Bornological Spectra and Bounded Cohomology

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Motivation: Spectral Geometry

Spectral algebraic geometry is geometry relative to the HA-context of symmetric spectra, i.e. the stable homotopy category.

Affine objects can be thought of as generalised multiplicative cohomology theories, including in particular elliptic cohomology theories tmf is the (spectral) ring of global sections on a geometric stack in this HA context.

We would like a similar theory in analytic geometry. This is based on current work developing analytic geometry as geometry relative to the symmetric monoidal category of complete bornological modules over a Banach ring.

Analytic geometry as relative geometry I

In algebraic geometry, a space is determined (locally) by its algebra of functions.

In analytic geometry, if X is a (finite dimensional) complex- or rigid- analytic space, then its algebra of functions has a canonical Fréchet algebra structure. The main idea is that an analytic space should be determined locally by its algebra of functions *as long as we remember the Fréchet structure*.

A 0th approximation is that analytic geometry should be geometry relative to the monoidal category of Fréchet spaces. For technical reasons, it is better to work with the category $\mathrm{CBorn}_k = Ind^m(\mathrm{Ban}_k)$ of complete bornological spaces. Here k can be any Banach ring. In particular one can consider the Banach ring of integers $\mathbb{Z}_{an}$. This is the initial Banach ring, and is just the integers equipped with the Euclidean norm.

Analytic Geometry as Relative Algebraic Geometry II

In this talk we will be more interested in the category $\mathrm{Born}_k^{\frac{1}{2}}$ of (neither complete nor separated) bornological k-modules of convex type. This is not abelian, however it is *quasi-abelian*. In particular it has a canonical 'left' t-structure, whose heart $LH(\mathrm{Born}_k^{\frac{1}{2}})$ is called the *left heart* of $\mathrm{Born}_k^{\frac{1}{2}}$.

Moreover there is an adjunction

$$C\colon LH(\mathrm{Born}_k^{\frac{1}{2}}) \rightleftarrows \mathrm{Born}_k^{\frac{1}{2}} : i$$

in which i is exact, and there is an induced adjoint equivalence of derived categories.

Analytic Geometry as Relative Algebraic Geometry III

The category $\mathrm{LH}(\mathrm{Born}_{\mathrm{k}}^{\frac{1}{2}})$ is a *monoidal elementary abelian category*, meaning that it is a 'good' setting for relative derived algebraic geometry.

In particular it has compact projective generators given by the the subcategory spanned by objects of the form $\overset{-1}{l}(V)$ where V is a normed set. This means there is an equivalence of categories

$$\mathcal{P}_{\Sigma}(\{\overset{-1}{l}(V)\}) \cong \mathrm{LH}(\mathrm{Born}_{\mathrm{k}}^{\frac{1}{2}})$$

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Spectra I: Generalised Cohomology Theories

A generalised cohomology theory is a pair (H^*, δ), where H^* is a contravariant functor from the category P of pairs of CW complexes into the category $\mathrm{Gr}(\mathrm{Ab})$ of graded abelian groups, and δ assigns to each pair (X, A) a collection of maps

$$\delta^n_{(X,A)} : H^n(A, \bullet) \to H^{n+1}(X, A)$$

which is natural in the obvious way, satisfying the following axioms.

1. Homotopy invariance: if $f, g : (X, A) \to (Y, B)$ are homotopic then $H^*(f) = H^*(g)$.
2. Exactness: for any pair (X, A) the sequence

   $$\ldots H^n(X, A) \to H^n(X) \to H^n(A) \to H^{n+1}(X, A) \to \ldots$$

   is exact
3. Excision: If X is the union of subcomplexes A and B then the map

   $$H^\bullet(X, B) \to H^\bullet(A, A \cap B)$$

   is an isomorphism.
4. Additivity: If (X, A) is the disjoint union of pairs (X_α, A_α) then the map $H^\bullet(X, A) \to \prod_\alpha H^\bullet(X_\alpha, A_\alpha)$ is an isomorphism.

Spectra II: Constructing Stable Categories

The category **Sp** of *spectra*, is the stabilisation $\mathbf{Stab}(\mathbf{Top}_*)$ of the $(\infty, 1)$-category of pointed topological spaces.

For any (presentable) pointed $(\infty, 1)$-category $\mathbf{C}$, $\mathbf{Stab}(\mathbf{C})$ is the homotopy limit

$$\mathbf{Stab}(\mathbf{C}) := \mathrm{holim}(\ldots \mathbf{C} \to^{\Omega} \mathbf{C} \to^{\Omega} \mathbf{C})$$

where $\Omega := 0 \times_{(-)} 0$ is the loop space functor. Informally an object consists of tuples

$$(C_n, f_n)_{n \in \mathbb{N}_0}$$

where $C_n \in \mathbf{C}$, and $f_n : \Sigma C_{n-1} \to C_n$ is a map such that

$$C_n \to \Omega C_{n+1}$$

is an equivalence. We usually write such an object as a sequence

$$(C_0, C_1, \ldots)$$

Spectra III: The t-Structure

$\mathbf{Stab}(\mathbf{C})$ is a stable $(\infty, 1)$-category, so in particular $\mathrm{Ho}(\mathbf{Stab}(\mathbf{C}))$ is triangulated.

Denote by $\Omega^\infty : \mathbf{Stab}(\mathbf{C}) \to \mathbf{C}$ the functors sending

$$(C_0, C_1, \ldots)$$

to C_0. For $\mathbf{C}$ presentable this functor has a left adjoint

$$\Sigma^\infty(C)_n = \mathrm{colim}_k \, \Omega^k \Sigma^{k+n} X$$

Moreover it has a t-structure: $X \in \mathbf{Stab}_{\le -1}(\mathbf{C})$ iff $\Omega^\infty X \cong 0$ in $\mathbf{C}$. $\mathbf{Stab}_{\ge 0}(\mathbf{C})$ can be described as the full subcategory of $\mathbf{Stab}(\mathbf{C})$ generated under sifted colimits by objects of the form $\Sigma^\infty C$.

If $\mathbf{C} = \mathbf{Top}_*$ then the heart is Ab. For $X \in \mathbf{Top}_*$ the homology of $\Sigma^\infty X$ for this t-structure are the stable homotopy groups of X.

Let S be an object of $\mathbf{Stab}(\mathbf{C})$, and consider the functor

$$\mathbf{Map}(-,S):\mathbf{Stab}(\mathbf{C})^{op}\to\mathbf{Stab}(\mathbf{C})$$

For any map $f:C\to D$ in $\mathbf{C}$, define

$$H^n_S(C,D):=\pi_n(\mathrm{fib}(\mathbf{Map}(\Sigma^\infty D,S)\to\mathbf{Map}(\Sigma^\infty C,S)))$$

If $\mathbf{Sp}_{\geq 0}(\mathbf{C})$ is closed under products, then $H^\bullet_S$ determines a '$\mathbf{C}^\heartsuit$-valued generalised cohomology theory'.

Brown representability says that all generalised cohomology theories on $\mathbf{Top}_*$ arise in this way.

All of the axioms follow by general nonsense of $(\infty,1)$-categories (or model categories). In particular excision is proven as follows: if $X \cong A\coprod_{A\cap B} B$ is a homotopy pushout in $\mathbf{C}$, then

$$\begin{tikzcd}
\mathbf{Map}(\Sigma^{\infty}X,S) \arrow[r] \arrow[d] & \mathbf{Map}(\Sigma^{\infty}A,S) \arrow[d] \\
\mathbf{Map}(\Sigma^{\infty}B,S) \arrow[r] & \mathbf{Map}(\Sigma^{\infty}A\cap B,S)
\end{tikzcd}$$

is a homotopy fibre product in $\mathbf{Stab}(\mathbf{C})$. Thus the fibres are equivalent.

Not Spectra I: Bounded Cohomology

Bounded cohomology is a homotopy invariant of spaces, useful in the study of simplicial volumes.
Let X be a CW complex, and consider

$$S_n(X) := \mathrm{Hom}_{cts}(\Delta^n, X)$$

We regard this as a *semi-normed set*, and define

$$\hat{C}^n(X) := \mathrm{Hom}_{bnd}(S_n(X), \mathbb{R})$$

where $\mathbb{R}$ is equipped with the Euclidean norm.
The *bounded cohomology of* X is

$$\hat{H}^n(X) := H^n(\mathrm{Hom}_{bnd}(S_\bullet(X), \mathbb{R}))$$

For a CW pair (X, A), the *relative bounded cohomology* is

$$\hat{H}^n(X, A) := H^n(\mathrm{Ker}(\hat{C}^n(X) \to \hat{C}^n(Y)))$$

$\hat{H}^\bullet$ satisfies axioms 1), 2), and 4) of a generalised cohomology theory, *but not axiom* 3).

Idea: we are computing bounded cohomology in the wrong category.

Not Spectra II: Bornological Bounded Cohomology

The space $\hat{C}^n(X)$ has a natural semi-normed (in fact Banach) structure, with

$$||\phi|| = \sup_{f \in S_n(X)} |\phi(f)|$$

We consider $\mathrm{Norm}_{\mathbb{R}}^{\frac{1}{2}} \subset \mathrm{Born}_{\mathbb{R}}^{\frac{1}{2}}$ and define *bornological bounded cohomology* by

$$L\hat{H}^{\bullet}(X, A) \coloneqq LH^{\bullet}(\mathrm{fib}(\hat{C}^{\bullet}(X) \to \hat{C}^{\bullet}(A))$$

Not Spectra III: Excision

'Theorem' (Excision)

Bornological bounded cohomology satisfies excision: if $A, B \subset X$ are bornological CW complexes (slide 22) with $\cancel{A \cup B = X}$ $X = A \coprod_{A \cup B} B$, then the map

$$L\hat{H}^n(X, B) \to L\hat{H}^n(A, A \cap B)$$

is an isomorphism.

This will be a consequence of bounded cohomology being representable by a bornological Eilenberg-Maclane spectrum.

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We are looking for a category of 'spaces' whose (stable) homotopy groups have a canonical bornological structure. This should be a stable category with a (left and right) complete t-structure whose heart is $\mathrm{LH}(\mathrm{Born}^{\frac{1}{2}}_{\mathbb{Z}_{\mathrm{an}}})$.

$$\mathbf{Sp} \cong \mathbf{Sp}(\mathbf{P}_\Sigma(\text{finite pointed sets})) \rightsquigarrow^{\pi_0} \mathcal{P}_\Sigma(\text{free abelian groups of finite rank}) \cong \mathrm{Ab}$$

$$??? \rightsquigarrow^{\pi_0} \mathcal{P}_\Sigma(\{\bar{l}^1(V) : V \text{ a normed set}\})$$

$\bar{l}^1(V)$ is in some sense a 'free' semi-normed abelian group on the semi-normed set V.
First idea: try $\mathbf{Sp}(\mathrm{Norm}^{\frac{1}{2}}_{\mathbb{F}_1})$.

Bornological Spectra II: Pointed normed sets

A **semi-normed pointed set** is a pointed set X equipped with a function $|-|: X \to \mathbb{R}_{\geq 0}$ such that $|0| = 0$.

X is **normed** if $|x| = 0 \Leftrightarrow x = 0$.

A map $f : (X, |-|) \to (X', |-|')$ of semi-normed pointed sets is said to be **bounded** if there is $C > 0$ such that for all $x \in X$, $|f(x)|' \leq C|x|$.

Semi-normed pointed sets and bounded maps between them arrange into a category $\mathrm{Norm}^{\frac{1}{2}}_{\mathbb{F}_1}$. We define the full subcategory $\mathrm{Norm}_{\mathbb{F}_1}$ in the obvious way.

These categories are Cartesian closed, and we regard them as closed symmetric monoidal categories with the Cartesian monoidal structure.

For V a pointed semi-normed set, $\overset{-1}{l}(V)$ is the free abelian group on V with semi-norm $||(r_v)||_{v \in V \setminus \{\bullet\}} = \sum |r_v||v|$.

Let C be a pointed 1-category with finite coproducts. Consider the category $\mathbf{P}_{\Sigma}(N(\mathrm{C}))$, the free sifted cocompletion of $N(\mathrm{C})$, i.e. the category of finite product preserving functors $N(\mathrm{C})^{op} \to \mathbf{sSet}$.

For each $n \geq 0$ and $F \in \mathbf{P}_{\Sigma}(N(\mathrm{C}))$ the functor

$$\pi_n \circ F : \mathrm{C}^{op} \to \mathrm{Set}$$

is finite-product preserving, and thus determines an object of $\mathcal{P}_{\Sigma}(\mathrm{C})$.

For $n = 1$ $\pi_n(F)$ is a group object, and for $n \geq 2$ it is an abelian group object.

$\mathbf{P}_{\Sigma}(\mathrm{C})$ has a model category presentation $\mathrm{s}\mathcal{P}_{\Sigma}(\mathcal{C})$.

A map f of simplicial objects is a weak equivalence/ fibration precisely iff $\mathrm{Hom}(c, f)$ is a weak equivalence/ fibration of simplicial sets for each $c \in \mathrm{C}$.

In particular a map between fibrant objects is a weak equivalence if and only if it induces an isomorphism of homotopy objects.

Categories of Generalised Spaces III: Spectra

Define

$$\mathbf{Sp}(\mathrm{C}) := \mathbf{Stab}(\mathbf{P}_{\Sigma}(N(\mathrm{C})))$$

For $X = (X_0, X_1, \dots) \in \mathbf{Sp}(\mathrm{C})$ we define

$$\pi_n^s(C) = \operatorname{colim}_k \pi_{n+k}(C_k)$$

By construction a map $f : X \to Y$ is an equivalence if and only if $\pi_n^s(f)$ is an equivalence for each $n \in \mathbb{Z}$.

The t-structure is characterised by the fact that $X \in \mathbf{Sp}_{\leq -1}$ iff $\mathrm{Map}(\Sigma^\infty c, X) \cong \mathrm{Map}(c, \Omega^\infty X)$ is trivial for all $c \in \mathrm{C}$

One has that $X \in \mathbf{Sp}_{\geq 0}(\mathrm{C})$ (resp. $X \in \mathbf{Sp}_{\leq 0}(\mathrm{C})$) if and only if $\pi_n^s(X) = 0$ for $n < 0$ (resp. $n > 0$). This follows from considering the exact triangle

$$\Sigma^\infty \Omega^\infty X \to X \to \mathrm{cofib}(\Sigma^\infty \Omega^\infty X \to X)$$

and using a Freudenthal/ Rezsk suspension theorem to show that $\pi_n^s(\Sigma^\infty \Omega^\infty X) \cong \pi_n^s(X)$ for $n \geq 0$.

Categories of Generalised Spaces IV: The Heart

Each $\pi_n^s(X) \in \mathrm{Fun}(\mathrm{C}^{op}, \mathrm{Ab})$ is finite-product preserving, and so we get functors

$$\pi_n^s : \mathbf{Sp}(\mathrm{C}) \to \mathrm{Fun}^{\times}(\mathrm{C}^{op}, \mathrm{Ab}) \cong \mathrm{Ab}(\mathcal{P}_{\Sigma}(\mathrm{C})) \cong \mathcal{P}_{\Sigma}(\{\mathrm{Free}(c)\})$$

An Eilenberg-Maclane type construction shows that the map

$$\pi_0^s : \mathbf{Sp}(\mathrm{C})^{\heartsuit} \to \mathrm{Ab}(\mathcal{P}_{\Sigma}(\mathrm{C}))$$

is an equivalence.

The Case of Semi-normed Sets

Consider the category of bornological CW complexes $\mathbf{P}_{\Sigma}(\mathrm{Norm}^{\frac{1}{2}}_{\mathbb{F}_1})$.

Via Yoneda, $\mathrm{Ab}(\mathcal{P}_{\Sigma}(\mathrm{C}))$ contains 'non-convex' semi-normed modules, i.e. satisfying $||x+y|| \leq A(||x|| + ||y||)$ for some A possibly > 1.

This category is too big. However there is a fully faithful functor $\mathrm{Born}^{\frac{1}{2}}_{\mathbb{Z}_{an}} \to \mathrm{Ab}(\mathcal{P}_{\Sigma}(\mathrm{Norm}^{\frac{1}{2}}_{\mathbb{F}_1}))$ which by general nonsense has a left adjoint. Thus $\mathrm{Born}^{\frac{1}{2}}_{\mathbb{Z}_{an}}$ is a localisation of $\mathrm{Ab}(\mathcal{P}_{\Sigma}(\mathrm{Norm}^{\frac{1}{2}}_{\mathbb{F}_1}))$, and we hope to 'boost' this localisation to the level of spectra.

Moore Spectra

For $c \in \mathcal{C}$ define $\mathbb{S}_c := \Sigma^\infty(c)$. One has $\pi_0^s(\mathbb{S}_c) \cong \mathrm{Free}(c)$.

We have

$$\begin{aligned}
\pi_0 \mathbf{Map}(\mathbb{S}_c, \mathbb{S}'_c) &\cong \pi_0 \mathbf{Map}(c, \Omega^\infty \Sigma^\infty(c')) \\
&\cong \mathrm{Hom}(c, \pi_0(\Omega^\infty \Sigma^\infty(c'))) \\
&\cong \mathrm{Hom}(c, \mathrm{Free}(c')) \\
&\cong \mathrm{Hom}(\mathrm{Free}(c), \mathrm{Free}(c'))
\end{aligned}$$

In particular any map $\mathrm{Free}(c) \to \mathrm{Free}(c')$ lifts to a map $\mathbb{S}_c \to \mathbb{S}_{c'}$ (though in no way functorially).
Pick a projective semi-simplicial resolution $\mathrm{Free}(c_\bullet) \to A$, and define

$$M(A) := |\mathbb{S}_{c_\bullet}|$$

The Localisation I

Although the Moore spectra construction is not functorial, it does define a functor

$$M : \mathrm{Ab}(\mathcal{P}_{\Sigma}(\mathrm{C})) \to \mathrm{Ho}(\mathbf{Sp}(\mathrm{C}))$$

in particular there is a well-defined map in $\mathrm{Ho}(\mathbf{Sp}(\mathrm{C}))$

$$M(\mathrm{Free}(V)) \to M(\overline{I}^{1}(V))$$

We define $\mathbf{Sp}^{\mathrm{Born}^{\frac{1}{2}}}$ to be the localisation of $\mathbf{Sp}(\mathrm{Norm}^{\frac{1}{2}}_{\mathbb{F}_1})$ at maps of the form $\Sigma^n M(\mathrm{Free}(V)) \to \Sigma^n M(\overline{I}^{1}(V))$.

The Localisation II

For $X \in \mathbf{Sp}^{\mathrm{Born}^{\frac{1}{2}}}$ we have

$$\pi_n^s(X(V)) \cong \pi_n^s \underline{\mathrm{Map}}(M(\mathbb{S}_V), X)) \cong \pi_n^s \underline{\mathrm{Map}}(M(\bar{l}^1(V)), X)$$

Thus π_n^s determines a product-preserving map $\{\bar{l}^1(V)\} \to \mathrm{Ab}$, i.e. an object of $\mathrm{Born}^{\frac{1}{2}}_{\mathbb{Z}_{an}}$.

Remark

One can define complete bornological spectra by replacing $\bar{l}^1(V)$ with $l^1(V)$.

The Discrete Functor

Let

$$\mathrm{disc} : \mathrm{Set}_* \to \mathrm{Norm}^{\frac{1}{2}}_{\mathbb{F}_1}$$

be the functor which equips a pointed set X with the semi-norm whose value away from the marked point is everywhere 1, and consider the induced functor

$$\mathrm{disc} : \mathrm{sSet}_* \to \mathrm{sNorm}^{\frac{1}{2}}_{\mathbb{F}_1}$$

This functor commutes with pushouts along monomorphisms, and preserves homotopy equivalences between fibrant objects.

Lemma

disc *preserves homotopy equivalences, cofibrations, and pushouts. In particular it ~~commutes with homotopy pushouts~~ sends pushout squares which are homotopy pushout squares to homotopy pushouts.*

Bornological Bounded Cohomology of bornological CW complexes (Added Slide)

Let $X_\bullet \in \mathbf{P}_\Sigma(\mathrm{Norm}^{\frac{1}{2}}_{\mathbb{F}_1})$. Via the model category presentation we regard this as an object of $\mathrm{s}\mathcal{P}_\Sigma(\mathrm{Norm}^{\frac{1}{2}}_{\mathbb{F}_1})$.

Define $\hat{C}^\bullet(X_\bullet)$ to be the chain complex in $\mathrm{Born}^{\frac{1}{2}}_{\mathbb{Z}_{an}}$ associated to the simplicial bornological abelian group

$$\mathrm{Hom}_{s\mathcal{P}_\Sigma(\mathrm{Norm}^{\frac{1}{2}}_{\mathbb{F}_1})}(X_\bullet, \mathbb{R})$$

via Dold-Kan, and $L\hat{H}^\bullet(X_\bullet)$ of $X_\bullet$ to be the homology (in $\mathrm{LH}(\mathrm{Born}^{\frac{1}{2}}_{\mathbb{Z}_{an}})$)

Remark

We get a functor

$$\mathrm{disc} \circ \mathrm{Hom}(\Delta^\bullet, -) : \mathrm{Top}_* \to s\mathcal{P}_\Sigma(\mathrm{Norm}^{\frac{1}{2}}_{\mathbb{F}_1})$$

such that

$$L\hat{H}^\bullet(\mathrm{disc} \circ \mathrm{Hom}(\Delta^\bullet, Y)) \cong L\hat{H}^\bullet(Y)$$

Bornological Bounded Cohomology

Let ~~(X, A)~~ X be a bornological CW complex. Then

$$\cancel{L\hat{H}^{\bullet}(X) \cong H^{\bullet}\mathrm{Map}(disc(X), K(\mathbb{R}))}$$

$$L\hat{H}^{\bullet}(X) \cong H^{\bullet}\mathrm{Map}(X, K(\mathbb{R}))$$

In particular bornological bounded cohomology satisfies excision.

Bornological Bounded Cohomology vs. Bounded Cohomology (Added Slide)

For a complex $X_\bullet$ in any quasi-abelian category, we have

$$C(LH^n(X_\bullet)) \cong \mathrm{coker}(d_{n+1} : X_{n+1} \to \mathrm{Ker}(d_n))$$

In particular we have for (X, A) a CW-pair, we have

$$C(L\hat{H}^\bullet(X, A)) \cong \hat{H}^\bullet(X, A)$$

(where the right-hand side is equipped with the quotient bornology).

However excision for bornological bounded cohomology *does not imply* excision for bounded cohomology (which does not hold), since $\overline{l}^{1}$ does not commute with all homotopy pushouts (only those homotopy pushout squares which are also pushout squares), while $\mathrm{Hom}(\Delta^\bullet, -) : \mathrm{Top} \to \mathrm{sSet}$ preserves homotopy pushouts, put not pushouts in general.