Seminario de Geometría No Comutativa del Atlántico Sur, 19/10/21, 3-3.50 pm (Paris), B. Keller

Singularity categories, Leavitt path algebras and Hochschild homology

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Plan:
1. The philosophy of non com. algebraic geometry (after Drinfeld, Kontsevich ...)
2. Hochschild homology: Reminder and complements
3. HH_* of derived and singularity categories
4. Application to HH_* of dg Leavitt path algebras

1. The philosophy of non com. algebraic geometry (after Drinfeld, Kontsevich ...)

Question: What is a non commutative scheme?

Answer 1 (Grothendieck, Gabriel, Manin, ...): It is an abelian category, namely the abelian category of "quasicoherent sheaves" on a hypothetical "non com. space". Each com. scheme X, yields the non com. scheme $\mathrm{Qcoh}(X)$.

Observation: The essential homological invariants of X resp. $\mathrm{Qcoh}X$ only depend on its derived category $\mathcal{D}\mathrm{Qcoh}X$, which is no longer abelian but still triangulated.

Answer 2 (Drinfeld, Kontsevich, ...): A non com. scheme is a triangulated category, namely the "derived category" of the cat. of quasicoherent sheaves on some hypothetical non com. space. Each com. scheme X yields the non com. scheme $\mathcal{D}\mathrm{Qcoh}X$.

Observation: There are serious technical problems with triangulated categories.

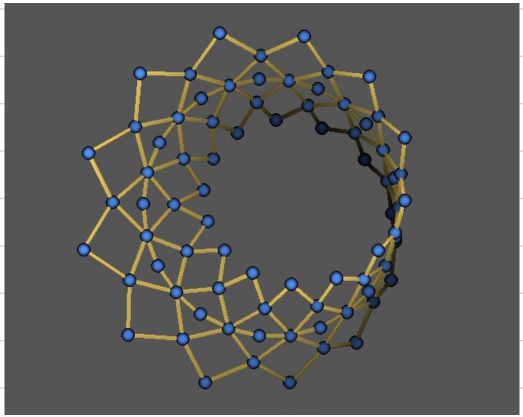
Most importantly, if $\mathcal{J}$ is triangulated and I a small category, then $\mathrm{Fun}(I, \mathcal{J})$ is no longer triangulated in general! (already for $I: 1 \to 2$).

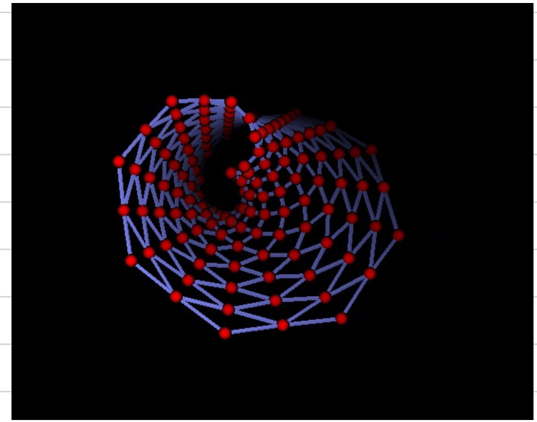
Solution: Replace triangulated categories with dg (= differential graded) categories, which are considered **up to derived Morita equivalence**: A **Morita functor** $F: \mathcal{A} \to \mathcal{B}$ is a dg functor yielding an equivalence $\mathcal{D}\mathcal{B} \xrightarrow{\text{res}} \mathcal{D}\mathcal{A}$. Two dg categories are **derived Morita equivalent** if they are linked by a zigzag of Morita functors.

Non com. algebraic geometry = study of (small) dg categories up to derived Morita equivalence and of their invariants, e.g. K-theory, Hochschild (co-) homology, cyclic homology, ...

Pictures of the type of categories we will consider:

Cluster category of type E_6





Cluster cat. of type A_{15}.

1. Hochschild homology

k a field, A a k-algebra (associative, with 1, non com.), $\operatorname{Mod} A = \{$all right A-modules$\}$,

$\mathcal{D}A = \mathcal{D}\operatorname{Mod} A =$ unbounded derived category : objects are complexes $\dots \to M^p \to M^{p+1} \longrightarrow \dots$

$A^e = A \underset{k}{\otimes} A^{op}$ so that A^e-module = A-bimodule, e.g. ${}_A A_A$ = identity bimodule

Hochschild homology of A = $HH_*(A)$ = $\mathrm{Tor}_*^{A^e}(A,A)$ = homology of the Hochschild complex

$$HH(A): \quad A \longleftarrow A\otimes A \longleftarrow A\otimes A\otimes A \longleftarrow \dots \longleftarrow A^{\otimes p+1} \longleftarrow \dots$$
$$ab - ba \longleftarrow\!\shortmid\; a\otimes b$$

Rks: 1) We see : $HH_0(A) = A/[A,A]$. derived cat. of the cat. of k-vector spaces

2) $HH_n(A)$ and $HH(A) \in \mathcal{D}k$ are functorial in A.

3) The definitions extend from k-algebras A to small k-categories $\mathcal{A}$, e.g.

the Hochschild complex becomes

$$\coprod_{A_0\in\mathcal{A}} \mathcal{A}(A_0,A_0) \longleftarrow \coprod_{A_0,A_1\in\mathcal{A}} \mathcal{A}(A_1,A_0)\otimes\mathcal{A}(A_0,A_1) \longleftarrow \coprod_{A_0,A_1,A_2} \mathcal{A}(A_2,A_0)\otimes\mathcal{A}(A_1,A_2)\otimes\mathcal{A}(A_0,A_1) \longleftarrow \dots)$$
$$a\circ b - b\circ a \longleftarrow\!\shortmid\; a \otimes b$$

$$\begin{tikzcd} A_0 \arrow[r, bend left, "b"] & A_1 \arrow[l, bend left, "a"] \end{tikzcd} \qquad \begin{tikzcd} & A_1 \arrow[dr, "a"] & \\ A_0 \arrow[ur, "b"] & & A_2 \arrow[ll, "c"] \end{tikzcd}$$

fin. gen. proj. A-modules

4) Can show : $HH(A) \underset{\text{in } \mathcal{D}k}{\xrightarrow{\sim}} HH(\mathrm{proj}\,A)$. This yields Morita invariance of HH.

5) The definitions further extend to small differential graded (= dg) k-categories $\mathcal{A}$

e.g. $\mathcal{A} = \mathcal{C}^b_{dg}(\mathrm{proj}A)$ = dg category of bounded complexes over $\mathrm{proj}A$ with morphism complexes: $\mathrm{Hom}^n_A(P^\cdot, Q^\cdot) = \{f : P^\cdot \to Q^\cdot$ A-lin. homog. of degree $n\}$, $n \in \mathbb{Z}$,

$$d(f) = d_{P^\cdot} \circ f - (-1)^n f \circ d_{Q^\cdot} .$$

Then we have $H^0 \mathcal{C}^b_{dg}(\mathrm{proj}A) = \mathcal{H}^b(\mathrm{proj}A)$ and

$$HH(A) \xrightarrow{\sim} HH(\mathrm{proj}A) \xrightarrow[K98]{\sim} HH(\mathcal{C}^b_{dg}(\mathrm{proj}A)) \quad \text{in } \mathcal{D}k.$$

This yields derived Morita invariance of HH:

$$\exists\ \mathcal{D}A \xrightarrow[\text{triangle equiv.}]{\sim} \mathcal{D}B \xRightarrow[\text{Rickard '89}]{} \exists\ \mathcal{C}^b_{dg}(\mathrm{proj}A) \xrightarrow[\substack{\text{zigzag of}\\ \text{quasi-equiv.}}]{\sim} \mathcal{C}^b_{dg}(\mathrm{proj}B) \Rightarrow HH(A) \cong HH(B)$$

$\mathcal{A}$ dg category:

$\mathcal{C}\mathcal{A} = \{$dg right $\mathcal{A}$-modules$\} = \{$dg functors $M : \mathcal{A}^{op} \to \mathcal{C}_{dg}k\}$, $\mathcal{D}\mathcal{A} = (\mathcal{C}\mathcal{A})[\mathrm{Qis}^{-1}]$.

Thm (Localization, K 1998): Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$

be an exact sequence of dg categories, i.e. the sequence of derived cat.

$$0 \to \mathcal{D}\mathcal{A} \to \mathcal{D}\mathcal{B} \to \mathcal{D}\mathcal{C} \to 0$$

is exact. Then we have a canonical triangle in $\mathcal{D}k$

$$HH(\mathcal{A}) \to HH(\mathcal{B}) \to HH(\mathcal{C}) \to \Sigma HH(\mathcal{A}).$$

suspension $= [1]$

path algebra

Let $A = kQ/I$, Q finite quiver, $I \triangleleft kQ$ admissible (i.e. $(Q_1)^N \subseteq I \subseteq (Q_1)^2$, $Q_1 = \{\text{arrows}\}$)

$\mathrm{Hom}_k(?, k)$

Thm (Van den Bergh '15): $HH(A^!) \xrightarrow{\sim} DHH(A)$,

dg algebra

where $A^! =$ Koszul dual of $A = \mathrm{RHom}_A(R, R)$

$$R = A/\mathrm{rad}\,A = \bigoplus_{i \in Q_0} S_i, \quad D = \mathrm{Hom}_k(?, k).$$

vertices of Q

"Beilinson quiver for $\mathbb{P}^2$"

Example: A given by Q:
$$\begin{tikzcd} 1 \arrow[r, shift left=3, "x_0"] \arrow[r, "x_1" description] \arrow[r, shift right=3, "x_2"'] & 2 \arrow[r, shift left=3, "x_0"] \arrow[r, "x_1" description] \arrow[r, shift right=3, "x_2"'] & 3 \end{tikzcd}$$
, I : commutativity: $x_i x_j - x_j x_i = 0$

$\Rightarrow$ $A^!$ given by Q^*:
$$\begin{tikzcd} 1 & 2 \arrow[l, shift right=3, "\xi_0"'] \arrow[l, "\xi_1" description] \arrow[l, shift left=3, "\xi_2"] & 3 \arrow[l, shift right=3, "\xi_0"'] \arrow[l, "\xi_1" description] \arrow[l, shift left=3, "\xi_2"] \end{tikzcd}$$

$|\xi_i| = 1$, I^* : anticommutativity: $\xi_i \xi_j + \xi_j \xi_i = 0$, $\forall i,j$, $d = 0$.

$\Longrightarrow$ $HH(A) \simeq k^3 \simeq DHH(A^!)$

2. HH_* of derived and singularity categories

Let $A = kQ/I$ admissible quotient, $\operatorname{mod} A = \{k\text{-fin.dim. right } A\text{-modules}\}$.

$\mathcal{D}^b A$ = bounded derived category of $\operatorname{mod} A$, $\operatorname{per} A$ = perfect derived category $= \operatorname{thick}(A) \subseteq \mathcal{D}^b A$.

$\mathcal{H}^b(\operatorname{proj} A) \xrightarrow{\sim} \operatorname{thick}(A)$

$\operatorname{sg}(A)$ = singularity category $= \mathcal{D}^b(\operatorname{mod} A)/\operatorname{per}(A)$ (Buchweitz 1986, Orlov 2003).

K 1999, Drinfeld 2004

Fact (based on dg quotients): We have a canonical exact sequence of dg categories

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{per}_{dg} A & \longrightarrow & \mathcal{D}^b_{dg} A & \longrightarrow & \operatorname{sg}_{dg} A & \longrightarrow & 0 \\ & & \wr\!\shortparallel & & & & & & \\ & & \mathcal{C}^b_{dg}(\operatorname{proj} A) & & & & & & \end{array}$$

can. dg enhancements

By the localization theorem, we obtain a triangle in $\mathcal{D}k$

$$\begin{array}{ccccccc} HH(\mathrm{per}_{dg} A) & \longrightarrow & HH(\mathcal{D}^b_{dg} A) & \longrightarrow & HH(sg_{dg} A) & \longrightarrow & \Sigma HH(\mathrm{per}_{dg} A) \\ \uparrow{\scriptstyle \wr} & & & & & & \uparrow{\scriptstyle \wr} \\ HH(A) & & & & & & \Sigma HH(A) \end{array}$$

Thm 1: We have $HH(\mathcal{D}^b_{dg} A) \simeq DHH(A)$.

Sketch of proof: Recall: $R = A/\mathrm{rad} A$, $A^! = R\mathrm{Hom}_A(R,R)$. We have $\mathcal{D}^b(\mathrm{mod} A) \xrightarrow[\sim]{R\mathrm{Hom}_A(R,?)} \mathrm{per} A^!$.

$$\Longrightarrow HH(\mathcal{D}^b_{dg} A) \xrightarrow{\sim} HH(\mathrm{per}_{dg} A^!) \xleftarrow[\text{Mor.}]{\sim} HH(A^!) \xleftarrow[\text{VdB}]{\sim} DHH(A). \quad \checkmark$$

Abbreviate $\mathcal{S} = sg_{dg}(A)$.

Thm 2: $HH(\mathcal{S})$ is canonically isomorphic (in $\mathcal{D}k$) to the double Hochschild complex of A:

$$(*) \qquad \underbrace{\ldots \longrightarrow A \otimes A \longrightarrow A}_{\text{Hochschild cplex}} \xrightarrow[\tau]{} \underbrace{\overset{\text{degree } 0}{DA} \longrightarrow D(A\otimes A) \longrightarrow \ldots}_{\text{dual of Hochsch. cplex}},$$

trace

where $(\tau(a))(b) = tr(\lambda_a \rho_b : A \to A)$, λ_a = left mult., ρ_b = right mult.

Cor.: $HH_n(\mathcal{S}) \simeq HH_{n-1}(A) \simeq DHH_{1-n}(\mathcal{S})$ for $n \geq 2$

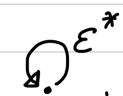
$$HH_1(\mathcal{S}) \simeq \mathrm{Ker}\left(HH_0(A) \xrightarrow{\tau} DHH_0(A) \right) \simeq DHH_0(\mathcal{S})$$

Rk: The thm generalizes to suitable proper dg algebras A.

4. Application: HH_* of dg Leavitt path algebras

Q a finite quiver, e.g. .

A = associated radical square 0 algebra $= kQ/(kQ_1)^2$, e.g. $k[\varepsilon]/(\varepsilon^2)$.

Q^* = opposite quiver with arrows $\alpha^* : j \to i$ of degree 1 for each $\alpha : i \to j$ of Q, e.g. .

Fix $i \in Q_0$. Consider the arrows $\alpha_s^* : i \longrightarrow t(\alpha_s^*)$, $1 \leq s \leq n_i$, starting in i.

target

Put $P_i = e_i kQ^* \in \mathrm{proj}(kQ^*)$.

suspension = [1]

Let $\varphi(i) = \begin{bmatrix} \alpha_1^* \\ \vdots \\ \alpha_t^* \end{bmatrix} : P_i \longrightarrow \bigoplus_{j=1}^{n_i} \Sigma P_{t(\alpha_s^*)}$, $P_i = e_i kQ^*$, e.g. $P_1 \xrightarrow{\alpha^*} \Sigma P_1$ for Q^*: $\begin{tikzcd} 1 \arrow[loop above, "\alpha^*"] \end{tikzcd}$

Note: If i is a sink of Q^* (i.e. $\nexists$ outgoing arrows at i), then $\bigoplus_{j=1}^{n_i} \Sigma P_{j_{is}} = 0$.

Let $\varphi(i)^{-1} = [\beta_{i1}, \dots, \beta_{it_i}] : \bigoplus_{s=1}^{n_i} \Sigma P_{j_{is}} \longrightarrow P_i$ be a formal inverse of $\varphi(i)$.

L_Q = Leavitt path algebra of Q = $kQ^*\langle \text{coeff. } \beta_{ij} \text{ of all } \varphi(i)^{-1}, i \in Q_0 \rangle$.

Endow L_Q with the grading inherited from Q^* and with $d = 0$, e.g. $L_Q = k[\alpha^*, \alpha^{*-1}]$.

Thm (Smith '12, Chen-Yang '15): We have $\mathrm{per}_{dg} L_Q \xrightarrow{\sim} \mathrm{sg}_{dg} A$, $e_i L_Q \mapsto S_i$.

Cor.: $HH_*(L_Q)$ is computed by the double Hochschild complex of A:

$$\cdots \longrightarrow A \otimes A \longrightarrow A \longrightarrow \underset{\deg 0}{DA} \longrightarrow DA \otimes DA \longrightarrow \cdots .$$

In particular, we have $\dim HH_p(L_Q) < \infty$, $\forall p$.

Rk: A different description of $HH_*(L_Q)$ is due to Ara–Cortiñas (Proc. AMS 2015).

Beyond radical square zero

Q a finite quiver, $I \triangleleft kQ$ admissible, $A = kQ/I$, $R = \prod ke_i \subseteq A$, $J = \operatorname{rad} A$ so $A = R \oplus J$.

tensor algebra

$A_0 = (T_R J)/(J \otimes J)$ = radical square 0 alg. assoc. with A

Rk: So $A_0 = R \oplus J = A$ as R-bimodules but $xy = 0$ in A_0, $\forall x, y \in J$.

Idea (Chen-Wang):

$$\begin{tikzcd}
A_0 \arrow[r, rightsquigarrow, "\text{deformation}"'] & A & \\
\Longrightarrow \quad sg(A_0) \arrow[r, rightsquigarrow, "\text{deformation}"'] \arrow[d, dash, "\simeq"] & sg(A) \arrow[d, dash, "\simeq"] & ? \\
per(L_{A_0}) \arrow[r, rightsquigarrow, "\text{deformation?}"'] & sg(L_A) & \\
L_{A_0} \arrow[r, rightsquigarrow, "\text{deformation?}"'] & L_A & ?
\end{tikzcd}$$

Thm (Chen-Wang '21): $\overset{d_{A_0}=0}{L_{A_0}}$ admits a can. differential $\overset{\neq 0}{d_A}$ s.th. for $L_A := (L_{A_0}, d_A)$

$$sg_{dg}(A) \xleftarrow{\sim} per_{dg} L_A \, .$$

Cor.: The Hochschild homology of L_A is computed by the double Hochschild complex of A.