to G/Z,$$

where we may assume without loss of generality that

$$\gamma_{ii} = Id \quad \text{and} \quad \gamma_{ij} = (\gamma_{ji})^{-1}.$$

According to our assumptions, the fibration $G \to G/Z$ is locally trivial. Therefore, if the covering $\mathcal{U}$ is fine enough, we can find continuous functions

$$g_{ji} : U_i \cap U_j \to G$$

such that the class of g_{ji} is γ_{ji}, and moreover $g_{ii} = Id$, $g_{ij} = (g_{ji})^{-1}$. From these identities, it follows that the following continuous function defined on $U_i \cap U_j \cap U_k$

$$\lambda_{kji} = g_{kj} \cdot g_{ji} \cdot g_{ik}$$

is a completely normalized 2-cocycle with values in Z. Therefore, it defines a twisted principal bundle P with transition functions (g_{ji}). Moreover, according to the previous considerations, the bundle of groups $\widetilde{G}$ is canonically isomorphic to $\mathrm{AUT}(P)$, with transition functions

$$u \longmapsto \overline{g}_{ji}(u) = g_{ji} \cdot u \cdot g_{ij}.$$

$\square$

This proposition enables us to relate the category of twisted principal bundles to more classical mathematical objects. We notice that if P and Q are twisted principal bundles with transition functions g_{ji} and h_{ji} respectively (with the same twist λ), we can define a locally trivial bundle $\mathrm{ISO}(P,Q)$ with fibre G, the transition functions being automorphisms of the underlying space G defined by

$$u \longmapsto h_{ji} \cdot u \cdot g_{ij} = \theta_{ji}(u).$$

Since we have $g_{kj} \cdot g_{ji} \cdot g_{ik} = h_{kj} \cdot h_{ji} \cdot h_{ik} = \lambda_{kji}$, the 1-cocycle condition is satisfied for the bundle $\mathrm{ISO}(P,Q)$, i.e. we have the relation

$$\theta_{kj} \cdot \theta_{ji} = \theta_{ki}.$$

In particular, if $P = Q$, we get the previous bundle of groups $\mathrm{AUT}(P)$.

Moreover, there is a bundle map

$$\mathrm{ISO}(P,Q) \times \mathrm{AUT}(P) \to \mathrm{ISO}(P,Q).$$

It is defined by

$$(u,v) \longmapsto u \circ v,$$

or by $(u_i, v_i) \mapsto u_i \circ v_i$ in local coordinates. Therefore, the bundle $\mathrm{ISO}(P,Q)$ inherits a right fibrewise $\mathrm{AUT}(P)$-action which is simply transitive on each fibre. In classical terminology[2], the bundle $\mathrm{ISO}(P,Q)$ is a "torsor" over the bundle of groups $\mathrm{AUT}(P)$, acting on the right.

Theorem 2.2. *Let $\widetilde{G}$ be a bundle of groups with fibre G and structural group G/Z acting on G by inner automorphisms. We assume the existence of a covering $\mathcal{U} = (U_i)$ such that $\widetilde{G}$ may be written as $\mathrm{AUT}(P)$, where P is a λ-twisted principal bundle. Then, any torsor M over $\widetilde{G}$ may be written as $\mathrm{ISO}(P,Q)$, where Q is a λ-twisted principal bundle. More precisely, the correspondence $Q \longmapsto \mathrm{ISO}(P,Q)$ induces an equivalence between the category of λ-twisted principal bundles and the category of $\widetilde{G}$-torsors.*

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[2]It is not the purpose of this paper to develop the theory of torsors. Roughly speaking, this notion is a generalization of the definition of a principal bundle P. Instead of having a topological group G acting on P as usual, we have a bundle of groups $\widetilde{G}$ acting fiberwise on P in a way which is simply transitive on each fiber. In our situation, the structural group of $\widetilde{G}$ is G/Z, acting on G by inner automorphisms.

Proof. Let γ_{ji} be the transition functions of M with fibre G and let g_{ji} be the transition functions of P. Then the transition functions of $AUT(P)$ are given by $\overline{g}_{ji}(u) = g_{ji} \cdot u \cdot g_{ij}$. Now we claim that the transition functions of M should be of type

$$\gamma_{ji}(u) = h_{ji} \cdot u \cdot g_{ij},$$

for some continuous functions h_{ji}. In order to prove this, we use the action of $\widetilde{G}$ on the right by writing

$$\gamma_{ji}(u) = \gamma_{ji}(1 \cdot u) = \gamma_{ji}(1) \cdot \overline{g}_{ji}(u) = \gamma_{ji}(1) \cdot g_{ji}.u \cdot g_{ij}.$$

We then put $h_{ji} = \gamma_{ji}(1) \cdot g_{ji}$. The fact that $\gamma_{kj} \cdot \gamma_{ji} = \gamma_{ki}$ implies the identity

$$h_{kj} \cdot h_{ji} \cdot u \cdot g_{ij} \cdot g_{jk} = h_{ki} \cdot u \cdot g_{ik}.$$

Since $g_{ij} \cdot g_{jk} = g_{ik} \cdot \lambda_{ijk}$, this implies that $h_{kj}.h_{ji} = h_{ki} \cdot (\lambda_{ijk})^{-1} = h_{ki} \cdot \lambda_{kji}$; therefore the (h_{ji}) are the transition functions of a λ-twisted principal bundle.

We have to check the coherence of the action of $\widetilde{G}$ on the right, i.e. the identity

$$\gamma_{ji}(u \cdot v) = \gamma_{ji}(u) \cdot \overline{g}_{ji}(v).$$

This follows from the simple calculation in local coordinates

$$\gamma_{ji}(u \cdot v) = h_{ji} \cdot (u \cdot v) \cdot g_{ij} = (h_{ji} \cdot u \cdot g_{ij}) \cdot (g_{ji} \cdot v \cdot g_{ij}) = \gamma_{ji}(u) \cdot \overline{g}_{ji}(v).$$

The previous computations show that we can define a functor backwards from the category of $\widetilde{G}$-torsors to the category of λ-twisted principal bundles. It remains to prove that the map

$$Hom(Q, Q') \to Hom(\mathrm{ISO}(P,Q), \mathrm{ISO}(P,Q'))$$

is an isomorphism. For this, we analyse the morphisms

$$\mathrm{ISO}(P,Q) \to \mathrm{ISO}(P,Q')$$

which are compatible with the structure of $AUT(P)$-torsor. Such a morphism

$$\mathrm{ISO}(P,Q) \to \mathrm{ISO}(P,Q')$$

is given in local coordinates by the formula

$$\Phi : u \longmapsto \beta_i \cdot u \cdot \alpha_i,$$

where (α_i) (resp. (β_i)) is associated to $AUT(P)$ (resp. $\mathrm{ISO}(Q,Q')$). We notice the formula

$$h'_{ji} \cdot \beta_i \cdot u \cdot \alpha_i \cdot g_{ij} = \beta_j \cdot h_{ji} \cdot u \cdot g_{ij} \cdot \alpha_j,$$

where h'_{ji} are the coordinate functions of Q'. In the same way, an element of $AUT(P)$ is given in local coordinates by

$$\Upsilon : g \longmapsto g \cdot \alpha_i.$$

Therefore, the equation

$$\Phi(u \cdot g) = \Phi(u) \cdot \Upsilon(g)$$

may be written

$$\beta_i \cdot (u \cdot g) \cdot \alpha_i = (\beta_i \cdot u \cdot \alpha_i) \cdot (g \cdot \alpha_i),$$

which is only possible if $\alpha_i = 1$. $\qquad\square$

Remark 2.3. An analog of this theorem in the framework of vector bundles will be proved in the next section (Theorem 3.5).

3. TWISTED VECTOR BUNDLES

One of the main aims of this paper is the theory of "twisted" vector bundles[3]. We essentially studied it in Section 1, with the structural group $G = GL_n(\mathbb{C})$. However, to keep track of the linear structure and because we want the "fibres" not to have the same dimension on each connected component of X, we change slightly the general definition as follows.

We start as before with a covering $\mathcal{U} = (U_i), i \in I$, together with a finite dimensional vector space E_i "over" U_i. Another piece of information is a completely normalized 2-cocycle λ_{kji} with values in $\mathbb{C}^\times$. A λ-twisted vector bundle E on X is then defined by transition functions

$$g_{ji} : U_i \cap U_j \to \text{Iso}(E_i, E_j),$$

such that

$$g_{ii} = 1, g_{ji} = (g_{ij})^{-1}$$

and

$$g_{kj} \cdot g_{ji} = g_{ki} \cdot \lambda_{kji},$$

as in the previous section. There is however a slight change for the definition of morphisms from a twisted vector bundle E to another one F, with the same twist λ. They are defined as continuous maps

$$u_i : U_i \to \text{Hom}(E_i, F_i),$$

such that

$$u_j \cdot g_{ji} = h_{ji} \cdot u_i.$$

The point is that we no longer require the u_i to be isomorphisms.

More generally, let E be a λ-twisted vector bundle on a covering $\mathcal{U}$ with transition functions (g_{ji}) and let F be a μ-twisted vector bundle on the same covering with transition functions (h_{ji}). We define a $\lambda^{-1} \cdot \mu$-twisted vector bundle in the following way: over each U_i we take as "fibre" $\text{Hom}(E_i, F_i)$ and as transition functions the isomorphisms

$$\text{Hom}(E_i, F_i) \to \text{Hom}(E_j, F_j),$$

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[3]For simplicity's sake, we shall only consider complex vector bundles. The theory for real or quaternionic vector bundles follows the same pattern. More generally, we may also consider vector bundles with fibres finitely generated projective modules over a Banach algebra. This remark will be useful in the next section for $\mathcal{A}$-bundles.

defined by

$$\theta_{ji} : f_i \longmapsto h_{ji} \circ f_i \circ g_{ij} = f_j.$$

We denote this twisted vector bundle by $\mathrm{HOM}(E, F)$. An interesting case is when E and F are associated to the same 2-cocycle λ. Then $\mathrm{HOM}(E, F)$ is a genuine vector bundle associated to $Hom(E, F)$ by the following proposition.

Proposition 3.1. *Let E and F be two λ-twisted vector bundles. Then the vector space of morphisms from E to F, i.e. $Hom(E, F)$, may be identified canonically with the vector space of sections of the vector bundle $\mathrm{HOM}(E, F)$.*

Proof. A section of this vector bundle is defined by elements f_i of $Hom(E_i, F_i)$ such that

$$\theta_{ji}(f_i) = f_j.$$

This relation is translated as

$$h_{ji} \circ f_i = f_j \circ g_{ji},$$

which is exactly the definition of morphisms from E to F. $\qquad \square$

An interesting case of the previous proposition is when $E = F$, so that $\mathrm{HOM}(E, E) = \mathrm{END}(E)$ is an algebra bundle A. The following theorem relates algebra bundles to twisted vector bundles.

Theorem 3.2. *Any algebra bundle A with fibre $\mathrm{End}(V)$, where V is a finite dimensional vector space of positive dimension, is isomorphic to some $\mathrm{END}(E)$, where E is a twisted vector bundle on a suitably fine covering of X.*

Proof. Let $V = \mathbb{C}^n$. According to the Skolem-Noether Theorem, the structural group of A is $\mathrm{PGL}_n(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})/\mathbb{C}^\times$, where $\mathrm{PGL}_n(\mathbb{C})$ acts on $M_n(\mathbb{C})$ by inner automorphisms. We may describe this bundle A by transition functions

$$\gamma_{ji} : U_i \cap U_j \to \mathrm{PGL}_n(\mathbb{C}),$$

for a suitable covering $\mathcal{U} = (U_i)$ of X. Without loss of generality, we may assume that $\gamma_{ii} = 1$ and that $\gamma_{ji} = (\gamma_{ij})^{-1}$. On the other hand, the principal fibration

$$\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$$

admits local continuous sections. Therefore, if we choose the covering $\mathcal{U} = (U_i)$ fine enough, we can lift these γ_{ji} to continuous functions

$$g_{ji} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C}).$$

Moreover, we may choose the g_{ji} such that $g_{ii} = 1, g_{ij} = (g_{ji})^{-1}$. Therefore, we have the identity $g_{kj} \cdot g_{ji} = g_{ki} \cdot \lambda_{kji}$, where

$$\lambda_{kji} : U_i \cap U_j \cap U_k \to \mathbb{C}^\times$$

is de facto a completely normalized 2-cocycle. If E is the twisted vector bundle associated to the $g's$, we see that the algebra bundle $\mathrm{END}(E)$ has transition functions which are

$$f \longmapsto g_{ji} \circ f \circ (g_{ji})^{-1},$$

i.e. the inner automorphisms associated to the g_{ji}. $\qquad\qquad\square$

Remark 3.3. We shall assume from now on that the coverings $\mathcal{U}$ we are considering are "good". This means that $\mathcal{U}$ has a finite number of elements and that all possible intersections of elements of $\mathcal{U}$ are either empty or contractible. This is always possible if X is a compact manifold for instance [5], [25]. In the previous theorem, we are then able to replace the words "suitably fine" by "good" since the fibration

$$\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$$

has the homotopy lifting property. In this case, we also have

$$H^*(X) \cong H^*(N(\mathcal{U})), K(X) \cong K(N(\mathcal{U})),$$

etc., where $N(\mathcal{U})$ is the nerve of the covering $\mathcal{U}$. Note that its geometric realization has the homotopy type of X.

Remark 3.4. For most spaces we are considering, good coverings are cofinal: any open covering as a good refinement. This is the case for finite polyedra and, more geometrically, for compact riemannian manifolds with open geodesic coverings [5].

The previous considerations also show that the cohomology class in

$$H^2(X; \mathbb{C}^\times) \cong H^3(X; \mathbb{Z})$$

associated to a twisted vector bundle is a torsion class (assuming that the covering is good as in Remark 3.3). To prove this, we consider the commutative diagram

$$
\begin{array}{ccccccc}
1 \to & \mu_n & \to & \mathrm{U}(n) & \to & \mathrm{PU}(n) & \to 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
1 \to & \mathbb{C}^\times & \to & \mathrm{GL}_n(\mathbb{C}) & \to & \mathrm{PGL}_n(\mathbb{C}) & \to 1
\end{array}
$$

The non abelian cohomologies $H^1(X; PU(n))$ and $H^1(X; PGL_n(C))$ are isomorphic and the coboundary map

$$H^1(X; \mathrm{PU}(n)) \cong H^1(X; \mathrm{PGL}_n(\mathbb{C})) \to H^2(X; \mathbb{C}^\times) \cong H^3(X; \mathbb{Z})$$

factors through $H^2(X; \mu_n)$ (also see Appendix 8.1). Therefore the cocycle (λ_{kji}) defines a torsion class in $H^3(X; \mathbb{Z})$. It is a theorem of Serre [17] that such an element comes from an algebra bundle such as we have described. Later on, we shall show how we can recover the full cohomology group $H^3(X; \mathbb{Z})$ from algebra bundles of infinite dimension, as it was observed by Rosenberg [29].

The following theorem is important for our dictionary relating twisted vector bundles to modules over suitable algebra bundles.

Theorem 3.5. *Let A be an algebra bundle which may be written as* $\mathrm{END}(E)$, *where E is a twisted vector bundle associated to a covering $\mathcal{U}$, transition functions g_{ji} and a completely normalized 2-cocycle λ with values in $\mathbb{C}^{\times}$. Let $\mathcal{E}_{\lambda}(\mathcal{U})$ be the category of λ-twisted vector bundles and $\mathcal{E}^{A}(\mathcal{U})$ be the category of finite dimensional vector bundles trivialized by the covering $\mathcal{U}$, which are right A-modules. Then the functor*

$$\psi : \mathcal{E}_{\lambda}(\mathcal{U}) \rightarrow \mathcal{E}^{A}(\mathcal{U})$$

defined by

$$F \mapsto \mathrm{HOM}(E, F)$$

is an equivalence of categories.

Proof. We first notice that if M, N and P are finite dimensional vector spaces with $M \neq 0$ and if $\Lambda = End(M)$, the obvious map

$$Hom(N, P) \rightarrow Hom_{\Lambda}(Hom(M, N), Hom(M, P))$$

is an isomorphism. Since N is a direct summand of some M^{r}, it is enough to check the statement for $N = M$, in which case it is obvious. This functorial isomorphism at the level of vector spaces may be translated into the framework of twisted vector bundles by the isomorphism

$$Hom(F, G) \xrightarrow{\cong} Hom_{A}(\mathrm{HOM}(E, F), \mathrm{HOM}(E, G)).$$

This shows that the functor Ψ is fully faithful.

On the other hand, we have a canonical isomorphism of vector spaces

$$Hom(M, N) \otimes_{A} M \rightarrow N,$$

defined by $(f, x) \longmapsto f(x)$ which can also be translated in the framework of twisted vector bundles. This shows that if we start with a bundle L which is a right A-module, where A is some $\mathrm{END}(E)$, we can associate to it a twisted vector bundle F by the formula

$$F = L \otimes_{A} E = \Psi'(L).$$

Since $\mathrm{HOM}(E, F) \otimes_{A} E$ is canonically isomorphic to F, ψ' induces a functor going backwards

$$\psi' : \mathcal{E}^{A}(\mathcal{U}) \rightarrow \mathcal{E}_{\lambda}(\mathcal{U}).$$

Finally, there is an obvious isomorphism

$$L \rightarrow \mathrm{HOM}(E, L \otimes_{A} E) = \Psi(\Psi'(L))$$

This shows that the functor Ψ is essentially surjective. $\qquad \square$

This module interpretation enables us to prove the following Theorem.

Theorem 3.6. *Let $\mathcal{U} = (U_{i}), i \in I$, be a good covering of X as in Remark 3.3 and let $\mathcal{V} = (V_{s})$, $s \in S$, be a refinement of $\mathcal{U}$ which is*

also good. Then for any $\tau : S \to I$ such that $V_s \subset U_{\tau(s)}$, the associated restriction map

$$R_\tau : H^1_\lambda(\mathcal{U}; GL_n(\mathbb{C})) \to H^1_{\tau^*(\lambda)}(\mathcal{V}; GL_n(\mathbb{C}))$$

is a bijection.

Proof. Since $\mathcal{U}$ is good, for any completely normalized cocycle λ, one can find a twisted vector bundle E of rank m on $\mathcal{U}$ and $\mathcal{V}$, such that $A = \mathrm{END}(E)$ is a bundle of algebras associated to λ. According to the previous equivalence of categories, the sets $H^1_\lambda(\mathcal{U}; GL_n(\mathbb{C}))$ and $H^1_{\tau^*(\lambda)}(\mathcal{V}; GL_n(\mathbb{C}))$ are in bijective correspondence with the set of A-modules which are locally of type $Hom(\mathbb{C}^m, \mathbb{C}^n)$. With this identification, the restriction map R is just an automorphism of this set. $\square$

Remark 3.7. We may prove the homotopy invariance of the category of twisted vector bundles thanks to this dictionary (at least if X is compact): a twisted vector bundle may be interpreted as a bundle of A-modules, or as a finitely generated projective module over the Banach algebra $\Lambda = \Gamma(X, A)$ of continuous sections of A. It is easy to show that modules over $\Lambda[0,1]$ can be extended from Λ (see e.g. [24]).

4. Twisted K-theory

Let $\mathcal{U}$ be a good covering (Remark 3.3) of a space X and let λ_{kji} be a completely normalized 2-cocycle with values in $\mathbb{C}^\times$. We consider the category of twisted vector bundles associated to $\mathcal{U}$ and to the cocycle λ. This is clearly an additive category which is moreover pseudo-abelian (every projection operator has a kernel). We denote by $K_\lambda(\mathcal{U})$ its Grothendieck group, which is also the K-group of the category of A-modules over X, where $A = \mathrm{END}(E)$, as explained at the end of the previous section. Since this definition is independent from $\mathcal{U}$ up to a non canonical isomorphism (see Appendix 8.3), we shall also call it $K_\lambda(X)$: this is the classical definition of (ungraded) twisted K-theory as detailed in many references, e.g. [15], [2], [23].

In this situation, the cocycle λ has a cohomology class $[\lambda]$ in the torsion subgroup of

$$H^2(X; \mathbb{C}^\times) \cong H^3(X; \mathbb{Z}).$$

as we saw in Section 2. When $[\lambda]$ is not necessarily a torsion class, we should consider "twisted Hilbert bundles" which are defined in the same way as twisted vector bundles but with a fibre which is an infinite dimensional Hilbert space[4] H. It is also more convenient to use the unitary group $U(H)$ instead of the general linear group as our basic

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[4]For simplicity's sake, we assume H to be separable, i.e. isomorphic to the classical l^2 space.

structural group. In other words, the (g_{ji}) in Sections 1 and 2 are now elements of $U(H)$. The 2-cocycle (λ_{kji}) takes its values in the topological group S^1.

From the fibration

$$S^1 \to U(H) \to PU(H)$$

and the contractibility of $U(H)$ (Kuiper's theorem), we see that $PU(H)$ is a model of the Eilenberg-Mac Lane space $K(\mathbb{Z}, 3)$. On the other hand, since $PU(H)$ acts on $\mathcal{L}(H) = \text{End}(H)$ by inner automorphisms, we deduce that any 2-cocycle $\lambda = (\lambda_{kji})$ defines an algebra bundle $\underline{\mathcal{L}}_\lambda$ with fibre $\mathcal{L}(H)$ which is well defined up to isomorphism. Therefore, as in the finite dimensional case, we have the following theorem.

Theorem 4.1. *Let $\underline{\mathcal{L}}_\lambda$ be the bundle of algebras with fibre $\mathcal{L}(H)$ associated to the cocycle λ. Then, if the covering $\mathcal{U}$ is good as in Remark 3.3, $\underline{\mathcal{L}}_\lambda$ may be written as $\text{END}(E)$, where E is a λ-twisted Hilbert bundle.*

Proof. We just copy the proof of Theorem 3.2 in the infinite dimensional case. In a more precise way, the structural group of $\underline{\mathcal{L}}_\lambda$ is $PU(H) = U(H)/S^1$ acting on $\mathcal{L}(H)$ by inner automorphisms. Therefore, we may describe the principal bundle by transition functions

$$\gamma_{ji} : U_i \cap U_j \to PU(H)$$

for a suitable covering $\mathcal{U} = (U_i)$ of X and we may assume that $\gamma_{ii} = 1, \gamma_{ji} = (\gamma_{ij})^{-1}$. Since the principal fibration

$$\pi : U(H) \to PU(H)$$

is locally trivial and the $U_i \cap U_j$ are contractible (if non empty), there are continuous maps

$$g_{ji} : U_i \cap U_j \to U(H),$$

such that $\pi \circ g_{ji} = \gamma_{ji}$. The proof now ends as the proof of Theorem 3.2. $\qquad\square$

Theorem 4.2. *Let $\underline{\mathcal{L}}_\lambda$ be the algebra bundle $\text{END}(E)$, where E is a λ-twisted Hilbert bundle on a covering $\mathcal{U}$. Let $\mathcal{E}_\lambda(\mathcal{U})$ be the category of λ-twisted Hilbert bundles with fibre H and, finally, let $\mathcal{E}^{\underline{\mathcal{L}}_\lambda}(\mathcal{U})$ be the category of bundles which are right $\underline{\mathcal{L}}_\lambda$-modules[5], trivialized over the elements of $\mathcal{U}$. Then, the functor*

$$\Psi : \mathcal{E}_\lambda(\mathcal{U}) \to \mathcal{E}^{\underline{\mathcal{L}}_\lambda}(\mathcal{U}),$$

defined by the formula

$$F \longmapsto \text{HOM}(E, F).$$

is an equivalence of categories.

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[5]More precisely, we assume that locally the module is isomorphic to $\mathcal{L}(H)$, with its standard $\mathcal{L}(H)$-module structure.

Proof. It is also completely analogous to the proof of Theorem 3.5. In a more precise way, instead of considering all finite dimensional vector spaces, we take Hilbert spaces $M, N, P, etc.$ of the same cardinality, i.e. isomorphic to the classical l^2-space. For instance, the isomorphism used in the proof of Theorem 3.5

$$Hom(F, G) \xrightarrow{\cong} Hom_A(\mathrm{HOM}(E, F), \mathrm{HOM}(E, G))$$

is a consequence of the fact that it is true at the level of Hilbert spaces since $Hom(M, N)$ is isomorphic to $End(M) = \mathcal{L}(H)$. The proof of the theorem again ends as in the case of finite dimensional vector spaces. $\square$

For $[\lambda] \in H^3(X; \mathbb{Z}) = H^1(X; PU(H))$ which is not necessarily a torsion class, we may define the associated twisted K-theory in many ways. The first definition is due to Rosenberg [29]: the class $[\lambda]$ is represented up to isomorphism by a principal bundle P with structural group $PU(H)$. Since $PU(H)$ is acting on the ideal of compact operators $\mathcal{K}$ in $\mathcal{L} = \mathcal{L}(H)$ by inner automorphisms, we get an associated bundle $\underline{\mathcal{K}}_\lambda$ of C*-algebras. The twisted K-theory is then the usual K-theory of the algebra of sections of $\underline{\mathcal{K}}_\lambda$. An equivalent way to define $\underline{\mathcal{K}}_\lambda$ is to consider a twisted Hilbert bundle E associated to the cocycle λ (it is unique up to isomorphism). Then, $\underline{\mathcal{K}}_\lambda$ is the subalgebra of the sections of the bundle $\underline{\mathcal{L}}_\lambda = END(E)$ which belong to $\mathcal{K}(H)$ over each open set of $\mathcal{U}$.

One unpleasant aspect of this definition is the non existence of a unit element in $\underline{\mathcal{K}}_\lambda$, which makes its K-theory slightly complicated to handle. However, we may replace $\mathcal{K}$ by the subalgebra $\mathcal{A}$ of $\mathcal{L} \times \mathcal{L}$ consisting of couples of operators (f, g) such that $f - g \in \mathcal{K}$. The group $PU(H)$ is acting on $\mathcal{A}$, so that we may also twist the algebra $\mathcal{A}$ by λ in order to get an algebra bundle $\underline{\mathcal{A}}_\lambda$. The obvious exact sequence of C*-algebras

$$0 \to \mathcal{K} \to \mathcal{A} \to \mathcal{L} \to 0$$

induces an exact sequence of algebra bundles

$$0 \to \underline{\mathcal{K}}_\lambda \to \underline{\mathcal{A}}_\lambda \to \underline{\mathcal{B}}_\lambda \to 0.$$

Here and elsewhere, using a variation of the Serre-Swan theorem, we shall often use the same terminology for an algebra bundle and its associated algebra of continuous sections. In particular the K-theory of $\underline{\mathcal{A}}_\lambda$ is canonically isomorphic to the K-theory of $\underline{\mathcal{K}}_\lambda$ since $\underline{\mathcal{B}}_\lambda$ is a flabby algebra[6] (in particular its K-groups are trivial).

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[6]A Banach algebra A is called flabby if there is a topological A-bimodule M which is projective of finite type as a right module, such that $M \oplus A$ is isomorphic to M. This is equivalent to saying that the Banach category $\mathcal{C} = \mathcal{P}(A)$ is flabby: there is a linear continuous functor τ from $\mathcal{C}$ to itself such that $\tau \oplus Id_\mathcal{C}$ is isomorphic to τ.

A comment is in order to make our previous definition more func-torial: the λ-twisted K-theory is defined precisely as the K-theory of bundles with fibres $\mathcal{A}$-modules which are finitely generated and pro-jective but twisted by the cocycle λ. How this depends only on the cohomology class $[\lambda]$ is discussed in Appendix 8.3. Our Section 3 on twisted vector bundles may now be rewritten by replacing the field of complex numbers $\mathbb{C}$ by the C*-algebra $\mathcal{A}$ and the finite dimensional bundles by "$\mathcal{A}$-bundles" as above. Theorem 3.5 adapted to this situ-ation shows that the category of λ-twisted $\mathcal{A}$-bundles is equivalent to the category of $\underline{\mathcal{A}}_\lambda$-modules if the covering $\mathcal{U}$ of X is good. This shows in particular that the theory of twisted $\mathcal{A}$-bundles is homotopically invariant (at least if X is compact).

However, one has to point out a main difference between $\mathbb{C}$-modules and $\mathcal{A}$-modules: a priori, the fibres of $\mathcal{A}$-bundles are not necessarily free[7]. However, since $K(\mathcal{A})$ is canonically isomorphic to $\mathbb{Z}$, each $\mathcal{A}$-bundle E induces a locally constant function (called the "rank")

$$Rk : X \to \mathbb{Z},$$

obtained by applying the K-functor to each fibre. This correspondence defines a group map

$$Ch_{(0)} : K(\underline{\mathcal{A}}_\lambda) \to H^0(X; \mathbb{Z}).$$

In Section 7 we shall see how to define "higher Chern characters" $Ch_{(m)}$, starting from this elementary step.

In the spirit of Section 1, we may also consider twisted principal G-bundles, where G is the group of invertible elements in the algebra $\mathcal{A}$. We note that the elements of G are couples of invertible operators (g, h) in a Hilbert space such that $g - h$ is compact. We get elements in the centre by considering $g = h \in \mathbb{C}^\times$. More accurately, one should replace $\mathcal{A}$ by the sub-algebra $\mathrm{End}(P)$, where P is a finitely generated projective $\mathcal{A}$-module which is the fibre of the bundles we are considering (assuming the base is connected; otherwise the fibre P may vary). Then G is not exactly $\mathcal{A}^\times$ but the subgroup $\mathrm{Aut}(P)$ of $\mathcal{A}^*$. This point of view will be exploited in Section 7 for the definition of the Chern character, whose target is twisted cohomology.

Finally, there is a third definition of twisted K-theory in terms of Fredholm operators, following the ideas in [1], [19] and [15]. We con-sider the set of homotopy classes of triples

$$(E_0, E_1, D),$$

————————

[7]However, we shall show in Section 7 that the fibres are free modules if the restriction of the cohomology class of λ to every connected component of X is of infinite order.

where E_0 and E_1 are λ-twisted Hilbert bundles on a good covering $\mathcal{U}$ and D is a family of Fredholm operators[8] from E_0 to E_1. With the operation induced by the direct sum of triples, we get a group denoted by $K_\lambda(\mathcal{U})$. We note that $K_\lambda(\mathcal{U})$ is a module over $K(\mathcal{U})$. Here $K(\mathcal{U})$ is a short notation for the usual K-theory of the nerve of $\mathcal{U}$. If $\mathcal{U}$ is good as in Remark 3.3, it is isomorphic to the classical topological K-group $K(X)$.

In order to prove that this last definition is consistent with the previous ones, we consider the Banach category of λ-twisted Hilbert bundles. It is equivalent to the category of bundles of $\underline{\mathcal{L}}_\lambda$-modules, where $\underline{\mathcal{L}}_\lambda$ is the algebra bundle above with fibre $\mathcal{L}(H)$ twisted by λ. Let $\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda$ be the quotient bundle with fibre the Calkin algebra $\mathcal{L}(H)/\mathcal{K}(H)$.

Lemma 4.3. *Let $\overline{D}$ be the class of D as a morphism between the associated $\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda$-modules. Then two triples (E_0, E_1, D) and (E_0', E_1', D') are homotopic if and only if the associated triples $(E_0, E_1, \overline{D})$ and $(E_0', E_1', \overline{D}')$ are homotopic.*

Proof. In general, let us denote also by $\overline{M}$ the class of M as an $\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda$-module. We have a continuous map

$$\mathcal{F}(E_0, E_1) \to \mathrm{Iso}(\overline{E}_0, \overline{E}_1),$$

where the notation $\mathcal{F}$ stands for continuous families of Fredholm maps. According to a classical theorem on Banach spaces, this map admits a continuous section. Therefore, we get a trivial fibration with contractible fibre which is the Banach space of sections of the bundle $\underline{\mathcal{K}}_\lambda$. The proposition follows immediately. $\square$

The philosophy of the lemma is that our third definition of twisted K-theory is equivalent to the Grothendieck group of the Banach functor

$$\varphi : \mathcal{P}'(\underline{\mathcal{L}}_\lambda) \to \mathcal{P}'(\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda),$$

as defined in [24, Section II]. Here the category $\mathcal{P}'(\underline{\mathcal{L}}_\lambda)$ (resp. $\mathcal{P}'(\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda)$) is equivalent to the category of free modules over $\underline{\mathcal{L}}_\lambda$ (resp. $\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda$). Since $K_0(\mathcal{P}'(\underline{\mathcal{L}}_\lambda)) = 0$, this Grothendieck group is canonically isomorphic to $K_0(\underline{\mathcal{K}}_\lambda)$ which is precisely our first definition since, as already mentioned, $\underline{\mathcal{K}}_\lambda$ is the algebra bundle with fibre $\mathcal{K}(H)$ associated to the cocycle λ.

Remark 4.4. Instead of the Grothendieck group of the functor φ, we could as well consider the group $K_1(\underline{\mathcal{L}}_\lambda/\underline{\mathcal{K}}_\lambda)$ which is isomorphic to $K(\varphi)$, since $\underline{\mathcal{L}}_\lambda$ is a flabby ring. We shall use this equivalent description of twisted K-theory in Appendix 8.2.

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[8]We note that $\mathrm{HOM}(E, F)$ is an ordinary bundle with fibre $Hom(H, H) = \mathcal{L}(H)$. The space of Fredholm operators "from E to F" is the subspace of $\mathrm{HOM}(E, F)$ which are Fredholm over each point of X.

Remarks 4.5. *If λ is of finite order, the Fredholm definition of twisted K-theory is detailed in [15, pg. 18]. If $\lambda = 1$, we recover the theorem of Atiyah and Janich [1], [19], in a slightly weaker form.*

As is shown in [15] and [23], there is a $\mathbb{Z}/2$-graded version of twisted K-theory. This version is needed for the Thom isomorphism in the general case of an arbitrary real vector bundle V (which is not necessarily oriented). It is also needed for the Poincaré pairing applied to arbitrary manifolds. We shall concentrate on the case of non torsion classes $[\lambda]$ in the third cohomology group of X. The case when $[\lambda]$ is a torsion class in $H^3(X; \mathbb{Z})$ has been extensively studied in [15].

The essential idea is to replace the previous structural group $U(H)$ by the group $\Gamma(H)$ of matrices in $U(H \oplus H)$ of type

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

or

$$\begin{pmatrix} 0 & h_1 \\ h_2 & 0 \end{pmatrix}.$$

The point here is that $\Gamma(H)$ acts by inner automorphisms on $\mathcal{L}(H \oplus H)$ with a degree shift which is either 0 or 1, the first copy of H being of degree 0 and the second one of degree 1. As in the previous Section, we may give a $\mathbb{Z}/2$-graded module interpretation of twisted Hilbert bundles modelled on $\Gamma(H)$. If E is such a graded twisted Hilbert bundle, $A = \mathrm{END}(E)$ is a bundle of graded algebras with fibre $\mathcal{L}(H \oplus H)$. Conversely, for any bundle of graded algebras A with fibre $\mathcal{L}(H \oplus H)$, there is a twisted Hilbert bundle E with structural group $\Gamma(H)$ such that A is isomorphic to $\mathrm{END}(E)$. According to [15], [23] and our previous computations, these graded algebras are classified by the following cohomology group

$$H^1(X; \mathbb{Z}/2) \times H^3(X; \mathbb{Z}),$$

with a twisted addition rule, as explained in [15, p. 10]. The first invariant in $H^1(X; \mathbb{Z}/2)$ is induced by the map

$$\Gamma(H) \to \mathbb{Z}/2,$$

which describes the type of matrices in $\Gamma(H)$ (diagonal or antidiagonal). The second invariant is defined as before for the underlying ungraded twisted Hilbert bundle.

If we consider the graded tensor product of the twisted Hilbert bundle E by the Clifford algebra $C^{0,1} = \mathbb{C}[x]/(x^2 - 1)$, we get another type of structural group we might call $\Gamma_1(H)$ which is simply $U(H) \times U(H)$. The elements of degree 0 are of type (g, g), while the ones of degree 1 are of type $(g, -g)$. Algebraicallly, this reflects the fact that over the complex numbers there are two types of $\mathbb{Z}/2$-graded Azumaya algebra,

up to graded Morita equivalence, which are $\mathbb{C}$ and $\mathbb{C} \times \mathbb{C}$. For simplicity's sake, in the following discussion, we shall restrict ourselves to the first case which is the group $\Gamma(H)$ above. We note however that for real graded vector bundles, there are eight types of graded algebras (up to graded Morita equivalence) to consider instead of two, as noticed in [15]. They correspond to the Clifford algebras $C^{0,n}$ for $n = 0, 1..., 7$, over the real numbers.

If E and F are two graded twisted Hilbert bundles of structural group $\Gamma(H)$, a morphism (g_i) is of degree 0 (resp. 1) if it is represented locally by a matrix of type

$$\begin{pmatrix} u_i & 0 \\ 0 & v_i \end{pmatrix} \qquad \text{resp.} \qquad \begin{pmatrix} 0 & u_i \\ v_i & 0 \end{pmatrix}.$$

From the previous category equivalences and the definitions in [23], we deduce the following theorem.

Theorem 4.6. *Let λ be a graded twist defined by two cocycles, with classes in $H^1(X; Z/2)$ and $H^3(X; Z)$ respectively. We consider the set of homotopy classes of couples (E, ∇), where E is a λ-twisted graded Hilbert bundle and ∇ a family of self-adjoint Fredholm operators on E which are of degree one. With the operation given by the direct sum of couples, the group obtained is isomorphic to the λ-twisted graded K-theory defined in [23].*

Remark 4.7. One should point out that there is a variant of this Fredholm definition of twisted K-theory on a base X which is locally compact: the family of Fredholm operators ∇ must be an isomorphism outside a compact set (see e.g. [1] or [24]). This remark will be important for the definition of the Thom isomorphism in Section 6.

Remark 4.8. Whatever definition of graded or ungraded twisted K-theory we choose, the group we obtain, denoted by $K_\lambda(X)$ in all cases, may be "derived". One nice way to see this is to notice that we are considering a K-group of special Banach algebras (or $\mathbb{Z}/2$-graded Banach algebras, see [23]), for instance $A = \underline{\mathcal{K}}_\lambda$. We then define $K_\lambda^{-n}(X)$ as $K_n(A)$. By Bott periodicity for complex Banach algebras, we have $K_\lambda^{-n}(X) \cong K_\lambda^{-n-2}(X)$. According to general theorems on K-theory, one shows that

$$K_\lambda^{-n}(X) \cong Coker(K_\lambda(X) \to K_{\pi^*\lambda}(X \times S^n)),$$

where $\pi : X \times S^n \to X$ is the canonical projection. We note here that the smash product $X \wedge S^n$ cannot be used to define $K_\lambda^{-n}(X)$, since there is no associated twist in the cohomology of $X \wedge S^n$ in general.

As a consequence, we may apply Mayer-Vietoris arguments to the direct sum $K_\lambda(X) \oplus K_\lambda^{-1}(X)$, as for the K-theory of general Banach algebras.

5. Multiplicative structures

Since we have defined twisted K-theory in three ways (at least in the non graded case), we should investigate the possible multiplicative structure from these different viewpoints and show that they coincide up to isomorphism. These multiplicative structures were also investigated in a more general framework in [20].

The end result is a "cup-product"

$$K_\lambda(X) \times K_\mu(X) \to K_{\lambda\mu}(X),$$

where λ and μ are two 2-cocycles[9] with values in S^1. Since $K_\lambda(X)$ is the K-theory of the Banach algebra $\underline{\mathcal{K}}_\lambda$ in general, it is enough to define a continuous bilinear pairing between nonunital Banach algebras

$$\varphi : \underline{\mathcal{K}}_\lambda \times \underline{\mathcal{K}}_\mu \to \mathcal{K}_{\lambda\mu},$$

such that $\varphi(aa', bb') = \varphi(a,b)\varphi(a',b')$. The implication that such a φ induces a pairing between K-groups is not completely obvious and relies on excision in K-theory.

To define the pairing φ, we observe that if E_λ is a twisted Hilbert bundle with twist λ and F_μ another one with twist μ, $E\widehat{\otimes}F$ is a twisted Hilbert bundle with twist $\lambda\mu$. Here, the fibres of $E_\lambda\widehat{\otimes}F_\mu$ are the Hilbert tensor product of the fibres of E and F respectively (we implicitly identify the Hilbert tensor product of $H \otimes H$ with H since it is infinite dimensional). Therefore, we have a pairing between Banach bundles

$$\mathrm{END}(E_\lambda) \times \mathrm{END}(F_\mu) \to \mathrm{END}(E_\lambda\widehat{\otimes}F_\mu),$$

which is bilinear and continuous. If we take continuous sections, we deduce the map φ required. We note that φ also induces a ring map

$$\underline{\mathcal{K}}_\lambda\widehat{\otimes}\underline{\mathcal{K}}_\mu \to \mathcal{K}_{\lambda\mu}.$$

The symbol $\widehat{\otimes}$ now denotes the completed projective tensor product of Grothendieck. The inclusion of Banach algebras

$$\underline{\mathcal{K}}_\lambda\widehat{\otimes}\underline{\mathcal{K}}_\mu \subset \mathcal{K}_{\lambda\mu}$$

is not an isomorphism. However, when X varies, both functors define a (twisted) cohomology theory which is the usual K-theory when X is contractible. Therefore, by a standard Mayer-Vietoris argument and Bott periodicity, this inclusion induces an isomorphism on K-groups.

We should note that this cup-product is much simpler to define if $[\lambda]$ and $[\mu]$ are torsion classes in the cohomology. According to Section 3, we may then assume that E and F are finite dimensional twisted vector bundles. The cup-product is simply the usual one[10]

$$K(\underline{A}) \times K(\underline{B}) \to K(\underline{A \otimes B})$$

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[9] See Appendix 8.3 for a possible pairing if we replace λ and μ by their cohomology classes in $H^2(X; S^1) \cong H^3(X; \mathbb{Z})$.

[10] As often, we underline the algebra of sections of the algebra bundles involved.

where $A = \text{END}(E_\lambda)$ and $B = \text{END}(F_\mu)$ are bundles of finite dimensional algebras, with matrix algebras as fibres.

Coming back to the general case, we now use our second definition of twisted K-theory in order to get a cup-product between K-groups of unital rings. According to Section 4, we have exact sequences of Banach algebras

$$0 \to \underline{\mathcal{K}}_\lambda \to \underline{\mathcal{A}}_\lambda \to \underline{\mathcal{B}}_\lambda \to 0$$
$$0 \to \underline{\mathcal{K}}_\mu \to \underline{\mathcal{A}}_\mu \to \underline{\mathcal{B}}_\mu \to 0,$$

which split as exact sequences of Banach spaces. Therefore, we deduce another exact sequence by taking completed projective tensor products of Banach algebras

$$0 \to \underline{\mathcal{K}}_\lambda \widehat{\otimes} \underline{\mathcal{K}}_\mu \to \underline{\mathcal{A}}_\lambda \widehat{\otimes} \underline{\mathcal{A}}_\mu \to \mathcal{D}_{\lambda,\mu} \to 0.$$

The Banach algebra $\mathcal{D}_{\lambda,\mu}$ is the following fibre product

$$\begin{array}{ccc} \mathcal{D}_{\lambda,\mu} & \to & \underline{\mathcal{A}}_\lambda \widehat{\otimes} \underline{\mathcal{B}}_\mu \\ \downarrow & & \downarrow \\ \underline{\mathcal{B}}_\lambda \widehat{\otimes} \underline{\mathcal{A}} & \to & \underline{\mathcal{B}}_\lambda \widehat{\otimes} \underline{\mathcal{B}}_\mu \end{array}$$

Since the algebras $\underline{\mathcal{B}}_\lambda$ and $\underline{\mathcal{B}}_\mu$ are flabby, the algebra $\mathcal{D}_{\lambda,\mu}$ has trivial K-groups. It follows that the map

$$\underline{\mathcal{K}}_\lambda \widehat{\otimes} \underline{\mathcal{K}}_\mu \to \underline{\mathcal{A}}_\lambda \widehat{\otimes} \underline{\mathcal{A}}_\mu$$

is a K-theory equivalence. Therefore, we may also define a cup-product

$$K(\underline{\mathcal{A}}_\lambda) \times K(\underline{\mathcal{A}}_\mu) \to K(\underline{\mathcal{A}}_{\lambda\mu}),$$

as the following composition

$$K(\underline{\mathcal{A}}_\lambda) \times K(\underline{\mathcal{A}}_\mu) \to K(\underline{\mathcal{A}}_\lambda \widehat{\otimes} \underline{\mathcal{A}}_\mu) \cong K(\underline{\mathcal{K}}_\lambda \widehat{\otimes} \underline{\mathcal{K}}_\mu) \overset{\cong}{\to} K(\underline{\mathcal{K}}_{\lambda\mu}) \cong K(\underline{\mathcal{A}}_{\lambda\mu}).$$

If we identity $K(\underline{\mathcal{A}}_\lambda \widehat{\otimes} \underline{\mathcal{A}}_\mu)$ with $K(\underline{\mathcal{A}}_{\lambda\mu})$ by the previous sequence of isomorphisms, we may take as the definition for our cup-product the pairing

$$K(\underline{\mathcal{A}}_\lambda) \times K(\underline{\mathcal{A}}_\mu) \to K(\underline{\mathcal{A}}_\lambda \widehat{\otimes} \underline{\mathcal{A}}_\mu).$$

We now come to the third definition of the cup-product in terms of Fredholm operators. As is well known (see e.g.[1], [15] or [23]), one advantage of this definition of twisted K-theory (for $[\lambda]$ of finite or infinite order) is a handy definition of the cup-product. In the ungraded case, it is more convenient still to view $E = E_1 \oplus E_1$ as a $\mathbb{Z}/2$-graded twisted bundle and replace[11] $D : E_0 \to E_1$ by the following operator ∇ which is self-adjoint and of degree 1:

$$\nabla = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}.$$

---

[11]More correctly, we should write D as a section of the bundle $\text{HOM}(E_0, E_1)$.

The cup-product of (E, ∇) with another couple of the same type (E', ∇') is simply defined by the formula

$$(E, \nabla) \smile (E', \nabla') = (E \widehat{\otimes} E', \nabla \widehat{\otimes} 1 + 1 \widehat{\otimes} \nabla').$$

Here the symbol $\widehat{\otimes}$ denotes the graded and Hilbert tensor product. We notice that if E is associated to the twist λ, E' to the twist λ', the cup-product is associated to the twist $\lambda \cdot \lambda'$, a cocycle whose cohomology class is the sum of the two related cohomology classes in $H^2(X; S^1)$.

It is not completely obvious that this third definition of the cup-product is equivalent to the previous one with the bundles $\underline{\mathcal{K}}_\lambda$ or $\underline{\mathcal{A}}_\lambda$. In order to prove this technical point, we use the results of Appendix 8.2 describing explicitly the isomorphism between $K(\underline{\mathcal{K}}_\lambda)$ and $K_1(\underline{\mathcal{B}}_\lambda / \underline{\mathcal{K}}_\lambda)$. In fact, any element of $K_1(\underline{\mathcal{B}}_\lambda / \underline{\mathcal{K}}_\lambda)$ is the cup-product of an element u of $K(\underline{\mathcal{K}}_\lambda)$ by a generator τ of

$$K_1(\mathcal{L}/\mathcal{K}) \cong \mathbb{Z}.$$

This generator is classically defined by the shift (as a Fredholm operator). Moreover, we may assume that u is induced by a self-adjoint involution on $M_2((\underline{\mathcal{K}}_\lambda)^+)$, where $(\underline{\mathcal{K}}_\lambda)^+$ is the algebra $\underline{\mathcal{K}}_\lambda$ with a unit added. On the other hand, both $K_*(\underline{\mathcal{K}}_\lambda)$ and $K_{1+*}(\underline{\mathcal{B}}_\lambda / \underline{\mathcal{K}}_\lambda)$ may be considered as (twisted) cohomology theories on X. Therefore, again by a Mayer-Vietoris argument, the formula above for the cup-product with Fredholm operators has to be compared with the previous one only when X is reduced to a point, a case which is obvious.

This Fredholm multiplicative setting has the advantage that it may be extended to the graded version of twisted K-theory by the same formula

$$(E, \nabla) \smile (E', \nabla') = (E \widehat{\otimes} E', \nabla \widehat{\otimes} 1 + 1 \widehat{\otimes} \nabla').$$

If $([\lambda_1], [\lambda_3])$ and $([\lambda_1'], [\lambda_3'])$ are the twists of E and E' respectively, the twist of $E \widehat{\otimes} E'$ in cohomology is $([\mu_1], [\mu_3])$, where

$$[\mu_1] = [\lambda_1] + [\lambda_1']$$

and

$$[\mu_3] = [\lambda_3] + [\lambda_3'] + \beta([\lambda_1] \cdot [\lambda_1']).$$

Here $\beta : H^2(X; \mathbb{Z}/2) \to H^3(X; \mathbb{Z})$ is the Bockstein homomorphism (compare with [15, p. 10]). Thanks to the Thom isomorphism which is proved in [23] (see also the next section and [10]), this graded cup-product is compatible with the ungraded one defined on the Thom space of the orientation bundle determined by the graded twist.

6. Thom isomorphism and operations in twisted K-theory

This Section is just a short rewriting of the corresponding sections 4 and 7 of [23], with the point of view of twisted Hilbert bundles. It is added here for completeness' sake.

In order to define the Thom isomorphism in twisted K-theory, as in [23] and [10] with our new point of view, we need to consider twisted Hilbert bundles E with a Clifford module structure. Such a structure is given by a finite dimensional real vector bundle V on X, provided with a positive metric q and an action of V on E, such that $(v)^2 = q(v).1$. Now let λ be a graded twist, given by a covering $\mathcal{U} = (U_i)$ together with a couple (λ_1, λ_3) consisting of a 1-cocycle with values in $\mathbb{Z}/2$ and a 2-cocycle with values in S^1. We define the Grothendieck group $K_\lambda^V(X)$ from the set of homotopy classes of couples

$$(E, \nabla),$$

as follows: E is a $\mathbb{Z}/2$-graded twisted Hilbert bundle which is also a graded $C(V)$-module, V acting by self-adjoint endomorphisms of degree 1. Moreover, the family of Fredholm operators ∇ must satisfy the following properties

1) ∇ is self-adjoint and of degree 1, as in the previous section,
2) ∇ anticommutes with the elements v in V.

This group is not entirely new. Using our dictionary relating twisted Hilbert bundles and module bundles, we described it in great detail in [23, § 4]. We should also notice that this structure of $C(V)$-module may be integrated into the twist λ: if $w_1 = w_1(V)$ and $w_2 = w_2(V)$ are the first two Stiefel-Whitney classes of V, one has to replace λ by the sum of λ and $C(V)$ in the graded Brauer group (this was one of the main motivations for the paper [15]). More precisely, the resulting cohomology classes are

$$[\lambda_1] + w_1(V)$$

in degree one and

$$[\lambda_3] + \beta([\lambda_1] \cdot w_1) + \beta(w_2)$$

in degree 3.

Using our previous reference [23], we are now able to define the Thom isomorphism

$$t : K_\lambda^V(X) \to K_{\pi^*\lambda}(V)$$

in simpler terms. If π denotes the projection $V \to X$, and if (E, ∇) defines an element of the group $K_\lambda^V(X)$, we define $t(E, \nabla)$ as the couple $(\pi^*(E), \nabla')$, where ∇' is defined over a point v of V, with projection x, by the formula

$$\nabla'_v = v + \nabla_x.$$

We recognize here the formula already given in [23]: we have just replaced module bundles by twisted Hilbert bundles.

Operations on twisted K-theory have already been defined in many references [15], [2], [23]. Twisted Hilbert bundles give a nice framework to redefine them. For simplicity's sake, we restrict ourselves to ungraded twisted K-groups.

If we start with an element (E, ∇) defining an element of $K_\lambda(X)$ as at the end of Section 4, its kth-power[12]

$$(E^{\widehat{\otimes}k}, \nabla \widehat{\otimes}...\widehat{\otimes}1 + ... + 1\widehat{\otimes}...\widehat{\otimes}\nabla)$$

has an obvious action of the symmetric group S_k. We should notice that the twist of the kth-power is λ^k. According to Atiyah's philosophy [1], the kth-power defines a map

$$K_\lambda(X) \to K_{\lambda^k}(X) \otimes_{\mathbb{Z}} R(S_k),$$

where $R(S_k)$ denotes the complex representation ring of S_k. Therefore, any $\mathbb{Z}$-homomorphism

$$R(S_k) \to \mathbb{Z}$$

gives rise to an operation in twisted K-theory. In particular, the Grothendieck exterior powers and the Adams operations may be defined in twisted K-theory, using Atiyah's method.

As an interesting $\mathbb{Z}$-homomorphism from $R(S_k)$ to $\mathbb{Z}$, one may choose the map which associates to a complex representation ρ the trace of $\rho(c_k)$, where c_k is the cycle $(1, 2, ..., k)$, a trace which is in fact an integer. The resulting homomorphism

$$K_\lambda(X) \to K_{\lambda^k}(X)$$

is quite explicit. We associate to $F = (E, \nabla)$ the "Gauss sum"

$$\sum (F^{\widehat{\otimes}k})_n \otimes \omega^n$$

in the group $K_{\lambda^k}(X) \otimes_{\mathbb{Z}} \Omega_k$, where Ω_k is the ring of k-cyclotomic integers. In this sum, ω is a primitive k^{th}-root of unity. The element $(F^{\widehat{\otimes}k})_n$ is the eigenmodule associated to the eigenvalue ω^n of a generator of the cyclic group C_k acting on $F^{\widehat{\otimes}k}$. This sum belongs in fact to $K_{\lambda^k}(X)$, as a subgroup of $K_{\lambda^k}(X) \otimes \Omega_k$. As shown by Atiyah [1], we get this way a nice alternative definition of the Adams operation Ψ^k.

Remark 6.1. If the class of λ in $H^3(X; \mathbb{Z})$ is of finite order, it is not necessary to consider twisted Hilbert bundles and Fredholm operators. One just deal with finite dimensional twisted vector bundles as in Section 3.

Remark 6.2. One should notice that operations are much more delicate to define in graded twisted K-theory, even for coefficients $[\lambda]$ of finite order in $H^3(X; \mathbb{Z})$. This was pointed out in [15] and recalled in [23]. Fredholm operators were already introduced in [15] in order to deal with this problem, before subsequent works on twisted K-theory.

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[12]where the symbol $\widehat{\otimes}$ denotes again the graded Hilbert tensor product.

7. Connections and the Chern homomorphism

Let us now assume that X is a manifold. The previous definitions make sense in the differential category. The fact that we get the same K-groups is more or less standard and relies on arguments going back to Steenrod [31]. As an illustrative example, the Čech cohomologies $H^1(X; GL_n(\mathbb{C}))$ and $H^1(X; PGL_n(\mathbb{C}))$ may be computed with differential cochains. Therefore the classification of topological algebra bundles (with fibre $M_n(C)$) is the same in the differential category. The same general result is true for module bundles and therefore for twisted K-theory, if we choose differential 2-cocycles λ with values in S^1 to parametrize the twisted K-groups.

In the differential category, the definition of the Chern homomorphism between twisted K-theory and "twisted cohomology" was given in many papers [3], [27], [8], [32], [11], and [4]. Our method is more elementary and is based on the classical definitions of Chern-Weil theory applied to twisted bundles[13]. We start with twisted finite dimensional bundles which are easier to handle. However, as we shall see later on, the same method may be applied to infinite dimensional bundles in the spirit of Section 4.

Let E be a twisted vector bundle of rank[14] n, defined on a covering $\mathcal{U}$ $= (U_i)$ by transition functions (g_{ji}), with the twisted cocycle condition

$$g_{ki} = g_{kj} \cdot g_{ji} \cdot \lambda_{kji},$$

as in Section 3. We assume that all functions are of class C^∞, which does not change the classification problem for twisted bundles as we have seen previously.

Definition 7.1. A connection Γ on E is given by $(n \times n)$-matrices Γ_i of 1-differential forms on U_i such that on $U_i \cap U_j$ we have the relation

$$\Gamma_i = g_{ji}^{-1} \cdot \Gamma_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} + \omega_{ji}.1.$$

Here ω_{ji} is a 1-differential form related to the λ_{kji} by the following relation

$$\omega_{ji} - \omega_{ki} + \omega_{kj} = \lambda_{kji}^{-1} \cdot d\lambda_{kji}.$$

Moreover, from the relation above with the Γ's, we deduce that $\omega_{ij} = -\omega_{ji}$. If we take the differential of the previous relation, we also get

$$d\omega_{ji} - d\omega_{ki} + d\omega_{kj} = 0.$$

In the applications below, ω will be a differential form with values in $i\mathbb{R}$, where[15] $i = \sqrt{-1}$ (if the g_{ji} are unitary operators).

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[13]For the classical computations, we refer to the books [26, pg. 78] and [22] for instance.

[14]The rank may vary above different connected component of X.

[15]The two different meanings of the symbol "i" are clear from the context.

Example 7.2. (which shows the existence of such connections). Let (α_k) be a partition of unity associated to the covering $\mathcal{U}$. We then consider the "barycentric connection" defined by the formula

$$\Gamma_i = \sum_k \alpha_k \cdot g_{ki}^{-1} \cdot dg_{ki}.$$

Since $g_{ki} = g_{kj} \cdot g_{ji} \cdot \lambda_{kji}$, we have the following expansion

$$g_{ki}^{-1} \cdot dg_{ki} = g_{ji}^{-1} \cdot (g_{kj}^{-1} \cdot dg_{kj}) \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} + \lambda_{kji}^{-1} \cdot d\lambda_{kji}.$$

Therefore, on $U_i \cap U_j$ we have the expected identity

$$\Gamma_i = g_{ji}^{-1} \cdot \Gamma_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} + \omega_{ji} \cdot 1,$$

where

$$\omega_{ji} = \sum_k \alpha_k \cdot \lambda_{kji}^{-1} \cdot d\lambda_{kji}.$$

Remark 7.1. It is clear from the definition that the space of connections on E is an affine space: if Γ and ∇ are two connections on E, for any real number t, $(1 - t)\Gamma + t\nabla$ is also a connection.

We have choosen a definition of a connexion in terms of "local coordinates". However, we have to check how connections correspond when we change them. In other terms, let (α) be an isomorphism from the coordinate bundle (h) to (g) as in Section 1. According to Formula(1), we then have the relation

$$g_{ji} \cdot \alpha_i = \alpha_j \cdot h_{ji}$$

Associated to this morphism, we define the pull back $\alpha^*(\Gamma)$ of the connection (Γ) as locally defined on the coordinate bundle (h) by the formula

$$\nabla_i = \alpha_i^{-1} \cdot \Gamma_i \cdot \alpha_i + \alpha_i^{-1} \cdot d\alpha_i$$

In order for this to make sense, we have to check the relation

$$\nabla_i = h_{ji}^{-1} \cdot \nabla_j \cdot h_{ji} + h_{ji}^{-1} \cdot dh_{ji} + \omega_{ji}.1.$$

which is slightly tedious. We start from the formula

$$\Gamma_i = g_{ji}^{-1} \cdot \Gamma_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} + \omega_{ji}.1,$$

where we replace g_{ji} by $\alpha_j \cdot h_{ji} \cdot \alpha_i^{-1}$. We also replace dg_{ji} by

$$dg_{ji} = d\alpha_j \cdot h_{ji} \cdot \alpha_i^{-1} + \alpha_j \cdot dh_{ji} \cdot \alpha_i^{-1} - \alpha_j \cdot h_{ji} \cdot \alpha_i^{-1} \cdot d\alpha_i \cdot \alpha_i^{-1}.$$

We then get

$$\nabla_i = \alpha_i^{-1} \cdot (g_{ji}^{-1} \cdot \Gamma_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} + \omega_{ji}.1.) \cdot \alpha_i + \alpha_i^{-1} \cdot d\alpha_i$$

$$= \alpha_i^{-1} \cdot (g_{ji}^{-1} \cdot \Gamma_j \cdot g_{ji}) \cdot \alpha_i$$

$$+ \alpha_i^{-1} \cdot g_{ji}^{-1} \cdot (d\alpha_j \cdot h_{ji} \cdot \alpha_i^{-1} + \alpha_j \cdot dh_{ji} \cdot \alpha_i^{-1} - \alpha_j \cdot h_{ji} \cdot \alpha_i^{-1} \cdot d\alpha_i \cdot \alpha_i^{-1}) \cdot \alpha_i$$

$$+ \omega_{ji}.1.$$

$$= h_{ji}^{-1} \cdot (\nabla_j - \alpha_j^{-1} \cdot d\alpha_j) \cdot h_{ji}$$

$$+ h_{ji}^{-1} \cdot \alpha_j^{-1} \cdot d\alpha_j \cdot h_{ji} + h_{ji}^{-1} \cdot dh_{ji} - \alpha_i^{-1} \cdot d\alpha_i + \alpha_i^{-1} \cdot d\alpha_i + \omega_{ji}.1.$$

$$= h_{ji}^{-1} \cdot \nabla_j \cdot h_{ji} + h_{ji}^{-1} \cdot dh_{ji} + \omega_{ji}.1,$$

which is the expected formula.

The "local curvatures" R_i associated to the Γ_i are given by the usual formula[16]

$$R_i = d\Gamma_i + (\Gamma_i)^2.$$

Unfortunately, the traces of these local curvatures do not agree on $U_i \cap U_j$, since a simple computation as above leads to the relation

$$R_i = g_{ji}^{-1} \cdot R_j \cdot g_{ji} + d\omega_{ji}.1.$$

However, using a partition of unity (α_i), as in the case of the barycentric connection, we may define a family of "twisted curvatures" by the following formula, where $m = 1, 2, ...$

$$R_{(m)} = \sum_i \alpha_i \cdot (R_i)^m.$$

We now define a family of "Chern characters" $Ch_{(m)}(E, \Gamma)$ as

$$Ch_{(m)}(E, \Gamma) = Tr(R_{(m)}).$$

We should notice that $Ch_{(m)}(E, \Gamma)$ belongs to the vector space of differential forms with values in $(i)^m \mathbb{R}$, since the g_{kl} are unitary matrices. By convention, we put

$$Ch_{(0)}(E, \Gamma) = n.$$

The differential of $Ch_{(1)}$ is

$$d(Ch_{(1)}(E, \Gamma)) = \sum_i \alpha_i \cdot .Tr(dR_i) + \sum_i d\alpha_i \cdot Tr(R_i).$$

It is well known (and easy to prove) that

$$Tr(dR_i) = Tr(d\Gamma_i \cdot \Gamma_i - \Gamma_i \cdot d\Gamma_i) = 0.$$

On the other hand, the relation between R_i and R_j above leads to the following identity between differential forms on U_j :

$$\sum_i d\alpha_i \cdot Tr(R_i) = (\sum_i d\alpha_i \cdot Tr(R_j)) + n \sum_i d\alpha_i \cdot d\omega_{ji} = n \sum_i d\alpha_i \cdot d\omega_{ji}.$$

The 3-differential form $\theta_j = \sum_k d\alpha_k \cdot d\omega_{jk}$ is clearly closed on U_j. Moreover, on $U_i \cap U_j$ we have

$$\theta_j - \theta_i = \sum_k d\alpha_k \cdot (d\omega_{jk} - d\omega_{ik}) = \sum_k d\alpha_k \cdot d\omega_{ji} = 0,$$

---

[16]As in classical Chern-Weil theory, one may also write $1/2\,[\Gamma_i, \Gamma_i]$ instead of $(\Gamma_i)^2$.

according to the relation above between various $d\omega's$. Therefore, the $\{\theta_i\}$ define a global 3-differential form θ on the manifold X with values in $i\mathbb{R}$. This 3-cohomology class is the opposite of the image of λ by the connecting homomorphism[17]

$$H^2(X; S^1) \to H^3(X; 2i\pi\mathbb{Z}) \cong H^3(X; \mathbb{Z}),$$

associated to the classical exact sequence of coefficient groups

$$0 \to 2i\pi\mathbb{Z} \to i\mathbb{R} \overset{\exp}{\to} S^1 \to 0.$$

Summarizing the above discussion, we get our first relation

$$d(Ch_{(1)}(E,\Gamma)) = n \cdot \theta.$$

Analogous computations can be made with $R_{(2)}, R_{(3)}$, etc.. For an arbitrary m,

$$d(Ch_{(m)}(E,\Gamma)) = \sum_i \alpha_i \cdot Tr(d(R_i)^m) + \sum_i d\alpha_i \cdot Tr(R_i)^m.$$

Since $Tr(d(R_i)^m) = 0$ for the same reasons as above, we have

$$d(Ch_{(m)}(E,\Gamma)) = \sum_i d\alpha_i \cdot Tr(R_i)^m.$$

On the other hand, from the relation

$$Tr(R_i)^m = Tr(R_j)^m + mTr(R_j)^{m-1} \cdot d\omega_{ji},$$

we deduce the following identity between differential forms on U_j :

$$\sum_i d\alpha_i \cdot Tr(R_i)^m = \sum_i d\alpha_i \cdot Tr(R_j)^m + \sum_i m \cdot d\alpha_i \cdot Tr(R_j)^{m-1} \cdot .d\omega_{ji}$$

$$= m \cdot Tr(R_j)^{m-1} \cdot \theta.$$

Therefore,

$$\sum_i d\alpha_i \cdot Tr(R_i)^m = \sum_j \alpha_j \sum_i d\alpha_i \cdot Tr(R_i)^m$$

$$= \sum_j m \cdot \alpha_j \cdot Tr(R_j)^m \cdot \theta = m \cdot Ch_{(m-1)}(E,\Gamma) \cdot \theta.$$

Summarizing again, we get the relation

$$d(Ch_{(m)}(E,\Gamma)) = m \cdot Ch_{(m-1)}(E,\Gamma) \cdot \theta.$$

We now define the total Chern character of (E,Γ) with values in the even de Rham forms[18]

$$\Omega^0(X) \oplus \Omega^2(X) \oplus ... \oplus \Omega^{2m}(X) \oplus ...$$

---

[17]See Appendix 8.1 for a proof of this fact.

[18]More precisely, $\Omega^{2k}(X)$ is the vector space of $2k$-differential forms with values in $(i)^k\mathbb{R}$.

by the following formula:

$$Ch(E, \Gamma) = Ch_{(0)}(E, \Gamma) + Ch_{(1)}(E, \Gamma) + \frac{1}{2!}Ch_{(2)}(E, \Gamma) + ...$$

$$+ \frac{1}{m!}Ch_{(m)}(E, \Gamma) + ...$$

We have choosen the coefficients in front of the $Ch_{(m)}$ such that $Ch(E, \Gamma)$ is a cycle in the even/odd de Rham complex with the differential given by $D = d - .\theta$, where $.\theta$ is the map defined by the cup-product with θ.

In Appendix 8.3, we prove by classical considerations that this total Chern character is well defined as a twisted cohomology class and does not depend on the connection Γ and on the partition of unity. This remark is also valid with the infinite dimensional case which will be studied later on.

For the time being, since we consider finite dimensional bundles, the class θ in $H^3(X; i\mathbb{R})$ is reduced to 0. Therefore, by classical considerations on complexes using exponentials of even forms [3], we see that the target of this special Chern character reduces to the classical one. Moreover, we may also consider twisted bundles over $X \times S^1$, which enables us to define a Chern character from odd twisted K-groups to odd twisted cohomology. From standard Mayer-Vietoris arguments and Bott periodicity, we deduce that the Chern character induces an isomorphism between $K_\lambda(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $H^{even}(X; \mathbb{R})$.

We want to extend the previous considerations to the case when the cohomology class $[\lambda]$ is of infinite order. For this, we use the second definition of twisted K-theory in terms of twisted principal bundles associated to the group G of couples (g, h) such that g and h are invertible operators in $\mathcal{L}(H)$ with $g - h$ compact. However, in order to be able to take traces, we have to modify slightly this group by assuming moreover that $g - h$ is a trace class operator, i.e. belongs to L^1. By abuse of notation, we still call $\mathcal{A}$ the algebra of couples $(g, h) \in \mathcal{L} \times \mathcal{L}$ such that $g - h \in L^1$. Using the classical density theorem[19] in topological K-theory [24, pg. 109], it is easy to show that we get the same twisted K-theory as for $g - h$ compact. We may also choose the transition functions to be C^∞, as we did in the finite dimensional case.

The computations in the finite dimensional case may now be easily transposed in this framework if we consider transition functions (g_{ji}, h_{ji}) in the group[20] $G = \mathcal{A}^*$ and take "supertraces" instead of traces. We just have to be careful that the fibres of our bundles are

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[19]The norm of an element (g, h) is the sum of the operator norm on g and the L^1-norm on $g - h$.

[20]More precisely, in the group $Aut(P) \subset G = \mathcal{A}^*$; see below.

not necessarily free[21]. Concretely, we define a rank map

$$Rk = Ch_{(0)} : K_\lambda(X) \to H^0(X; \mathbb{Z})$$

as follows: if E is a finitely generated projective module over $\underline{\mathcal{A}}_\lambda$, it is defined by a family of two projection operators (p_0, p_1) in the algebra $\underline{\mathcal{A}}_\lambda$. Then the trace of $p_0 - p_1$ is a locally constant integer, defining the rank function, since

$$K(\mathcal{A}) \cong K(\mathcal{K}) \cong K(\mathbb{C}) = \mathbb{Z}.$$

If we look at E as a twisted $\mathcal{A}$-bundle over X with fibre P (which is a finitely generated projective $\mathcal{A}$-module), we may consider $\mathrm{End}(P)$ as included in $M_n(\mathcal{A}) \cong \mathcal{A}$ and restrict the supertrace defined on $\mathcal{A}$ to $\mathrm{End}(P)$. For instance, the suspertrace of the identity on P is just the rank of P. By abuse of notations, we shall identify $\mathrm{End}(P)$ and its image in $\mathcal{A}$.

We now define a connection on E as a family of differential forms $\Gamma_i = (\Gamma_i^0, \Gamma_i^1)$ with values in $\mathrm{End}(P) \subset \mathcal{A} \subset \mathcal{L} \times \mathcal{L}$, such that $\Gamma_i^0 - \Gamma_i^1$ is a differential form with values in L^1, satisfying the same compatibility condition as above

$$\Gamma_i = g_{ji}^{-1} \cdot \Gamma_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji} + \omega_{ji}.1.$$

We choose the transition functions $g_{ji} = (g_{ji}^0, g_{ji}^1)$ to be in $\mathrm{Aut}(P)$ rather than $\mathrm{GL}_n(\mathbb{C})$. Such connections exist, for instance the barycentric connection considered in the finite dimensional case

$$\Gamma_i = \sum \alpha_k \cdot g_{ki}^{-1} \cdot dg_{ki},$$

where (α_k) is a partition of unity associated to the covering. The only difference with the finite dimensional case is that n is replaced by $Rk(E) = Ch_{(0)}(E)$ and the usual trace by the supertrace[22]. If we denote by str this supertrace, we have define:

$$Ch(E, \Gamma) = Ch_{(0)}(E) + \sum_{m=1}^{\dim(X)/2} \frac{1}{m!} str(\sum_i \alpha_i(R_i)^m),$$

where the R_i are the local curvatures as functions of the Γ_i defined above, and where (α_i) is a partition of unity associated to the given covering $\mathcal{U}$.

The computations made before in the finite dimensional case show as well that $Ch(E, \Gamma)$ is a cocycle for the differential $D = d - .\theta$. As in the finite dimensional case, standard homotopy arguments also show that

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[21]As we mentioned already in Section 4, the fibres should be free if λ does not define a torsion class in the cohomology of each connected component of X; see below.

[22]Note again that the supertrace of "1" is the rank of P which is positive or negative.

the cohomology class of $Ch(E, \Gamma)$ is independent from the connection Γ and from the partition of unity (α_i) (see Appendix 8.3 for the details).

Therefore, for any λ, the Chern character induces an isomorphism between $K_\lambda(X) \otimes_Z \mathbb{R}$ and the twisted cohomology which is the cohomology of the even part of the even/odd de Rham complex[23] with the twisted differential $D = d - .\theta$. It is proved in [3], in a computation involving again the exponential of even forms, that this twisted cohomology depends only on the class of θ in the cohomology group $H^3(X; i\mathbb{R})$.

Summarizing the previous discussion, we get the following theorem:

Theorem 7.2. *Let $\mathcal{U}$ be a good covering of X, λ be a completely normalized 2-cocycle with values in S^1 associated to this covering. Let (α_i) be a partition of unity associated to this covering and let θ be the 3-differential form associated to $-\lambda$, according to Appendix 8.1. Then the Chern character*

$$Ch : K_\lambda(X) \to H_\theta^{ev}(X; \mathbb{R})$$

from twisted K-theory to even twisted cohomology induces an isomorphism

$$K_\lambda(X) \otimes_Z \mathbb{R} \cong H_\theta^{ev}(X; \mathbb{R}).$$

Remark 7.3. The functoriality of the Chern character is discussed in Appendix 8.3. Its multiplicative properties will be studied later (Theorem 7.2).

Remark 7.4. One may also normalize the Chern character by putting a factor $(1/2\pi i)^r$ in front of $Ch_{(r)}(E, \Gamma)$ and replace θ by $\theta/2\pi i$. Then we have to consider the usual de Rham complex, contrarily to our convention in the Note 23.

If the space X is formal in the sense of rational homotopy theory [16], we may replace the de Rham complex by its cohomology viewed as a graded vector space (with the differential reduced to 0). In that case, the (even) twisted cohomology is isomorphic to the even part of the cohomology of the complex

$$\left[\oplus H^{2k}(X; (i)^k \mathbb{R}) \right] \oplus \left[\oplus H^{2k+1}(X; (i)^k \mathbb{R}) \right],$$

with the differential given by the cup-product with the cohomology class of θ in $H^3(X; i\mathbb{R})$. By a well known and deep theorem of Deligne, Griffith, Morgan and Sullivan [13], this computation is valid when X is a simply connected compact Kähler manifold.

In the particular case when θ is not 0 in all the cohomology groups $H^3(X_r; i\mathbb{R})$, where the X_r are the connected components of X, we see

---

[23]Note that $\Omega^{2k}(X)$ and $\Omega^{2k+1}(X)$ are the real vector spaces of differential forms of degree $2k$ or $2k+1$ with values in $(i)^k \mathbb{R}$. The differential is the usual one d on $\Omega^{2k}(X)$ and *id* on $\Omega^{2k+1}(X)$.

by a direct computation that $Ch_{(0)}(E, \Gamma)$ is necessarily 0, which implies that the fibres of E should be free $\mathcal{A}$-modules. This also implies that $Ch_{(1)}(E, \Gamma)$ is a closed differential form. Therefore, for any λ, one can define the first Chern character[24] $Ch_{(1)}(E, \Gamma)$ in the (non twisted) cohomology group $H^2(X; i\mathbb{R})$. However, we need the twisted differential cycles for the total Chern character of E.

Let now $\mathcal{U} = (U_i)$ and $\mathcal{V} = (V_j)$ be a covering of X and Y respectively. Let (α_i) (resp. (β_j)) be a partition of unity associated to $\mathcal{U}$ (resp. $\mathcal{V}$). The products $(\alpha_i \cdot \beta_j)$ define a partition of unity associated to the covering $\mathcal{W} = (U_i \times V_j)$ of $X \times Y$.

Theorem 7.5. *Let E be a λ-twisted $\mathcal{A}$-bundle on X and let F be a μ-twisted $\mathcal{A}$-bundle on Y. Here λ and μ are explicit Čech cocycles λ_{tsr} and μ_{wvu} with values in S^1, associated to the coverings $\mathcal{U}$ and $\mathcal{V}$ respectively. Let $\overline{\lambda}$ and $\overline{\mu}$ be the closed differential forms defined on each $U_i \times V_j$ by the formulas*

$$\overline{\lambda} = \sum_{t,s} d\alpha_t \cdot d\alpha_s \cdot \lambda_{tsi}^{-1} \cdot d\lambda_{tsi}$$

$$\overline{\mu} = \sum_{w,v} d\beta_w \cdot d\beta_v \cdot \mu_{wvj}^{-1} \cdot d\mu_{wvj},$$

as in Appendix 8.1. Then we have the commutative diagram[25]

$$
\begin{array}{ccc}
K_\lambda(X) \times K_\mu(Y) & \rightarrow & K_{\lambda\mu}(X \times Y) \\
\downarrow & & \downarrow \\
H_{\overline{\lambda}}^{ev}(X) \times H_{\overline{\mu}}^{ev}(Y) & \longrightarrow & H_{\overline{\lambda}+\overline{\mu}}^{ev}(X \times Y)
\end{array}
$$

Proof. Let $\Gamma = (\Gamma_i)$ (resp. $\nabla = (\nabla_j)$) be a connection on E (resp. F). Then $\Delta = \Gamma \otimes 1 + 1 \otimes \nabla$ is a connection on $E \otimes F$. Therefore, if R_E (resp. R_F) is the curvature associated to Γ (resp. ∇),

$$R_{E \otimes F} = R_E \otimes 1 + 1 \otimes R_F$$

is the curvature associated to Δ over each open subset $U_i \times V_j$ of $X \times Y$. Using the partition of unity $(\alpha_i \cdot \beta_j)$ associated to the covering $(U_i \times V_j)$ and the binomial identity, we find the relation

$$\frac{1}{m!} Ch_{(m)}(E \otimes F, \Delta) = \sum_{p+q=m} \frac{1}{p!q!} Ch_{(p)}(E, \Gamma) Ch_{(q)}(F, \nabla),$$

from which the theorem follows. $\square$

---

[24]which is also a Chern class.

[25]According to the computations in Section 5, we identify the K-theory of $\mathcal{A}_\lambda \widehat{\otimes} \mathcal{A}_\mu$ with the K-theory of $\mathcal{A}_{\lambda\mu}$. However, in these computations, one has to replace $\mathcal{K}$ by the ideal L^1 of trace class operators.

Finally, we should add a few words concerning graded twisted K-theory which is indexed essentially by elements

$$\left[\widetilde{\lambda}\right] \in H^1(X; \mathbb{Z}/2) \times H^2(X; S^1).$$

If we apply Theorem 4.4 of [23], this group (at least rationally) is isomorphic to the ungraded twisted K-theory of Y, where Y is the Thom space of the orientation real line bundle L. This L corresponds to the image of $\left[\widetilde{\lambda}\right]$ in $H^1(X; \mathbb{Z}/2)$. In more precise terms, the graded twisted K-group tensored with the field of real numbers is isomorphic to the odd twisted relative cohomology group of the pair (P, X). Here $P = P(L \oplus 1)$ denotes the real projective bundle of $L \oplus 1$ (with fibre $P^1 \cong S^1$), and the 3-dimensional cohomology twist is induced by the projection $P \to X$ from the one on X. This (graded) twisted cohomology is different in general from the twisted cohomology associated to the image of $\left[\widetilde{\lambda}\right]$ in $H^3(X; i\mathbb{R})$. This is not surprising since the usual real cohomology of a manifold with a coefficient system in $H^1(X; \mathbb{Z}/2)$ also depends on this system.

Remark 7.6. If A is not a commutative Banach algebra, there is no internal product

$$K_n(A) \times K_p(A) \to K_{n+p}(A)$$

in general. Therefore, it is remarkable that such a product exists for twisted K-groups which are $K_*(\underline{\mathcal{K}}_\lambda)$, where $\underline{\mathcal{K}}_\lambda$ is a noncommutative Banach algebra..

8. Appendix

8.1. Relation between Čech cohomology with coefficients in S^1 and de Rham cohomology.
This section does not claim any originality. It may be easily deduced from the classical books [5], [25] for instance, the basic ideas going back to André Weil. It is added for completeness' sake and a normalization purpose.

Our first task is to make more explicit the cohomology isomorphism

$$H^r(\mathcal{U}) \cong H^r_{dR}(X),$$

where $\mathcal{U}$ is a good covering of X. The Čech and de Rham cohomologies are here taken with coefficients in a real vector space of finite dimension V.

Let us denote by $\Omega^r(X)$ the vector space of differential forms on X with values in V and let (α_i) be a partition of unity associated to the covering $\mathcal{U}$. We define a morphism[26]

$$f_r : C^r(\mathcal{U}; V) \to \Omega^r(X)$$

---

[26]With V provided with the discrete topology.

in the following way. For $r = 0$, we send a cochain (c_i) to the C^∞ function

$$x \longmapsto \sum \alpha_i(x) \cdot c_i,$$

which we simply write $\sum\limits_i \alpha_i \cdot c_i$. For general $r > 0$, we send the r-cochain $(c_{i_0 i_1 \dots i_r})$ to the sum

$$\sum_{(i_0,\dots,i_r)} \alpha_{i_0}.d\alpha_{i_1} \cdot \dots d\alpha_{i_r} \cdot c_{i_0 i_1 \dots i_r}.$$

We have to check that this correspondence is compatible with the coboundaries, i.e. that

$$f_{r+1}(\partial c) = d(f_r(c)).$$

The cochain ∂c, which we call v, is defined by the usual formula

$$v_{i_0 i_1 \dots i_{r+1}} = \sum_{m=0}^{r+1} (-1)^m c_{i_0 \dots \widehat{i_m} \dots i_{r+1}}.$$

Therefore,

$$f_{r+1}(v) = \sum_{(i_0,\dots,i_{r+1})} \alpha_{i_0}.d\alpha_{i_1} \cdot \dots d\alpha_{i_{r+1}} \cdot \sum_{m=0}^{r+1} (-1)^m c_{i_0 \dots \widehat{i_m} \dots i_{r+1}}.$$

In the previous sum, the terms corresponding to an index $m > 0$ are reduced to 0 since the sum of the corresponding $d\alpha$ is 0. The previous identity may then be written

$$f_{r+1}(v) = \sum_{(i_0,\dots,i_{r+1})} \alpha_{i_0}.d\alpha_{i_1} \cdot \dots d\alpha_{i_{+1}} \cdot c_{i_1 \dots i_{r+1}}$$

$$= \sum_{(i_1,\dots,i_{r+1})} d\alpha_{i_1} \cdot \dots d\alpha_{i_{r+1}} \cdot c_{i_1 \dots i_{r+1}},$$

which is $d(f_r(c))$, if we reindex the components of this sum: notice that the $c's$ are constant functions.

The maps (f_r) define a morphism of complexes which is a quasi-isomorphism over any intersection of the U_i since the covering $\mathcal{U}$ is good. Therefore, by a classical Mayer-Vietoris argument, they induce an isomorphism between the Čech and de Rham cohomologies.

We take a step further and now compare the Čech cohomology $H^{r-1}(X : S^1)$ with $H_{dR}^r(X)$ via a map

$$H^{r-1}(\mathcal{U}; S^1) \to H^r(\mathcal{U}; V) \cong H_{dR}^r(X).$$

This is the coboundary map associated to the exact sequence

$$0 \to 2i\pi\mathbb{Z} \to i\mathbb{R} \xrightarrow{e} S^1 \to 0,$$

where e is the exponential function and V the real vector space $i\mathbb{R}$. If $\lambda_{i_0 i_1 .. i_{r-1}} \in Z^{r-1}(\mathcal{U}; S^1)$, there is a cochain $u = u_{i_0 i_1 .. i_{r-1}}$ such that $e(u) = \lambda$. The classical definition of the coboundary map

$$H^{r-1}(\mathcal{U}; S^1) \to H^r(\mathcal{U}; 2i\pi\mathbb{Z})$$

is as follows. We first consider the coboundary of u in $C^r(\mathcal{U})$, which we look as a cocycle with values in $2i\pi\mathbb{Z}$, defined by

$$c_{i_0 i_1 ... i_r} = \sum_{m=0}^{r} (1)^m u_{i_0 ... \widehat{i_m} ... i_r}.$$

According to the previous considerations, the associated de Rham class with values in $i\mathbb{R} = V$ is defined by

$$\omega = \sum_{(i_0,...,i_r)} \alpha_{i_0}.d\alpha_{i_1} \cdot ...d\alpha_{i_r} \cdot c_{i_0...i_r}$$

$$= \sum_{(i_0,...,i_r)} \alpha_{i_0}.d\alpha_{i_1} \cdot ...d\alpha_{i_r}. \sum_{m=0}^{r} (1)^m u_{i_0...\widehat{i_m}...i_r}.$$

Using the same argument as above, this sum may be written

$$\omega = \sum_{(i_1,...,i_r)} d\alpha_{i_1} \cdot ...d\alpha_{i_r} \cdot u_{i_1...i_r}.$$

We notice that ω is a closed form since $c_{i_0 i_1 ... i_r} \in 2i\pi\mathbb{Z}$. On the other hand, it is cohomologous up to the sign $(-1)^r$ to the form

$$\theta = \sum_{(i_1,...,i_r)} \alpha_{i_1} \cdot d\alpha_{i_2}...d\alpha_{i_r} \cdot du_{i_1 i_2...i_r}.$$

Using again the fact that $c_{i_0 i_1...i_r} \in 2i\pi\mathbb{Z}$, we see that θ is equal on U_{i_0} to the following differential form

$$\sum_{(i_1,...,i_r)} \alpha_{i_1} d\alpha_{i_2}...d\alpha_{i_r} \cdot du_{i_1 i_2...i_r} = \sum_{(i_2,...,i_r)} d\alpha_{i_2}...d\alpha_{i_r} \cdot du_{i_0 i_2...i_r}.$$

We observe that $du_{i_0 i_2...i_r}$ is the logarithmic differential of $\lambda_{i_0 i_2...i_r}$. Therefore, if we change the indices and r to $r+1$, we get the following theorem.

Theorem 8.1. *Let $\lambda_{i_0...i_r}$ be an r-cocycle on a good covering $\mathcal{U}$ with values in S^1 and let (α_i) be a partition of unity associated to $\mathcal{U}$. Then the closed de Rham form ω of degree $r+1$ with values in $V = i\mathbb{R}$ which is associated to λ by the coboundary map[27]*

$$H^r(\mathcal{U}; S^1) \to H^{r+1}(\mathcal{U}; 2i\pi\mathbb{Z}) \to H^{r+1}(\mathcal{U}; i\mathbb{R})$$

is given by the following formula on each open set U_{i0} :

$$\omega = (-1)^{r+1} \sum_{(i_1,...,i_r)} d\alpha_{i_1}...d\alpha_r \cdot (\lambda_{i_0...i_r})^{-1} \cdot d\lambda_{i_0...i_r}.$$

---

[27]Notice that $\mathbb{R}$ is provided with the discrete topology.

Example 8.3. If we choose $r = 2$ as in our paper, those formulas may be simply written as

$$\omega = \sum_{(i,j,k)} d\alpha_i \cdot d\alpha_j \cdot d\alpha_k \cdot c_{ijk}$$

which is cohomologous to

$$-\sum_{(i,j,k)} \alpha_i \cdot d\alpha_j \cdot d\alpha_k \cdot dc_{ijk}.$$

On the other hand, for a fixed l, if we consider the sequence (l, i, j, k), and the fact that

$$c_{ijk} - c_{ljk} + c_{lik} - c_{lij} \in 2i\pi\mathbb{Z},$$

we may replace dc_{ijk} by $dc_{ljk} - dc_{lik} + dc_{lij}$. Therefore, the restriction of ω to U_l may be written as

$$\omega = -\sum_{(i,j,k)} \alpha_i \cdot d\alpha_j \cdot d\alpha_k \cdot dc_{ljk} + \sum_{(i,j,k)} \alpha_i \cdot d\alpha_j \cdot d\alpha_k \cdot dc_{lik}$$

$$- \sum_{(i,j,k)} \alpha_i \cdot d\alpha_j \cdot d\alpha_k \cdot dc_{lij}.$$

This expression may also be written

$$-\sum_{(i,j,k)} \alpha_i \cdot d\alpha_j \cdot d\alpha_k \cdot dc_{ljk} = -\sum_{(j,k)} d\alpha_j \cdot d\alpha_k \cdot dc_{ljk}$$

$$= -\sum_{(j,k)} d\alpha_j \cdot d\alpha_k \cdot \lambda_{ijk}^{-1} d\lambda_{ljk},$$

as a differential form on U_l. If we assume the cocycle λ completely normalized, we find again the formula given in Section 7.

8.2. **Some key isomorphisms between various definitions of twisted K-groups.** We want to make more explicit the isomorphisms between the various definitions of twisted K-theory given in Section 4. This is especially relevant to the proof of the multiplicativity of the Chern character in Section 7.

  With the notations of Section 4, the more basic one is probably the following

$$K(\underline{\mathcal{K}}_\lambda) \xrightarrow{\cong} K_1(\underline{\mathcal{B}}_\lambda/\underline{\mathcal{K}}_\lambda).$$

We recall that the first group $K(\underline{\mathcal{K}}_\lambda)$ is the original definition of Rosenberg [29]. The second group may be interpreted as the Fredholm definition of twisted K-theory as in [2] (or [15] if λ defines a torsion class in $H^3(X; \mathbb{Z})$). More precisely, if E is a λ-twisted Hilbert bundle and if $\mathcal{F}(E)$ is the space of Fredholm maps in $\mathrm{END}(E)$, the map

$$\mathcal{F}(E) \to (\underline{\mathcal{B}}_\lambda/\underline{\mathcal{K}}_\lambda)^*$$

is a locally trivial fibration with contractible fibres, as we pointed out in Section 4. Therefore, we have the identifications

$$K_\lambda(X) \cong K(\underline{\mathcal{K}}_\lambda) \cong K_1(\underline{\mathcal{B}}_\lambda/\underline{\mathcal{K}}_\lambda).$$

Theorem 8.2. *Let τ be the generator of $K_1(\mathcal{L}/\mathcal{K}) \cong \mathbb{Z}$, associated to the Fredholm operator given by the shift. Then the cup-product with τ induces an isomorphism*

$$\varphi : K(\underline{\mathcal{K}}_\lambda) \xrightarrow{\cong} K_1(\underline{\mathcal{B}}_\lambda/\underline{\mathcal{K}}_\lambda).$$

Proof. In this statement, we implicitly identify the Hilbert tensor product $H \otimes H$ with H. If we forget the twisting, there is a well defined ring map

$$\mathcal{K} \otimes \mathcal{L}/\mathcal{K} \to \mathcal{L}/\mathcal{K}.$$

For the same reasons, there is a ring map

$$\underline{\mathcal{K}}_\lambda \widehat{\otimes} \mathcal{L}/\mathcal{K} \to \underline{\mathcal{B}}_\lambda/\underline{\mathcal{K}}_\lambda.$$

When the base space X varies, the cup-product with the element τ induces a morphism between the (twisted) K_*-theories associated to $\underline{\mathcal{K}}_\lambda$ and $\underline{\mathcal{B}}_\lambda/\underline{\mathcal{K}}_\lambda$ respectively (with a shift for the second one). By a standard Mayer-Vietoris argument (and Bott periodicity), we reduce the theorem to the case when X is contractible, a case which is obvious. $\square$

Although we don't really need it in this paper, it might be interesting to define explicitly the isomorphism backwards:

$$\psi : K_1(\underline{\mathcal{B}}_\lambda/\underline{\mathcal{K}}_\lambda) \xrightarrow{\cong} K(\underline{\mathcal{K}}_\lambda) \cong K(\underline{\mathcal{A}}_\lambda).$$

Such a map ψ is simply the connecting homomorphism in the Mayer-Vietoris exact sequence in K-theory associated to the cartesian square

$$\begin{array}{ccc} \underline{\mathcal{A}}_\lambda & \to & \underline{\mathcal{B}}_\lambda \\ \downarrow & & \downarrow \\ \underline{\mathcal{B}}_\lambda & \to & \underline{\mathcal{B}}_\lambda/\underline{\mathcal{K}}_\lambda \end{array}.$$

In more detail: if α is an invertible element in the ring $\underline{\mathcal{B}}_\lambda/\underline{\mathcal{K}}_\lambda$, we consider the 2×2 matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

By the Whitehead lemma (or analytic considerations: see below), this matrix may be lifted as an invertible 2×2 matrix with coefficients in $\underline{\mathcal{B}}_\lambda$, say γ. Let ε be the matrix defining the obvious grading

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the couple $(\varepsilon, \gamma \cdot \varepsilon \cdot \gamma^{-1})$ defines an involution J on $M_2(\underline{\mathcal{A}}_\lambda) \cong \underline{\mathcal{A}}_\lambda$, hence a finitely generated projective module over $\underline{\mathcal{A}}_\lambda$ which is simply

the image of $(J + 1)/2$. It is easy to show that the class in $K(\underline{\mathcal{A}}_\lambda)$ is independent from the choice of the lifting γ : this is the classical definition of the connecting homomorphism ψ (see e.g. [28]).

Instead of working with invertible elements α, we may as well consider families of Fredholm maps D mapping to α, which are already in $\underline{\mathcal{L}}_\lambda$. Without loss of generality, we may also assume α unitary which implies that a lifting of α^{-1} may be choosen to be the adjoint D^*. We now write the identity

$$\left(\begin{array}{cc} D & 0 \\ 0 & D^* \end{array} \right) = \left(\begin{array}{cc} 0 & D \\ -D^* & 0 \end{array} \right) \cdot \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right).$$

If we define ∇_D as

$$\nabla_D = \left(\begin{array}{cc} 0 & D \\ -D^* & 0 \end{array} \right)$$

in general, we see that we may choose the element γ above to be $\exp(\pi\nabla_D/2) \cdot \nabla_{-1}$. Therefore,

$$\gamma \cdot \varepsilon \cdot \gamma^{-1} = \exp(\pi\nabla_D/2) \cdot \nabla_{-1}.\varepsilon \cdot \nabla_1 \cdot \exp(-\pi\nabla_D/2)$$
$$= -\exp(\pi\nabla_D/2).\varepsilon \cdot \exp(-\pi\nabla_D/2).$$

On the other hand, it is clear that ∇_D and ε anticommute. Therefore, the previous formula may be written as

$$\gamma \cdot \varepsilon \cdot \gamma^{-1} = \exp(\pi\nabla_D).\varepsilon.$$

The couple

$$J = (\varepsilon, \exp(\pi\nabla_D).\varepsilon)$$

defines the required element of $K(\underline{\mathcal{A}}_\lambda)$. By construction, we see that J also defines an element of the relative group associated to the augmentation map

$$(\underline{\mathcal{K}}_\lambda)^+ \to \mathbb{C}.$$

Here $(\underline{\mathcal{K}}_\lambda)^+$ is the ring $\underline{\mathcal{K}}_\lambda$ with a unit added and the relative K-group is the usual one:

$$K(\underline{\mathcal{K}}_\lambda) = Ker(K((\underline{\mathcal{K}}_\lambda)^+) \to K(\mathbb{C}) = \mathbb{Z})$$

which is canonically isomorphic to $K(\mathcal{A}_\lambda)$.

8.3. Some functorial properties of twisted K-theory and of the Chern character.
In this paper, we have indexed twisted K-theory by completely normalized 2-cocycles λ with values in S^1. Of course, such a cocycle determines a cohomology class $[\lambda]$ in $H^2(X; S^1) \cong H^3(X; 2i\pi\mathbb{Z})$ as we have seen in 8.1 and we would like to index twisted K-theory by elements of this smaller group. There is an obstruction to doing so however as we shall see. If we apply Proposition 1.2 to $\mathbb{C}$-bundles (if $[\lambda]$ is a torsion class) or to $\mathcal{A}$-bundles in general, we see that if μ is cohomologous to λ, the equivalence Θ in this last proposition,

between the categories of λ-twisted bundles and μ-twisted bundles, depends on the choice of a cochain η such that

$$\mu_{kji} = \lambda_{kji} \cdot \eta_{ji} \cdot \eta_{ki}^{-1} \cdot \eta_{kj}.$$

If η' is another choice, $\eta_{ji}.\eta_{ji}'^{-1}$ is a one-dimensional cocycle with values in S^1. Since a one-dimensional coboundary does not change λ, we see that the ambiguity in the definition of the previous category equivalence lies in the cohomology group[28] $H^1(\mathcal{U}; S^1) \cong H^2(X; 2i\pi\mathbf{Z}) \cong H^2(X; \mathbf{Z})$. In particular, the definition of twisted K-theory with coefficients in $H^3(X; \mathbf{Z})$ has a well-defined meaning only if $H^2(X; \mathbf{Z}) = 0$.

This remark is important for the definition of the product

$$K_\lambda(X) \times K_\mu(X) \to K_{\lambda\mu}(X)$$

which is detailed in many ways in Section 5. The Hilbert bundle E_λ, defined at the beginning of this section, depends on the cocycle λ. It depends on its cohomology class $[\lambda]$ up to a non canonical isomorphism as we have just seen (except if $H^2(X; \mathbf{Z}) = 0$). Therefore, strictly speaking, we cannot define in a functorial way a cup-product

$$K_{[\lambda]}(X) \times K_{[\mu]}(X) \to K_{[\lambda\mu]}(X).$$

Another remark is the choice of a good covering to define twisted K-theory via twisted bundles. There is also a functorial problem since many choices are possible. One way to deal with this is to show that the categories of twisted bundles associated to different coverings give the same twisted K-theory if we choose two 3-Čech cocycles which are cohomologous. This is again included in the contents of Proposition 1.2. As we already pointed out, this identification is not canonical, except if $H^2(X; \mathbb{Z}) = 0$.

Let us now turn our attention to the definition of the Chern character. If we fix the good covering $\mathcal{U}$, our definition depends heavily on the choice of a partition of unity (α_i). If (β_i) is another choice, there is a homotopy between them which is $t \longmapsto (1-t)\alpha_i + t\beta_i$. If λ is a completely normalized 2-cocycle with values in S^1, the associated closed differential forms θ_α and θ_β are homotopic and therefore cohomologous: they define the same class in $H^3(X; i\mathbb{R})$. However, it is not completely obvious that the associated twisted cohomologies $H^{ev}_{\theta_\alpha}(X)$ and $H^{ev}_{\theta_\beta}(X)$ are isomorphic in a way compatible with the Chern character. One way to deal with this problem is to consider λ-twisted bundles over $X \times [0,1]$ with the partition of unity given by $(1-t)\alpha_i + t\beta_i$ as above. We then have a commutative diagram where the horizontal arrows are isomorphisms.

$$
\begin{array}{ccccc}
K_\lambda(X \times \{0\}) & \longleftarrow & K_\lambda(X \times [0,1]) & \longrightarrow & K_\lambda(X \times \{1\}) \\
\downarrow & & \downarrow & & \downarrow \\
H^{ev}_{\theta_\alpha}(X \times \{0\}) & \longleftarrow & H^{ev}_\theta(X \times [0,1] & \longrightarrow & H^{ev}_{\theta_\beta}(X \times \{1\})
\end{array}
$$

---

[28] We assume the covering good as in 3.3.

This diagram shows that the Chern character does not depend on the choice of partition of unity up to canonical isomorphisms given by the horizontal arrows.

We cannot expect the Chern character to be functorial with respect to the cohomology class of λ in $H^3(X;\mathbb{Z})$. However, it is "partially functorial" in the following sense: if we choose a good refinement $\mathcal{V} = (V_s)$ of $\mathcal{U} = (U_i)$ as in Section 1, any restriction map of type

$$\Theta_\tau : K_\lambda(\mathcal{U}) \to K_\mu(\mathcal{V})$$

(where $V_s \subset U_{\tau(s)}$) is an isomorphism. This isomorphism is not unique and depends on τ, as it was pointed out in the proof of Propositon 1.3. If (β_s) if a partition of unity associated to the covering $\mathcal{V}$ and (α_i) a partition of unity associated to the covering $\mathcal{U}$, the functions $(\alpha_i \cdot \beta_s)$ define a partition of unity associated to $\mathcal{U} \cap \mathcal{V}$ which is just a reindexing of the covering $\mathcal{V}$. On the other hand, we may also reindex $\mathcal{U}$, in such a way that the functions $(\alpha_i \cdot \beta_s)$ define also a partition of unity of $\mathcal{U}$. Since the twisted cohomology is homotopically invariant, it follows that the "restriction map"

$$H^{ev}_{\overline{\lambda}}(X) \to H^{ev}_{\overline{\mu}}(X)$$

is also well defined and that the diagram

$$
\begin{array}{ccc}
K_\lambda(\mathcal{U}) & \to & K_\mu(\mathcal{V}) \\
\downarrow & & \downarrow \\
H^{ev}_{\overline{\lambda}}(X) & \to & H^{ev}_{\overline{\mu}}(X)
\end{array}
$$

is commutative (with the notations of Theorem 7.2).

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