K-theory of noncommutative Bernoulli shifts

South Atlantic Noncommutative Geometry Seminar

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Motivation: Wreath products

Definition

Let G, H be discrete groups, $H^{\oplus G} := \bigoplus_{g \in G} H$, and let $G \curvearrowright H^{\oplus G}$ by left translation.

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Goal

Calculate $K_*(C_r^*(H \wr G))$.

The structure of $C_r^*(H \wr G)$

Recall: $C_r^*(H_1 \oplus H_2) \cong C_r^*(H_1) \otimes C_r^*(H_2)$.

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$$\Rightarrow C^*_r(H^{\oplus G}) \cong \varinjlim_{\substack{F \subseteq G \\ \text{finite}}} C^*_r(H^{\oplus F}) \cong \varinjlim_{\substack{F \subseteq G \\ \text{finite}}} C^*_r(H)^{\otimes F} =: C^*_r(H)^{\otimes G}$$

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Definition (Crossed products)

Let A be a unital C^*-algebra with G-action.

$$A \rtimes G := C^*\langle A, G \mid gag^{-1} = g.(a)\rangle$$

and $A \rtimes_r G \subseteq \mathcal{L}(\ell^2(G, A))$ via the regular representation.

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Goal (Noncommutative Bernoulli shifts)

Calculate $K_*(A^{\otimes G} \rtimes_r G)$.

The Baum–Connes conjecture with coefficients (BCC)

Conjecture (Baum–Connes–Higson)

Let G be a discrete group and let A be a G-C^*-algebra. Then the assembly map

$$\mu_*\colon K_*^G(E_{\mathrm{Fin}}G, A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

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Examples

- a-T-menable groups satisfy BCC (Higson–Kasparov)
- hyperbolic groups satisfy BCC (Lafforgue)
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We will always assume that our groups satisfy BCC!

The Going-Down-principle

Theorem (Chabert–Echterhoff–Oyono-Oyono, Meyer–Nest)

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Let A, B *be* G*-*C^**-algebras and* $\varphi \in KK^G(A, B)$*. If*

$$K_*(A \rtimes_r H) \to K_*(B \rtimes_r H)$$

is an isomorphism for every finite subgroup $H \subseteq G$*, then*

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Strategy

Use BCC to calculate $K_*(A^{\otimes G} \rtimes_r G)$ from $K_*(B^{\otimes G} \rtimes_r G)$ where B is easier to understand!

Izumi's Lemma

Lemma (Izumi)

Let G be a finite group, let Z be a finite G-set. Then then there is a functor $()^{\otimes Z}\colon KK \to KK^G$ making the following diagram commute:

$$\begin{array}{ccc} C^*alg & \xrightarrow{()^{\otimes Z}} & GC^*alg \\ \downarrow{\scriptstyle KK} & & \downarrow{\scriptstyle KK^G} \\ KK & \overset{()^{\otimes Z}}{\dashrightarrow} & KK^G \end{array}$$

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In particular, for every two C^*-algebras A, B, there is a (non-linear!) map

$$KK(A,B) \to KK^G(A^{\otimes Z}, B^{\otimes Z})$$

mapping equivalences to equivalences.

From finite to infinite groups

Corollary

Let G be a group satisfying BCC and let $\varphi\colon A \to B$ be a unital $*$-homomorphism which is a KK-equivalence. Then φ induces an isomorphism

$$K_*(A^{\otimes G} \rtimes_r G) \xrightarrow{\cong} K_*(B^{\otimes G} \rtimes_r G).$$

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If G satisfies BCC, then we have

$$K_*(\mathcal{O}_\infty^{\otimes G} \rtimes_r G) \cong K_*(\mathcal{Z}^{\otimes G} \rtimes_r G) \cong K_*(C_r^*(G))$$

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$$K_*(\mathcal{O}_2^{\otimes G} \rtimes_r G) = 0$$

Assume that G satisfies Baum–Connes with coefficients!

Theorem (Chakraborty–Echterhoff–K–Nishikawa)

Let A be a unital UCT C^*-algebra such that the unital inclusion $\iota\colon \mathbb{C} \to A$ induces a split injection $\iota_*\colon K_*(\mathbb{C}) \to K_*(A)$. Let B be a UCT C^*-algebra such that $K_*(B) = \mathrm{coker}(\iota_*)$. Then

$$K_*(A^{\otimes G} \rtimes_r G) \cong K_*((B^+)^{\otimes G} \rtimes_r G) \cong \bigoplus_{[F]\in G\backslash \mathrm{FIN}(G)} K_*(B^{\otimes F} \rtimes_r G_F).$$

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- *place where Baum–Connes is used*

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Theorem (Higson–Kasparov, Tu)

Let H be an a-T-menable group. Then $C^*_r(H)$ satisfies the UCT and the map $C^*(H) \to C^*_r(H)$ is a KK-equivalence.

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$$\operatorname{coker}\Big(K_*(\mathbb{C}) \to K_*(C_r^*(H))\Big) \cong K_*(B), \quad B := \ker(C^*(H) \xrightarrow{1_H} \mathbb{C}).$$

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Corollary

Let H be an a-T-menable group and let G be a group satisying BCC. We have

$$K_*(C_r^*(H \wr G)) = K_*(C_r^*(H)^{\otimes G} \rtimes_r G) \cong \bigoplus_{[F] \in G \backslash \mathrm{FIN}(G)} K_*(B^{\otimes F} \rtimes_r G_F)$$

Finite dimensional examples

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If G is torsion-free, then

$$K_*(A^{\otimes G} \rtimes_r G) \cong K_*(C_r^*(G)) \oplus \bigoplus_{i=1}^{\infty} K_*(\mathbb{C}).$$

$A = C(S^1)$

Let $A = C(S^1)$ and let G be a group satisfying BCC. Then $\mathrm{coker}\Big(K_*(\mathbb{C}) \to K_*(A)\Big) \cong K_*(C_0(\mathbb{R}))$ and thus

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By equivariant Bott periodicity, we have

$$C_0(\mathbb{R})^{\otimes F} = C_0(\mathbb{R}^{G_F})^{\otimes |G_F\backslash F|} \simeq_{KK^{G_F}} \begin{cases} \mathbb{C}, & |G_F\backslash F| \text{ even}, \\ C_0(\mathbb{R}^{G_F}), & |G_F\backslash F| \text{ odd}. \end{cases}$$

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and thus

$$K_*(C_0(\mathbb{R})^{\otimes F} \rtimes_r G_F) = \begin{cases} K_*(C^*(G_F)), & |G_F\backslash F| \text{ even}, \\ K^*_{G_F}(\mathbb{R}^{G_F}), & |G_F\backslash F| \text{ odd}. \end{cases}$$

This can be computed explicitely (Karoubi, Echterhoff–Pfante)!

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Let A be a unital C^*-algebra. Then we have

$$K_*((A \otimes M_n)^{\otimes G} \rtimes_r G) \cong K_*(A^{\otimes G} \rtimes_r G)[1/n], \quad \textit{if } |G| = \infty,$$
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Example (finite-dimensional algebras)

Let $A = M_{k_0} \oplus \ldots \oplus M_{k_N}$ with $\gcd(k_0, \ldots, k_N) = n$. Then

$$K_*(A^{\otimes G} \rtimes_r G) \cong \bigoplus_{[F]\in G\backslash \mathrm{FIN}(G)} \bigoplus_{[S]\in G_F\backslash\{1,\ldots,N\}^F} K_*(C_r^*(G_S))[1/n].$$

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Corollary

Let $\mathcal{O}_{n+1}$ be the Cuntz algebra on $n+1$ generators and let G be a group satisfying BCC. Then

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Lemma

If G is finite and $K_*(A)$ is finitely generated, then $K_*(A^{\otimes G} \rtimes_r G)$ is finitely generated as well.

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Corollary

If G is finite and p a prime, then $K_*(\mathcal{O}_{p+1}^{\otimes G} \rtimes_r G)$ is a finite direct sum of groups of the form $\mathbb{Z}/p^k$.

The orbit category

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Warning

The map

$$\operatorname*{colim}_{G/H \in \mathrm{Or}_{\mathrm{Fin}}(G)} K_*(A \rtimes_r H) \to K_*(A \rtimes_r G)$$

*is **not** the Davis–Lück assembly map*

$$H_*^G(E_{\mathrm{Fin}}G, \mathbb{K}_A^G) \to K_*(A \rtimes_r G).$$

AF-algebras

Theorem (Chakraborty–Echterhoff–K–Nishikawa)

Let G be an infinite group satisfying Baum–Connes with coefficients, let A be a unital AF-algebra and let

$$S := \{n \in \mathbb{N} \mid [1_A] \in K_0(A) \text{ divisible by } n\}.$$

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Then we have a pushout diagram

$$\begin{array}{ccc} \operatorname*{colim}_{G/H \in \mathrm{Or}_{\mathrm{Fin}}(G)} K_*(C_r^*(H))[S^{-1}] & \longrightarrow & \operatorname*{colim}_{G/H \in \mathrm{Or}_{\mathrm{Fin}}(G)} K_*(A^{\otimes G} \rtimes_r H) \\ \downarrow & & \downarrow \\ K_*(C_r^*(G))[S^{-1}] & \longrightarrow & K_*\left(A^{\otimes G} \rtimes_r G\right). \end{array}$$

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When G is torsion-free, we have

$$K_*(A^{\otimes G} \rtimes_r G) \cong \tilde{K}_*(C_r^*(G))[S^{-1}] \oplus K_*(A^{\otimes G})_G.$$

Rational computations

Let A be a unital, stably finite C^*-algebra.

$$\Rightarrow \iota_* \otimes \mathbb{Q} \colon K_*(\mathbb{C}) \otimes \mathbb{Q} \to K_*(A) \otimes \mathbb{Q} \text{ splits!}$$

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Theorem (Chakraborty-Echterhoff-K-Nishikawa)

Let A be a unital, stably finite C^*-algebra satisfying the UCT and let G be a group satisfying Baum–Connes with coefficients. Let B be a UCT C^*-algebra satisfying $K_*(B) \cong \operatorname{coker}(\iota_* \otimes \mathbb{Q})$. Then we have

$$K_*(A^{\otimes G} \rtimes_r G) \otimes \mathbb{Q} \cong \bigoplus_{[F] \in G \backslash \mathrm{FIN}(G)} K_*(B^{\otimes F} \rtimes_r G_F) \otimes \mathbb{Q}.$$

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Obstructions to the Rokhlin property

Corollary

Let $G \neq \{1\}$ be a finite group and let A be a unital, stably finite C^*-algebra satisfying the UCT. Then $G \curvearrowright A^{\otimes G}$ does not have the Rokhlin property.

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Proof by contradiction.

By our rational computations, the map

$$K_*(C_r^*(G)) \otimes \mathbb{Q} \to K_*(A^{\otimes G} \rtimes_r G) \otimes \mathbb{Q}$$

is injective.

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Proof by contradiction.

By our rational computations, the map

$$K_*(C_r^*(G)) \otimes \mathbb{Q} \to K_*(A^{\otimes G} \rtimes_r G) \otimes \mathbb{Q}$$

is injective. However, by the Rokhlin property, it can be factored as the composition

$$K_*(C_r^*(G)) \otimes \mathbb{Q} \to K_*(C(G) \rtimes_r G) \otimes \mathbb{Q} \to K_*(A^{\otimes G} \rtimes_r G) \otimes \mathbb{Q}.$$

The first map is never injective. □

Thank you very much!