LECTURES ON K-THEORY.

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Lecture 1

The beginning of K-theory : Grothendieck, Atiyah and Hirzebruch

In order to formalize his work on the Riemann-Roch theorem (in the spirit of Hirzebruch), Grothendieck introduced a new contravariant functor [BS] defined on the category of non singular algebraic varieties X. He named this functor $K(X)$, the "K-theory" of X. It seems that the terminology "K" came out from the German word "Klassen", since $K(X)$ may be thought of as a group of "classes" of vector bundles on X. Grothendieck could not use the terminology $C(X)$ since his thesis (in functional analysis) made an heavy use of the ring $C(X)$ of continuous functions on a space X.

In order to define $K(X)$, one considers first the free abelian group $L(X)$ generated by the isomorphism classes $[E]$ of vector bundles E on X. The group $K(X)$ is then the quotient of $L(X)$ by the subgroup generated by the following relations

$$[E'] + [E''] = [E]$$

when we have an exact sequence of vector bundles

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

Exercise: compute $K(X)$ when X is a Riemann surface in complex Algebraic Geometry.

This group $K(X)$ has nice algebraic properties, very much related to the usual ones in cohomology. As a matter of fact, the theory of characteristic classes, e.g. the Chern character denoted by Ch - which we shall review later on - might be thought of as defining a functor from $K(X)$ to suitable cohomology theories. For instance, if $f : X \longrightarrow Y$ is a morphism of projective varieties, we have a "Gysin map" $f_*^H : K(X) \longrightarrow K(Y)$ which is related to the usual Gysin map f_*^H in rational cohomology by the following formula

$$Ch(f_*^H(x)) = f_*^H(Ch(x).Todd(f))$$

where x is an arbitrary element of $K(X)$ and $Todd(f)$ is the "Todd class" (see the third lecture for more details) of the normal bundle of f, as introduced by Hirzebruch [AH2]. More precisely, the Chern character Ch is a natural transformation from $K(X)$ to the even Betti cohomology groups $H^{ev}(X;\mathbb{Q})$ (which depend only on the topology of X) and $Todd(f)$ is well defined in terms of the Chern classes of the tangent bundles associated to X and Y. If $x \in K(X), Ch(x)$ is the sum of homogeneous elements $Ch_0(x), Ch_1(x), \cdots$, where $Ch_n(x) \in H^{2n}(X;\mathbb{Q})$.
The formula above shows that $Todd(f)$ is the defect of commutativity of the following diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{f_*^K} & K(Y) \\ {\scriptstyle Ch}\big\downarrow & & {\scriptstyle Ch}\big\downarrow \\ H^{ev}(X;\mathbb{Q}) & \xrightarrow{f_*^H} & H^{eV}(Y;\mathbb{Q}) \end{array}$$

The group $K(X)$ is hard to compute in general : it is just a nice functor used for a better understanding of characteristic classes. This "Algebraic K-theory" was investigated deeply by other mathematicians Bass, Milnor, Quillen... at the end of the 60's and will be studied in detail during the lectures 4 and 5.

Just after the discovery of $K(X)$, Atiyah and Hirzebruch considered a topological analog named $K^{top}(X)$ - now for an arbitrary compact space X - by considering topological vector bundles instead of algebraic vector bundles. The main difference between the two definitions lies in the fact that an exact sequence of topological vector bundles[1]

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

always splits. Therefore, $K^{top}(X)$ is just the symmetrized group of the semi-group of isomorphism classes $[E]$ of vector bundles, with the addition rule $[E] + [F] = [E \oplus F]$, where $E \oplus F$ is the "Whitney sum" of the vector bundles E and F. It is clearly a contravariant functor of X. Moreover, every vector bundle is a direct summand in a trivial vector bundle (i.e. of the type $X \times \mathbb{C}^n$, with an easily guessed vector bundle structure). This fact enables us to compute $K^{top}(X)$ in purely homotopical terms. More precisely, let us define $\widehat{K}^{top}(X)$, the reduced K-theory of X, as the cokernel of the obvious map $\mathbb{Z} \approx K^{top}(Point) \longrightarrow K^{top}(X)$ induced by the projection $X \longrightarrow Point$. Then $\widetilde{K}(X)$ may be identified with the group of <u>*stable*</u> isomorphism classes of vector bundles : two vector bundles E and E' are stably isomorphic if $E \oplus T \approx E' \oplus T'$ for some trivial bundles T and T'. It follows that $K^{top}(X)$ is isomorphic to the set of homotopy classes of maps from X to $\mathbb{Z} \times BU$, where BU is the "infinite Grassmannian" , i.e. the direct limit of the Grasmannians $G_n(\mathbb{C}^m)$ with obvious inclusions between them.

Here is another important difference between the two groups : $K^{top}(X)$ is much easier to compute that its algebraic counterpart $K(X)$. For instance, $K^{top}(X) \otimes \mathbb{Q}$ is isomorphic to $H^{ev}(X; \mathbb{Q})$, where $H^{ev}(X; \mathbb{Q})$ denotes the rational even Čech cohomology groups of X via the topological Chern character. There is in fact an obvious factorization of the algebraic Chern character

$$K(X) \longrightarrow K^{top}(X) \longrightarrow H^{ev}(X; \mathbb{Q})$$

Therefore, if X is a finite CW complex, we find that $K^{top}(X)$ is isomorphic to $G \oplus \mathbb{Z}^d$, where d is the dimension of the vector space $H^{ev}(X; \mathbb{Q})$ and G is a finite group.

Exercise: compute $K^{top}(X)$ when X is a sphere S^n for $n \leq 4$.

A famous theorem of Bott which we shall see in the next lecture is the following : $\widetilde{K}^{top}(S^n)$ is isomorphic to $\mathbb{Z}$ if n is even and $= 0$ if n is odd. We shall see at the end how the Chern character enables us to construct a non trivial stable bundle over S^n when n is even. The idea of the proof of the isomorphism

[1] <u>*complex*</u> vector bundles, in order to fix the ideas.

$K^{top}(X) \otimes \mathbb{Q} \approx H^{ev}(X;\mathbb{Q})$ is to notice that $K^{top}(X) \otimes \mathbb{Q}$ and $H^{ev}(X;\mathbb{Q})$ are half-exact functors[2] and that the Chern character induces an isomorphism when X is a sphere. It is now a general statement that in such a situation the two half-exact functors are indeed isomorphic (for the details see for instance the Cartan-Schwartz seminar 1963/64, exposé 16).

In order to make the Chern character $K^{top}(X) \longrightarrow H^{2n}(X;\mathbb{Q})$ more explicit, let us describe it when X is a C^∞-manifold of finite dimension and let us imbed $\mathbb{Q}$ in $\mathbb{C}$ via the group homomorphism $\lambda \longrightarrow (2i\pi)^n\lambda$. The vector space $H^{2n}(X;\mathbb{Q})$ is then isomorphic to the de Rham cohomology of differential forms of degree n. On the other hand, it is not difficult to show that the classification of topological bundles on X is equivalent to the classification of C^∞ -bundles. Moreover, any C^∞-bundle E is a direct summand in a trivial bundle as in the topological situation : this means there is a C^∞-map $p : X \longrightarrow M_n(\mathbb{C})$ such that for x in X, the fiber E_x is the image of $p(x)$, with $p(x)^2 = p(x)$.
Now, as popularized a few years ago in the framework of cyclic homology and non commutative de Rham homology, the Chern character of E, denoted by $Ch_nE)$, is the cohomology class of the trace of the following product of matrices which entries are differential forms

$$Ch_n(E) = \frac{1}{n!(2i\pi)^n} Tr(p \cdot dp \cdots dp)$$

($2n$ factors dp). It is a nice exercise to show that $Ch_n(E)$ is a closed differential form and that its cohomology class is just a function of the class of E in $K^{top}(X)$. This is the "modern" version of the Chern character which we shall see in lecture 7.

In order to check the non triviality of the Chern character as defined above, let us choose the example of the Hopf bundle over the sphere $X = S^2$ which we write as the set of point (x, y, z) such that $x^2 + y^2 + z^2 = 1$. We then put $p = (1 + J)/2$, where J is the involution of the trivial bundle $X \times \mathbb{C}^2$ defined over the point (x, y, z) by the following matrix

$$\begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix}$$

The Hopf bundle is then the image of the projection operator p. The explicit computation of $Ch_1(E)$ gives the canonical volume form on the sphere S^2. Therefore the Chern character (as a cohomology class) is non trivial in general.

As it was pointed out by Hirzebruch, the Chern character can be caracterized by the following axioms (on the category of differentiable manifolds for instance)

1) Naturality : if $f : X \longrightarrow Y$ is a C^∞-map and if E is a vector bundle over

[2] According to Dold, a homotopy (contravariant) functor on pointed spaces is called half-exact if we have an sequence $F(X/Y) \longrightarrow F(X) \longrightarrow F(Y)$ each time we have a closed subspace Y of the compact space X.

Y, then $Ch_r(f^*(E)) = f^*(Ch_r(E))$.
2) Additivity : $Ch_r(E \oplus F) = Ch_r(E) + Ch_r(F)$
3) Normalisation : if L is the canonical line bundle over $P^n(\mathbb{C})$ (which restricts to the Hopf bundle over the sphere $S^2 = P_1(\mathbb{C})$), one has

$$Ch_r(L) = x^r/r!$$

where x is the unique element of $H^2(P^n(\mathbb{C}))$ which restricts to the volume form of S^2.

More generally, using the Chern character and Clifford algebras, we are going to show that Ch_n is non trivial when X is the sphere S^{2n} (cf. [K2], chapter 1) by choosing an explicit vector bundle on S^{2n} (as a matter of fact the generator of $\widetilde{K}(S^{2n}) \cong \mathbb{Z}$. More precisely, let E be a complex vector space of finite dimension and let $e_1, ..., e_{2n+1}$ be automorphisms of E such that
$(e_\alpha)^2 = 1$
$e_\alpha e_\beta = -e_\beta e_\alpha$ for $\alpha \neq \beta$
$e_1 ... e_{2n+1} = i^n$

An example of such a data is given by the exterior algebra $\Lambda\mathbb{C}^n = \Lambda\mathbb{C}\widehat{\otimes}...\widehat{\otimes}\Lambda\mathbb{C}$ (n factors), where the couple (e_{2r-1}, e_{2r}) is acting on the r^{th}-factor $\Lambda\mathbb{C} = \mathbb{C}^2$ via the (2×2) matrix defined above. The last automorphism e_{2n+1} is of course determined by the equation $e_1 ... e_{2n+1} = i^n$. It can be shown, using the theory of Clifford modules, that any such E is isomorphic to a direct sum of copies of this example.

Let now X be the sphere $S^{2n} = \{x_1, ..., x_{2n+1} | \Sigma(x_i)^2 = 1\}$ and let V be the vector bundle which is the image of the projection operator $p = (1 + J)/2$ on the trivial bundle $X \times E$, where

$$J = x_1 e_1 + ... + x_{2n+1} e_{2n+1}$$

Modulo an exact form, we have

$$n!(2i\pi)^n Ch_n(V) = 2^{-2n-1} Tr(J.(dJ)^{2n})$$

with $(dJ)^{2n} = (dx_1 e_1 + ... dx_{2n+1} e_{2n+1})^{2n}$

Since the entries $(dx_\alpha e_\alpha)$ commute which each other, we may write the previous expression as the following sum

$$(2n)! \Sigma_\alpha (dx_1 e_1) ... (\widehat{dx_\alpha e_\alpha}) ... (dx_{2n+1} e_{2n+1})$$

where the symbol $(\widehat{dx_\alpha e_\alpha})$ means that we omit $dx_\alpha e_\alpha$. If we expand it, we find

$$-i^n (2n)! \Sigma_\alpha (-1)^\alpha dx_1 ... \widehat{dx_\alpha} ... dx_{2n+1} e_\alpha)$$

Therefore,

$$Trace[J.(dJ)^{2n}] = -i^n (2n)! \Sigma_\alpha (-1)^\alpha x_\alpha dx_1 ... \widehat{dx_\alpha} ... dx_{2n+1}$$

modulo an exact form. Since the expression $\Sigma_\alpha(-1)^\alpha x_\alpha dx_1 ... \widehat{dx_\alpha} ... dx_{2n+1}$ is the volume form of S^{2n}, which we denote by $Vol(S^{2n})$, we finally find that

$$Ch_n(V) = \frac{(2n)!}{2^{2n+1} n! (2\pi)^n} Vol(S^{2n}).dim(E)$$

In our example $E = \Lambda\mathbb{C}^n$ above, we have $dim(E) = 2^n$. Therefore, the integral of $Ch_n(V)$ on the sphere S^{2n} is equal to 1 which implies that $Ch_n(V) \neq 0$ in the cohomology of S^{2n} (and is in fact a generator of the integral class).

Let us come back now to the general theory $K(X)$ of Grothendieck where, for simplicity, we assume X to be a complex projective variety. We may view X as a subvariety of the complex projective space $P^n(\mathbb{C})$ for a certain n. As it is well known, $P^n(\mathbb{C})$ may be written as the quotient of the group $GL_{n+1}(\mathbb{C})$ by the subgroup consisting of matrices of the type

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where a is an $(n \times n)$ matrix, c is a (1×1) matrix and finally b is an $(n \times 1)$ matrix.

Over $P^n(\mathbb{C})$, there is an affine variety $P = Proj_1(\mathbb{C}^{n+1})$ consisting of projection operators p in $M_{n+1}(\mathbb{C})$ such that $Tr(p) = 1$, which may also be written as the homogeneous space $GL_{n+1}(\mathbb{C})/H$, where H is the subgroup $GL_n(\mathbb{C}) \times GL_1(\mathbb{C})$. Note that P is an affine variety in $M_{n+1}(\mathbb{C}) = \mathbb{C}^{(n+1)(n+1)}$ since it is defined by the following equations

$$\Sigma_j p_{ij} p_{jk} = p_{ik} \text{ and } \sum_j p_{jj} = 1$$

Let now X' be the pull-back variety defined by the cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & GL_{n+1}(\mathbb{C})/GL_n(\mathbb{C}) \times GL_1(\mathbb{C}) = P \\ \downarrow & & \downarrow \\ X & \longrightarrow & P_n(\mathbb{C}) \end{array}$$

[the second vertical map associates to a projection operator p its image $Im(p)$]

Then X' is called a "torsor" over X (there is a vector bundle acting simply transitively on the fibers) and Grothendieck proved that $K(X) \simeq K(X')$. But now X' is an affine variety as a closed subvariety of P and Serre has shown that the category of algebraic vector bundles over X' is equivalent to the category $\mathcal{P}(A)$ of finitely generated projective modules over the ring A of coordinates of X'. For example, if $X = P_n(\mathbb{C})$ itself, A is the algebra generated by the elements p_{ij} with $1 \leq i, j \leq n+1$, subject to the relations $\Sigma p_{ij} \cdot p_{jk} = p_{ik}$ and $\Sigma p_{jj} = 1$. There is a topological analogy which we shall see in the next lecture, but already this point of view gives rise to several questions :

1) What will be the analog of the Chern character if we work with A-modules instead of vector bundles? As we shall see in Lecture 7, this opens a wide range of applications belonging to the new domain of "noncommutative geometry", linked with homology and cohomology theories for arbitrary rings.

2) Since the target of the Chern character is even cohomology, this suggests that there might be "derived functors" $K_n(A), n \in \mathbb{Z}$, for which $K_0(A)$ is just the first group. However, we shall see later on that this question is more complicated that it looks at the first glance. For instance, the topology of A (if any) plays an important role : this is not seen of course in the definition of $K(A)$.

Finally, K-theory has nice cohomology operations coming from exterior powers of bundles or modules (more geometric in nature than their cohomology counterparts which are the Steenrod operations). These operations gave spectacular applications of K-theory in the 60's. For instance, J.-F. Adams was able to compute the maximum number of independent tangent vector fields on the sphere S^n using these methods. Other applications were found in global analysis and in the theory of C^*-algebras. We shall review all these applications in lectures 2 and 3.

At this point in history we have two theories, quite different in nature, coming from Algebraic Geometry and Algebraic Topology. We shall analyse the topological theory first in the spirit of noncommutative geometry. However, we shall see during this historical sketch that the algebraic and topological methods are in fact deeply linked one to another.

Lecture 2

K-theory of Banach algebras. Bott Periodicity theorems

Let E be a complex vector bundle over a compact space X et let A be the Banach algebra $C(X)$ of continuous functions $f : X \longrightarrow \mathbb{C}$ (with the Sup norm). If $M = \Gamma(X, E)$ denotes the vector space of continuous sections $s : X \longrightarrow E$ of the vector bundle E, M is clearly a right A-module if we define $s.f$ to be the continuous section $x \longmapsto s(x)f(x)$.

As a matter of fact, since X is compact, we may find another vector bundle E' such that the Whitney sum $E \oplus E'$ is trivial, say $X \times \mathbb{C}^n$. Therefore, if we put $M' = \Gamma(X, E')$, we have $M \oplus M' \cong A^n$ as A-modules, which means that M is a finitely generated projective A-module. The theorem of Serre and Swan [K1] says precisely that the correspondance $E \longmapsto M$ induces a functor from the category $\mathcal{E}(X)$ of vector bundles over X to the category $\mathcal{P}(A)$ of finitely generated projective A-modules, which is in fact an equivalence of categories. In particular, isomorphism classes of vector bundles correspond bijectively to isomorphism classes of modules.

These considerations lead to the following definition of the K-theory of a ring with unit A : we just mimic the definition of $K^{top}(X)$ by replacing vector bundles by (projective finitely generated) A-modules. We call this group $K(A)$ by abuse of notation. It is clearly a covariant functor of the ring A (by extension of scalars). We have of course $K^{top}(X) \approx K(A)$, when $A = C(X)$.
As it was done in usual cohomology theory, one would like to "derive" this functor $K(A)$ and try to define $K_n(A), n \in \mathbb{Z}$, with some nice formal properties with $K_0(A) = K(A)$. This task is in fact more difficult than it looks for general rings A, as we shall see in Lectures 4 and 5.

On the other hand, if we avoid too much generality by working in the category of (complex) Banach algebras, there is essentially one way to do it (at least for $n \in \mathbb{N}$).

Firstly, we extend the definition of $K(A)$ to non necessarily unital algebras A by adding a unit to A. For this purpose, we consider the vector space $\mathbb{C} \oplus A = \overline{A}$ provided with the "twisted" multiplication defined by

$$(\lambda, a).(\lambda', a') = (\lambda.\lambda', \lambda.a' + a.\lambda' + a.a')$$

The algebra has now a unit $= (1, 0)$. There is an obvious augmentation

$$\overline{A} \longrightarrow \mathbb{C}$$

and we define $K(A)$ as the kernel of the induced map $K(\overline{A}) \longrightarrow K(\mathbb{C}) = \mathbb{Z}$. It is easy to see that if A already has a unit, we recover the previous definition $K(A)$.

Exercise: Let $\mathcal{K}$ be the ideal of compact operators in a Hilbert space H. Prove that $K(\mathcal{K}) \cong K(\mathbb{C}) \cong \mathbb{Z}$.

Secondly, for $n \in \mathbb{N}$, we define $K_n(A)$ as $K(A_n)$, where $A_n = A(\mathbb{R}^n)$ is the Banach algebra of continuous functions $f = f(x)$ from $\mathbb{R}^n$ to A which vanish when x goes to ∞.

Exercise: show that $K_1(A) \cong injlim \pi_0(GL_n(A)) \cong \pi_0(GL(A))$, where $GL(A)$ is the direct limit of the $GL_n(A)$ with respect to the obvious inclusions $GL_n(A) \subset GL_{n+1}(A)$.

Exercise: let $A = C(X)$, where X is locally compact. Let X^+ be the one point compactification of X (X with one point added outside) and let $Y = S^n(X^+)$, the n^{th} suspension of X^+. Show that $K^{top}(Y)$ is isomorphic to $K_n(A) \oplus \mathbb{Z}$ (hint : notice that $C(Y)$ is isomorphic to $\overline{C}(X \times \mathbb{R}^n)$. In particular $K^{top}(S^n)$ is isomorphic to $K_n(\mathbb{C}) \oplus \mathbb{Z}$.

The following theorem is not too difficult to prove. It can be extracted easily from [K1]

THEOREM . *The functors $K_n(A), n \in \mathbb{N}$, are characterized by the following properties*

1) Exactness: *for any exact sequence of Banach algebras (where A'' has the quotient norm and A' the induced norm)*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

we have an exact sequence of K-groups

$$K_{n+1}(A) \longrightarrow K_{n+1}(A'') \longrightarrow K_n(A') \longrightarrow K_n(A) \longrightarrow K_n(A'')$$

2) Homotopy : *$K_n(A(I)) \cong K_n(A)$, where $A(I)$ is the ring of continuous functions on the unit interval I with values in A.*

3) Normalization : *$K_0(A)$ is the group $K(A)$ defined above.*

Another type of K-groups which will be quite useful later on, although more technical, is the <u>*relative*</u> K-group associated to a functor. If $\phi : \mathcal{P}(A) \longrightarrow \mathcal{P}(A'')$ is an additive functor (with some extra topological conditions), one can also define groups $K_n(\phi)$ which fit into exact sequences

$$K_{n+1}(A) \longrightarrow K_{n+1}(A'') \longrightarrow K_n(\phi) \longrightarrow K_n(A) \longrightarrow K_n(A'')$$

For instance, if ϕ is asssociated to an epimorphism of Banach algebras $A \longrightarrow A''$ as above, $K_n(\phi)$ may be identified with $K_n(A')$. Another case of interest is when A (resp. A'') is the ring of continuous functions on a compact space X (resp. a closed subspace X"). Then, the group $K_n(\phi)$ is also written $K_n(X, X'')$ or $K^{-n}(X, X'')$.

The functors $K_*(A)$ have some other nice properties like the following : a continuous bilinear pairing of Banach algebras

$$A \times C \longrightarrow B$$

induces a "cup-product"

$$K_i(A) \otimes K_j(C) \longrightarrow K_{i+j}(B)$$

In particular, if $C = \mathbb{C}$, the field of complex numbers and if $A = B$, we have a pairing

$$K_i(A) \otimes K_j(C) \longrightarrow K_{i+j}(A)$$

Exercise: compute this pairing when $A = \mathbb{C}$ with i and $j \leq 2$.

We can now state the Bott periodicity theorem in the Banach algebras setting.

THEOREM . *The group $K_2(\mathbb{C})$ is isomorphic to $\mathbb{Z}$ and the cup-product with a generator u_2 induces an isomorphism $\beta : K_n(A) \longrightarrow K_{n+2}(A)$ for any Banach algebra A.*

This theorem has a spectacular application in homotopy : let $GL(A)$ be the infinite general linear group which is the direct limit of the $GL_r(A)$ with respect to the obvious inclusions. Then an analysis of the group $K(A(\mathbb{R}^n))$ shows that it is isomorphic to the homotopy group $\pi_{n-1}(GL(A))$. More precisely, if $A = \mathbb{C}$ for instance, the group $K(A(\mathbb{R}^n))$ is linked with the classification of stable complex vector bundles over the sphere S^n which are determined by homotopy classes of "gluing functions" $f : S^{n-1} \longrightarrow GL(A)$. For a general Banach algebra A, one has to consider bundles over the sphere whose fibers are A^r instead of $\mathbb{C}^r$. Then the previous theorem has the following corollary:

COROLLARY . *Let A be any complex Banach algebra. Then we have isomorphisms*

$\pi_i(GL(A)) \cong \pi_{i+2}(GL(A)) \cong K_1^{top}(A)$ *if i is even;*
$\pi_i(GL(A)) \cong \pi_{i+2}(GL(A)) \cong K(A)$ *if i is odd;*
In particular, we have $\pi_1(GL(A)) \cong K(A)$

As a matter of fact, the last isomorphism (inspired by Atiyah and Bott) is the basis of the proof of the theorem. The idea is to show that any loop in $GL(A)$ can be deformed into a loop of the following type

$$\theta \longmapsto pz + 1 - p$$

where p is an idempotent matrix and $z = e^{i\theta}$. Such an idempotent matrix p is of course associated to the projective module $Im(p)$. A more conceptual proof will be sketched at the end of this lecture.

Remark. If $A = \mathbb{C}$, the groups $\pi_i(GL_r(\mathbb{C})) = \pi_i(U(r))$ stabilize and are equal to $K_{i+1}^{top}(\mathbb{C})$ if $r > i/2$.

Exercise. Prove the following facts :

1) $K^{top}(S^n)$ is isomorphic to $\mathbb{Z}$ if n is odd and to $\mathbb{Z} \oplus \mathbb{Z}$ if n is even.
2) $\pi_i(GL_r(\mathbb{C})) \cong \pi_i(U(r)) \cong \mathbb{Z}$ if i is odd and $i > r/2$ and $= 0$ if i is even and $i > r/2$
3) $K^{top}(X) \oplus K^{top}(X) \cong K^{top}(X \times S^0) \cong K^{top}(X \times S^2) \cong K^{top}(X \times S^{2n})$ for any compact space X.

In the previous considerations, we were working with modules over *complex* Banach algebras and their topological counterparts which are complex vector bundles. We could as well consider *real* Banach algebras and real vector bundles.

Most considerations are valid in the real case, with the notable exception of the last theorem : in this case, we have an eight-periodicity and not a two-periodicity. More precisely :

THEOREM . *The group $K_8(\mathbb{R})$ is isomorphic to $\mathbb{Z}$ and the cup-product with a generator induces an isomorphism between the groups $K_n(A)$ and $K_{n+8}(A)$ for any real Banach algebra A.*

For instance, the groups $K_n(\mathbb{R})$ are equal to $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0$ respectively for $n = 0, 1, .., 7$ mod. 8.

The original proof of this theorem used Morse theory [Bott]. Other elementary proofs (whom we mentioned already) were found by Atiyah and Bott, Wood and the author, the last one in the framework of Banach algebras and Banach categories. See [At], [K1] and [Wo] for references.

As a matter of fact, Bott proved other theorems in the real case using loop spaces of homogeneous spaces. He gave the following impressive list of homotopy equivalences :

$$\Omega(\mathbb{Z} \times BGL(\mathbb{R})) \approx GL(\mathbb{R})$$
$$\Omega(GL(\mathbb{R})) \approx GL(\mathbb{R})/GL(\mathbb{C})$$
$$\Omega(GL(\mathbb{R})/GL(\mathbb{C})) \approx GL(\mathbb{C})/GL(\mathbb{H})$$
$$\Omega(GL(\mathbb{C})/GL(\mathbb{H})) \approx \mathbb{Z} \times BGL(\mathbb{H})$$
$$\Omega(\mathbb{Z} \times BGL(\mathbb{H})) \approx GL(\mathbb{H})$$
$$\Omega(GL(\mathbb{H})) \approx GL(\mathbb{H})/GL(\mathbb{C})$$
$$\Omega(GL(\mathbb{H})/GL(\mathbb{C})) \approx GL(\mathbb{C})/GL(\mathbb{R})$$
$$\Omega(GL(\mathbb{C})/GL(\mathbb{R})) \approx \mathbb{Z} \times BGL(\mathbb{R})$$

One way to understand these eight homotopy equivalences is to use Clifford algebras as it was pointed out for the first time by Atiyah, Bott and Shapiro. As a matter of fact, if we denote by C_n the Clifford algebra of $\mathbb{R}^n$ provided with the quadratic form $(x_1)^2 + \ldots + (x_n)^2$, there is a kind of "periodicity" of the C_n[3] : we have graded algebra isomorphisms

$$C_{n+8} \approx M_{16}(C_n)$$

On the other hand, the complexified Clifford algebras have a 2-periodicity

$$C_{n+2} \otimes \mathbb{C} \approx M_2(C_n) \otimes \mathbb{C}$$

[3]Here and until p. 21, we shall write K instead of K^{top}

These remarks gave rise to an elementary proof of the previous 8 homotopy equivalences and the fact that $\pi_7(GL(A)) \approx K_0(A)$ [K1] [Wo]. As a matter of fact, a uniform way to state these homotopy equivalences is to write (up to connected components)

$$GL(C_n)/GL(C_{n-1}) \approx \Omega[GL(C_{n+1})/GL(C_n)]$$

A second way to interpret Bott periodicity is to use Hermitian K-theory where one studies finitely generated projective modules provided with non degenerate Hermitian forms. This theory is associated to other classical Lie groups like the orthogonal group or the symplectic group. We shall investigate this theory in the sixth lecture and extend it to a discrete context (which is not possible with the Clifford algebra interpretation).

Before going back to classical Algebraic Topology, it might be interesting to give a conceptual proof of Bott periodicity (at least in the complex case).

Let us assume we have defined $K_n(A)$, not only for $n \in \mathbb{N}$, as we did before, but also for $n \in \mathbb{Z}$. Let us assume also that the pairing

$$K_i(A) \otimes K_j(C) \longrightarrow K_{i+j}(B)$$

roughly described above, extends to all values of i and j in $\mathbb{Z}$ and has some obvious "associative" properties. Finally, let us suppose the existence of a "negative" Bott element u_{-2} in $K_{-2}(\mathbb{C})$ such that the cup-product with u_2 gives the unit element 1 in $K_0(\mathbb{C}) \cong \mathbb{Z}$. With these hypotheses, we may define an inverse

$$\beta' : K_{n+2}(A) \longrightarrow K_n(A)$$

of the Bott map

$$\beta : K_n(A) \longrightarrow K_{n+2}(A)$$

It is defined by the cup-product with u_{-2}. It is clear that the compositions of β with β' both ways are the identity.
The price to pay for this proof is of course the construction of this "negative" K-theory $K_n(A)$ for $n < 0$. This may be done, using the notion of "suspension" of a ring which is in some sense dual to the notion of the suspension of a space. More precisely, we define the "cone" CA of a ring A to be the set of all infinite matrices $M = (a_{ij}), i, j \in \mathbb{N}$, such that each row and each column only contains a finite number of non zero elements in A. This is clearly a ring for the usual rule of matrix multiplication. We make CA into a Banach algebra by completing it with respect to the following norm

$$\|M\| = \sup_j \sum_i \|a_{ij}\|$$

Finally we define $\widetilde{A}$, the "stabilization" of A as the closure of the set of finite matrices[4] in CA. It is a closed 2-sided ideal in CA and the suspension of A -

[4] An infinite matrix is called finite if all its elements are 0, except a finite number.

denoted by SA - is the quotient ring $CA/\widetilde{A}$.

DEFINITION/THEOREM . *Let A be a Banach algebra. Let us define the groups $K_{-n}(A)$ to be $K(S^nA)$, where S^nA is the $n^{th}-$suspension of A. Then $BGL(SA)$ is a "delooping" of $K_0(A) \times BGL(A)$, i.e. we have a homotopy equivalence*

$$\Omega(BGL(SA)) = K_0(A) \times BGL(A)$$

Therefore, to any exact sequence of Banach algebras as above

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

we can associate an exact sequence of K-groups

$$K_{n+1}(A) \longrightarrow K_{n+1}(A'') \longrightarrow K_n(A') \longrightarrow K_n(A) \longrightarrow K_n(A")$$

for $n \in \mathbb{Z}$.

For a better understanding of SA, it is interesting to notice that the ring of Laurent series $A < t, t^{-1} >$ is a good approximation of the suspension. Any element of $A < t, t^{-1} >$ is a series

$$S = \Sigma_{n \in Z} a_n t^n$$

such that $\Sigma_{n\in\mathbb{Z}} \mid a_n \mid< +\infty$. We define a ring homomorphism

$$A < t, t^{-1} > \longrightarrow SA$$

which associates to the series above the class of the following infinite matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 \ldots\ldots \\ a_{-1} & a_0 & a_1 \ldots\ldots \\ a_{-2} & a_{-1} & a_0 \ldots\ldots \\ \ldots & \ldots & \ldots \ldots\ldots \end{pmatrix}$$

For instance, in order to define the element u_{-2} mentionned above, it is enough to define a projective module over the Banach algebra $\mathbb{C} < t, u, t^{-1}, u^{-1} >$, i.e. a non trivial complex vector bundle on the torus $S^1 \times S^1$, as we can see easily, using Fourier analysis.

Let us come back now to usual Algebraic Topology and define $\overline{K}^n(X), n \in \mathbb{Z}$, for a compact space with base point $*$ as $K_{-n}(A)$ where A is the Banach algebra of continuous functions on X which vanish on the base point. The (contravariant) functors $X \longmapsto \overline{K}^n(X)$ define a (reduced) cohomology theory for compact spaces. One may also define a relative K-theory by putting $K^n(X, Y) = \overline{K}^n(X/Y)$ which is also isomorphic to the group $K_{-n}(\phi)$ defined p. 2, where the functor ϕ is associated to the surjection of rings

$$\theta : C(X) \longrightarrow C(Y)$$

To sum up, one has therefore an exact sequence

$$K^{n-1}(X) \longrightarrow K^{n-1}(Y) \longrightarrow K^n(X,Y) \longrightarrow K^n(X) \longrightarrow K^n(Y)$$

and an excision isomorphism

$$K^n(X,Y) \cong K^n(X/Y,*) \simeq K^n(X/Y)$$

Finally, it is easy to show that $K^0(X,\Phi) \cong K(X)$, the original K-theory of Grothendieck-Atiyah-Hirzebruch.

What we have defined above is a “cohomology theory” satisfying all the Eilenberg-Steenrod axioms, except the dimension axiom. This is just an example of what mathematicians called an “extraordinary” cohomology theory in the 60’s (other famous examples are coming from cobordism). As we shall see in the next lecture, this theory has many interesting topological applications.

There are many variants of topological K-theory which were considered in the 60’s. One of appealing interest is equivariant K-theory $K_G(X)$ where G is a compact Lie group acting on X which was introduced by Atiyah and Segal. It is defined as the K-theory of G-equivariant vector bundles on X. The analog of Bott periodicity in this context is the “Thom isomorphism” : one considers a *complex* G-vector bundle V on X and we would like to compute the equivariant K-theory of V (viewed as a locally compact space), i.e. $\overline{K}_G(V^+)$, where V^+ is the one-point compactification of V. If we denote this group simply by $K_G(V)$, we have a Thom isomorphism (due to Atiyah)

$$K_G(X) \longrightarrow K_G(V)$$

More generally, if V is a real vector bundle, we can define on V a *positive* definite metric invariant under the action of G and consider the associated Clifford bundle $C(V)$; the group G also acts naturally on $C(V)$. We denote by $\mathcal{E}_G(X)$ the category of real vector bundles where G and $C(V)$ act simultaneously ; these two actions are linked together by the formula

$$g * (a.e) = (g * a).(g * e)$$

where the symbol $*$ (resp. .) denotes the action of G (resp. of $C(V)$). For instance, if G is finite, it is easy to show that $\mathcal{E}_G^V(X)$ is equivalent to the category of finitely generated projective modules over the crossed product algebra $G \ltimes \overline{C}(V)$, where $\overline{C}(V)$ is the algebra of continuous sections of the bundle $C(V)$. On the other hand, if “1” denotes the trivial bundle of rank one (with trivial action of G), we have a “restriction” functor

$$\varphi : \mathcal{E}_G^{V\oplus 1}(X) \longrightarrow \mathcal{E}_G^V(X)$$

A generalization of the theorem of Atiyah quoted above is the following : the relative group $K(\varphi)$ is isomorphic to the equivariant K-theory of the Thom space, i.e. $K_G(V)$, with the notations above.

We should notice that the last statement is true in both real and complex K-theory and implies Bott periodicity for the K-theory of vector bundles!

It is also worth mentioning that Atiyah's theorem and its generalization are not easy to prove even if X is reduced to a point. A key ingredient in the proof is an index map which Atiyah constructed with suitable elliptic operators.

There is another generalization of topological K-theory which was considered by Donovan and the author at the beginning of the 70's and which became "fashionable" recently : this is "K-theory with twisted coefficients" $K^{\alpha}(X)$[5] defined for $\alpha \in H^3(X;\mathbb{Z})$ (we limit ourselves to the complex case). The simplest way to define it is to notice that the usual complex K-theory is also the K-theory of the Banach algebra of continuous functions

$$f : X \longrightarrow \mathcal{K}$$

where $\mathcal{K}$ is the ideal of compact operators in a Hilbert space H (we just enlarge $\mathbb{C}$ to $\mathcal{K}$, which is quite usual in functional analysis). This is also the algebra of sections of the trivial bundle

$$\begin{array}{c} E = X \times \mathcal{K} \\ \big\downarrow \\ X \end{array}$$

viewed as a bundle of algebras. The idea is now to "twist" this algebra bundle by an automorphism. More precisely, using the fact that $Aut(H)$ is contractile by Kuiper's theorem, we have the following principal bundles with contractible total spaces (where $P(H) = Aut(H)/C^*$).

$$B\mathbb{C}^* \longrightarrow B(Aut(H)) \longrightarrow B(P(H))$$

$$K(\mathbb{Z},2) \longrightarrow B\mathbb{C} \longrightarrow B\mathbb{C}^*$$

This shows $BP(H)$ is the Eilenberg-Mac Lane space $K(\mathbb{Z},3)$. In other words, any element α of $H^3(X;\mathbb{Z})$ is associated to a principal bundle with fiber $P(H)$ which may be defined by transition functions g_{ji} over an open cover (U_i) of X. We can use this $P(H)$ bundle to twist the previous <u>*algebra*</u> bundle E in the following way : the twisted algebra bundle $\overline{E}$ is obtained by gluing $U_i \times \mathcal{K}$ with $U_j \times \mathcal{K}$ over $U_i \cap U_j$: we identify (x_i, k_i) with (x_j, k_j) whenever $x = x_i = x_j \in U_i \cap U_j$ and $k_j = g_{ji}(x)k_i(g_{ji}(x))^{-1}$.

The space of sections of $\overline{E}$ is a Banach algebra whose K-theory is precisely the twisted K-theory of X we wanted to define.

Exercise. Let V be an oriented real vector bundle of even rank with second Stiefel-Whitney class $w_2 \in H^2(X;\mathbb{Z}/2)$ and a the Bockstein of w_2 in $H^3(X;\mathbb{Z})$.

[5] now called "twisted" K-theory

Show that $K^{\alpha}(X)$ is isomorphic to the K-theory of the Thom space of V.

Let us mention finally that in the classical case of untwisted K-theory $K^*(X)$, there is a spectral sequence due to Atiyah and Hirzebruch whose E_2 term is $H^p(X; K^q_{top}(*))$, where $*$ is a point. This spectral sequence converges to $K^{p+q}(X)$. There is a generalization of this spectral sequence in the case of twisted K-theory which will appear in a forthcoming paper by Atiyah and Segal.

Lecture 3

Some applications of topological K-theory. The Atiyah-Singer index theorem

There are many applications of topological K-theory, most of them due to Atiyah and Adams. More recent applications in the K-theory of Banach algebras are also given.

1) Riemann-Roch theorems for differentiable manifolds [AH2][K1].

As it was explained in the first lecture, Grothendieck's original aim was to generalize the Hirzebruch-Riemann-Roch theorem in the context of projective algebraic smooth varieties. Soon after the discovery of topological K-theory, Atiyah and Hirzebruch achieved the same result in the category of differentiable manifolds. The simplest case is the following : let us consider two $C^{\infty}-$ manifolds X and Y, with an almost complex structure given on the tangent bundles, together with a smooth <u>proper</u> map

$$f : X \longrightarrow Y$$

Since X and Y are oriented, there is a Gysin map in cohomology

$$H^*(X) \longrightarrow H^*(Y)$$

which increases the degrees by the number $dim(Y) - dim(X)$. On the other hand, as briefly stated in the first lecture, any complex vector bundle V has a "Todd class" $Todd(V)$ which can be formally defined as follows. We express the "total Chern class"

$$c(V) = 1 + c_1(V) + c_2(V) + ... + c_n(V) + ...$$

as a formal product $\prod(1 + x_i)$, so that the $c_i(V)$ are the elementary symmetric functions of the x_j. Then we consider the formal product $\prod x_i/(1 - e^{-x_i})$ which can be written as

$$Todd(V) = 1 + Todd_1(V) + Todd_2(V) + ... + Todd_n(V) + ...$$

For instance[6], one has the following formulas (with $c_i = c_i(V)$, and $T_i = Todd_i(V)$)

[6]Note that this definition of $Todd(V)$ can be extended to elements x of $K(X)$: if $x = V - W$, we put formally $Todd(x) = Todd(V)/Todd(W)$.

$T_1 = 1/2c_1$
$T_2 = 1/12(c_2 + (c_1)^2)$
$T_3 = 1/24c_2c_1$
$T_4 = 1/720[-c_4 + c_3c_1 + 3(c_2)^2 + 4c_2(c_1)^2 - (c_1)^4]$
....

Using (complex) Bott periodicity and Thom isomorphism in K-theory, one may also define a Gysin map in K-theory

$$f_*^K : K^{top}(X) \longrightarrow K^{top}(Y)$$

The Atiyah-Hirzebruch theorem may then be stated as follows : for any element x in $K(X)$, we have the formula

$$Ch(f_*^K(x)).Todd(TY) = f_*^H(Ch(x).Todd(TX))$$

If Y is reduced to a point, $f_*^H$is given by integration over the compact manifold X and we find that for any element x in the K-theory of X, $Ch(x).Todd(TX)$ is always an $\underline{integral}$ class of degree $2n = dim(X)$ (a priori it is only a rational class). For instance, $Todd_n(TX)$ is an integral class when X is a manifold of dimension $2n$ with an almost complex structure.
There are many interesting generalizations of this formula (all due to Atiyah and Hirzebruch) when Tf (the tangent bundle of f "along the fibers") is provided with a ^{c}spinorial structure. There is an analog of the *Todd* class, called the $\widehat{A}$-genus which can be expressed in terms of the Pontrjagin classes of TX. Here is the type of formula we get :

$$Ch(f_*^K(x)) = f_*^H(ch(x).\widehat{A}(Tf))$$

This relation implies other integrality theorems in terms of the Pontrjagin classes of the tangent bundles involved. Here is a typical example : if X is a manifold of dimension 8 which is spinorial, the value of $7(p_1)^2 - 4p_2$ is divisible by 11720. Here p_1 and p_2 are the first two Pontjagin classes of TX (see [K1] for more examples and details].

2) The Atiyah-Singer index theorem

There is an extensive literature about this famous theorem and the basic references are the series of papers of the authors themselves [AS1][AS2]. Therefore, we limit ourselves to the general ideas underlying this theorem. Let X be a compact oriented manifold, E and F two C^∞-vector bundles on X and $D : \Gamma(X, E) \longrightarrow \Gamma(X, F)$ an elliptic differential operator. Then $Ker(D)$ and $Coker(D)$ are finite dimensional vector spaces and one would like to compute the index of D.

$$Index(D) = dim(KerD) - dim(CokerD)$$

in purely topological terms.

Since the operator D is elliptic, its symbol may be viewed as a morphism of vector bundles

$$\sigma : \pi^*(E) \longrightarrow \pi^*(F),$$

where π is the projection of the cotangent bundle T^*X over X, which is an isomorphism outside the zero section of T^*X. If $B(X)$ (resp. $S(X)$) denotes the ball bundle (resp. the sphere bundle) of T^*X with respect to a metric, the symbol gives rise to an element, called $[\sigma]$, of the relative group $K^0(B(X), S(X))$. Let us denote $C(\sigma)$ the image of $[\sigma]$ under the composition of the Chern character : $K^0(B(X), S(X)) \longrightarrow H^{ev}(B(X), S(X))$ and the Thom isomorphism $H^{ev}(B(X), S(X)) \longrightarrow H^*(X)$. The product of $C(\sigma)$ by the Todd class of the complexified bundle $T'(X)$ of TX is a cohomology class $C(\sigma).Todd(T'(X))$ which we can evaluate on the fundamental class $[X]$ of the manifold. This evaluation is written classically as $C(\sigma).Todd(T'(X))[X]$. The Atiyah-Singer index theorem can be stated as follows :

$$Index(D) = C(\sigma).Todd(T'(X))[X]$$

In particular, $C(\sigma).Todd(T'(X))[X]$ is an integer[7]. On the other hand, various classical elliptic differential operators enable us to recover classical previous results : the Riemann-Roch theorem in the differentiable category and the signature theorem of Hirzebruch among many others [AS1][AS2]. We may also notice that the integrality theorems are a consequence of the index theorem. However, as it was shown in [AH2] (see also [K1]), these theorems are topological in nature and do not require the pseudo-differential machinery needed in the proof of the index theorem.

3) The Hopf invariant one problem

Before attacking this topological problem, we recall the operations λ^k of Grothendieck, defined by the exterior powers of bundles. As Adams has pointed out, there exist more convenient operations, called ψ^k, which are ring endomorphisms of $K(X)$ (and also $K^{top}(X)$). Moreover, we have the remarkable relation $\psi^k\psi^1 = \psi^{k1}$. In order to define this operation ψ^k, we notice the universal formula expressing the "fundamental" symmetric functions

$$S_K = \sum (x_i)^k$$

in terms of the elementary symmetric functions

$$\sigma_k = \sum_{i_1 < \ldots < i_k} x_{i_1} \ldots x_{i_k}$$

For instance, $S_1 = \sigma_1$, $S_2 = (\sigma_1)^2 - 2\sigma_2$,$S_3 = \sigma_1)^3 - 3\sigma_1\sigma_2 + 3\sigma_3$, etc. In general $S_k = Q_k(\sigma_1, ..., \sigma_k)$, where the Q_k are the so-called Newton polynomials. Then, for any element x of $K(X)$, we define $\varphi^k(x)$ as the expression $Q_k(\lambda^1(x), ..., \lambda^k(x))$.

[7] As a matter of fact, one has to be careful about signs, according to the chosen conventions.

The Hopf invariant one problem in Algebraic Topology is linked with another problem of more appealing interest : for which values of n the sphere S^{n-1} may be provided with an H-space structure ? It is not difficult to show (using Künneth theorem in rational cohomology for instance) that this is not possible if n is odd > 1. On the other hand, S^0, S^1, S^3, S^7 are easily seen to have a structure of H-spaces, in fact topological groups, except for S^7. It is a remarkable theorem of Adams that these spheres are the only ones with an H-space structure. The K-theory proof (due to Adams and Atiyah) is as follows (see [K1] for more details). Starting with the given product $S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$, with n even, and following Hopf, we construct a map $f : S^{2n-1} \longrightarrow S^n$ to which we associate the "Puppe sequence"

$$S^{2n-1} \longrightarrow S^n \longrightarrow Cf \longrightarrow S^{2n} \longrightarrow S^{n+1}$$

More precisely, Cf, the cone of f, is a CW-complex with 2 cells of dimensions n and $2n$ respectively : we attach a $2n$-cell to S^n using the map f. Now the reduced K-theory of Cf is easily computed : it is a free group with 2 generators u and v and we have $u^2 = hv$ for a certain scalar h which is precisely ± 1 (for a general map $f, h = h(f)$ is called the $\underline{\textit{Hopf invariant}}$ of f).

We now follow the pattern given by Atiyah and Adams to show that $h(f)$ cannot be odd in general (except if $n = 2, 4$ *or* 8). From the general properties of the Adams operations ψ^k, we have $\psi^k(v) = k^{2r}v$ and $\psi^k(v) = k^r u + \sigma(k)v$, where $n = 2r$ and $\sigma(k) \in \mathbb{Z}$. On the other hand, since $\psi^2 = (\lambda^1)^2 - 2\lambda^2$, we have $\psi^2(u) = u^2 \ mod.\ 2 = h(f)v \ mod.\ 2$. Hence $\sigma(2)$ must be odd, since $h(f)$ is odd by hypothesis. On the other hand, from the relation $\psi^k\psi^l = \psi^l\psi^k$, we deduce (for $l \neq 1$)

$$k^r(k^r - 1)\sigma(l) = l^r(l^r - 1)\sigma(k)$$

If we choose $l = 2$ and k odd, we see that 2^r should divide $k^r - 1$ for all odd integers k, a property of r which is only true if $r = 1, 2$ *or* 4 (an easy exercise left to the reader).

4) The vector field problem on the sphere.

Let M be a compact connected oriented manifold. Here is a classical result due to Hopf : there exists a non zero tangent vector field on M if and only if $\chi(M)$ - the Euler Poincaré characteristic of M - vanishes. A much harder problem is to find the maximum number of linearly independent tangent vector fields on M, even if M is the sphere S^{t-1}. The Gram-Schmidt orthonormalization procedure may be used to replace any field of $(n-1)-$ linearly independent tangent vectors by a field of $n-1$ tangent vectors of norm one which are orthogonal to each other. Therefore, if $O_{n,t}$ denotes the Stiefel manifold $O(t)/O(t-n)$, the existence of a field of $(n-1)$ linearly independent tangent vectors on S^{t-1} is equivalent to the existence of a continuous section $\sigma : O_{1,t} = S^{t-1} \longrightarrow O_{n,t}$ of the natural projection

$$O(n,t) \longrightarrow O_{1,t}$$

Note that each element a of $O(n,t)$ defines a linear map $\varphi_a : \mathbb{R}^n \longrightarrow \mathbb{R}^t$ which is injective and depends continuously on a.

Let us introduce now the (real) Hopf bundle ζ on $RP^{n-1} = S^{n-1}/\mathbb{Z}^2$ as the quotient of $S^{n-1} \times \mathbb{R}^t$ by the equivalence relation $(x,\lambda) \approx (-x, -\lambda_1, ..., -\lambda_t)$. In the same way, the Whitney sum of t copies of ζ (denoted by $t\zeta$) is the quotient of $S^{n-1} \times \mathbb{R}^t$ by the identification

$$(x, \lambda_1, ..., \lambda_t) \approx (-x, -\lambda_1, ..., -\lambda_t)$$

If we write $S(t\zeta)$ (resp. $S(t\varepsilon)$) for the sphere bundle of $t\zeta$ (resp. $t\varepsilon$), the existence of σ shows the existence of a commutative diagram

$$\begin{tikzcd}[column sep=small] S(t\varepsilon) \arrow[rr] \arrow[dr] & & S(t\varepsilon) \arrow[dl] \\ & X = RP^{n-1} & \end{tikzcd}$$

Here θ is induced by the map $(x,v) \longmapsto (x, \varphi_{\sigma(v)}(x))$ from $S^{n-1} \times S^{t-1}$ into itself, where φ_a is the linear map $\mathbb{R}^n \longrightarrow \mathbb{R}^t$ defined above. Moreover, θ induces a homotopy equivalence on each fiber.

We are now in the general situation of two vector bundles V and W (here $t\varepsilon$ and $t\zeta$) such that the associated sphere bundles $S(V)$ and $S(W)$ have the same <u>*fiber*</u> homotopy type. The same property is true if we compactify the fibers instead of considering the spheres. Therefore, the map θ (extended to the vector bundles by radial extension) has the property that for each point x in X we have the commutative diagram

$$\begin{tikzcd} K_{\mathbb{R}}(W) \arrow[r, "\theta^*"] \arrow[d] & K_{\mathbb{R}}(V) \arrow[d] \\ K_{\mathbb{R}}(W_x) \arrow[r, "\theta^*_x"] & K_{\mathbb{R}}(V_x) \end{tikzcd}$$

where $K_{\mathbb{R}}$ means <u>*real*</u> K-theory. As it was shown by Adams, this condition implies the existence of an element y of $\overline{K}_{\mathbb{R}}(X)$ such that for each k we have the relation

$$\rho^k(V) = \rho^k(W)\frac{\psi^k(1+y)}{1+y}$$

under the condition that V and W are spinorial bundles of rank $\equiv 0$ mod. 8.

Here $\rho^k(V)$ is the characteristic class in K-theory[8] introduced by Bott : it is defined as the image of 1 under the composition of the following maps

$$K_{\mathbb{R}}(X) \xrightarrow{\varphi} K_{\mathbb{R}}(V) \xrightarrow{\psi^k} K_{\mathbb{R}}(V) \xrightarrow{\varphi^{-1}} K_{\mathbb{R}}(X)$$

[8]Note that this class is constructed in complete analogy with the Stiefel-Whitney classes in cohomology mod.2: one just has to replace the Adams operations by the Steenrod squares.

where φ is the Thom isomorphism in $K_{\mathbb{R}}$-theory. These remarks, together with the computation of the order of the group $\overline{K}(RP^{n-1})$ are the key ingredients for the solution in two steps of our original problem.

PROPOSITION . *Let a_n be the order of the group $\overline{K}_{\mathbb{R}}(RP^{n-1})$, that is $a_n = 2^f$, where f is the number of integers i such that $0 < i < n$ and $i = 0, 1, 2$ or 4 mod 8. If S^{t-1} admits $n-1$ linearly independent tangent vector fields, then t is a multiple of a_n.*

From this proposition and the theory of Clifford algebras over the real numbers, we deduce the following fundamental theorem of Adams

THEOREM . *Let us write each integer t in the form $t = (2\alpha - 1).2^{\beta}$ where $\beta = \gamma + 4\delta$, with $0 \leq \delta \leq 3$. Then the maximum number of linearly independent tangent vector fields on the sphere S^{t-1} is $\sigma(t) = 2^{\gamma} + 8\delta - 1$.*

5) Applications to C^*-algebras.

Here is a theorem due to Connes : if $A = C^*_r(G)$ is the reduced C^*-algebra of a free group on n generators, then the only idempotents of A are 0 and 1. The proof is too technical to give the details here (see [R] p. 358-361) ; K-theory comes into the picture since one has to consider in the proof extensions of C^*algebras of the type

$$0 \longrightarrow \mathcal{K} \longrightarrow D \longrightarrow A \longrightarrow 0$$

where $\mathcal{K}$ is the ideal of compact operators in a Hilbert space. A splitting of this extension induces a homomorphism $K(A) \longrightarrow K(\mathcal{K}) \approx \mathbb{Z}$ which Connes identifies with the usual positive normalized trace on A.

Another beautiful application in Connes's book [C1] is a description of the Penrose tiling in terms of inductive limits of finite dimensional algebras (so-called AF-algebras). Such an inductive limit A (or rather its closure in the algebra of bounded operators in a Hilbert space) has of course a Grothendieck group $K(A)$. Moreover, $K(A)$ is an *ordered group* (an element of $K(A)$ is called ≥ 0 if it comes from a genuine projective module). In the case of the Penrose tiling, one has $K(A) \cong \mathbb{Z}^2$, the positive cone being given by the set of all (m, n) such that $n(1 + \sqrt{5})/2 + m \geq 0$. As a matter of fact, all AF-algebras A are classified by their ordered Grothendieck group $K(A)$.

In order to get more applications of the theory to C^*-algebras, a considerable generalization was made by Kasparov [Kas], in relation with Atiyah-Singer theorem. In Kasparov's theory, one associates to two C^*algebras A and B a new group called $KK(A, B)$ with remarkable formal properties. According to Higson $[H]$ for instance, this new theory $KK(A, B)$ - at least for separable C^*-algebras, not necessarily unitary - is characterized by the following properties

1. Homotopy invariance : the group is unchanged (functorially) if we replace A or B by the ring of continuous function on $[0, 1]$ with values in A or B.

2. The group is C^*-stable with respect to the second variable, i.e. we have the functorial isomorphism

$$KK(A, B) \approx KK(A, \mathcal{K} \otimes B),$$

where $\mathcal{K}$ is the C^*-algebra of compact operators in a separable Hilbert space.

3. If

$$0 \longrightarrow J \longrightarrow \mathcal{E} \longrightarrow B \longrightarrow 0$$

is a split exact sequence of C^*-algebras, we have $KK(A, \mathcal{E}) \approx KK(A, J) \oplus KK(A, B)$.

4. A "composition" (called Kasparov product)

$$KK(A, B) \times KK(B, C) \longrightarrow KK(A, C)$$

may be defined.

Therefore, the correspondence $(A, B) \longmapsto KK(A, B)$ defines a functor from the category of C^*-algebras couples (A, B) to the the category of abelian groups, which is an extended definition of the usual K-theory in the following sense. We have $KK(\mathbb{C}, A) \approx K(A)$ and every morphism $f : A \longrightarrow B$ gives rise to an element of $KK(A, B)$. By Kasparov's product, this induces a morphism from $K(A) = KK(\mathbb{C}, A)$ to $KK(\mathbb{C}, B) = K(B)$ which is the usual one : to every A-module M we associate the B-module $f_*(M) = B \otimes_A M$.

A concrete definition of $KK(A, B)$ and of the Kasparov product has been given by Cuntz [Cu]. He describes $KK(A, B)$ as the set of homotopy classes of $*$-homomorphisms

$$qA \longrightarrow \mathcal{K} \otimes B$$

Here qA is the ideal in the free product $A * A$ generated by the elements of type $x * 1 - 1 * x$. One can prove that this definition is equivalent to the first one of Kasparov and also to the definition of Connes-Skandalis ([C1] p. 428-436).

The relation with the Atiyah-Singer theorem is the following. Let $B = \mathbb{C}$ be the field of complex numbers and let A be the algebra of continuous functions on a compact manifold of even dimension. Then, an elliptic operator D on X defines a "Fredholm module" on A [At2] which gives rise to an element of $KK(A, \mathbb{C})$. The image of $1 \in K(A)$ by the associated morphism

$$K(A) \longrightarrow K(\mathbb{C}) = \mathbb{Z}$$

is the index of the operator D.

An open problem is to define a group $KK(A, B)$ for general algebras A and B and to find the relation with Algebraic K-theory on one side (lecture 5) and cyclic homology on the other side (lecture 7).

Lecture 4

Algebraic K-theory of Bass and Milnor Applications

After the success of topological K-theory, the algebraists felt challenged to "derive" the K_0 group for ANY ring A, as it was done for Banach algebras and, hopefully, prove some kind of "Bott periodicity" for these hypothetic $K_n(A)$.

The first problem was of course to define the K_1 group in a purely algebraic context. Remember that for a Banach algebra A, $K_1^{top}(A)$ (called before $K_1(A)$) was defined either as $K(A(\mathbb{R}))$ or $\pi_0(GL(A)) = injlim_n \pi_0(GL_n(A))$. The first definition does not make any sense a priori for an abstract ring A. However, Bass managed to give a meaning to the second definition thanks to the notion of an elementary matrix. Such a matrix - called e_{ij}^{λ} - has all entries $= 0$, except for the diagonal entries $= 1$ and the entry at the spot $(i, j), i \neq j$, which is equal to the scalar λ. These matrices satisfy the following identities

$e_{ij}^{\lambda} e_{ij}^{\mu} = e_{ij}^{\lambda+\mu}$
$[e_{ij}^{\lambda}, e_{jk}^{\mu}] = e_{ik}^{\lambda\mu}$ if $i \neq k$
$[e_{ij}^{\lambda}, e_{kl}^{\mu}] = 1$ if $j \neq k$ and $i \neq l$

We call $E(A)$ the subgroup generated by these elementary matrices. One has to think of the elements of $E(A)$ as those "homotopic to the identity matrix".

DEFINITION/THEOREM . *The subgroup $G' = E(A)$ is equal to the commutator subgroup $[G, G]$ of $G = GL(A)$. Moreover it is perfect, i.e. $[G', G'] = G'$. We define $K_1(A)$ as the quotient group G/G', in other words the group G made abelian.*

This theorem is quite easy to prove, using the relations above between elementary matrices. One of the key point is to show that a commutator $ghg^{-1}h^{-1}$ is a product of elementary matrices if we stabilize 3 times the size of the matrix, say n. Indeed, let us write $Diag(u, v, w)$ for a matrix of size $3n$ with u, v and w as n-diagonal blocks. Then we have $Diag(ghg^{-1}h^{-1}, 1, 1) = [u, v]$, where $u = Diag(g, 1, g^{-1})$ and $v = Diag(h, h^{-1}, 1)$. On the other hand, a matrix of type

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

may be written as the product

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}$$

We notice finally that the matrix

$$\begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}$$

may be written as the product of 3 matrices, each of them being a product of elementary matrices, i.e.

$$\begin{pmatrix} 1 & -g^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} \begin{pmatrix} 1 & -g^{-1} \\ 0 & 1 \end{pmatrix}$$

Caution: if A is a Banach algebra, this group $K_1(A)$ does not coincide with its topological analog. For instance, if A is the field of complex numbers $\mathbb{C}$, we have $K_1(\mathbb{C}) \approx \mathbb{C}^*$, whereas the topological analog $\pi_0(GL(\mathbb{C}))$ is equal to 0.

THEOREM . *Let us consider a cartesian square of rings with units*

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \big\downarrow & & \big\downarrow{\scriptstyle \phi_1} \\ A_2 & \xrightarrow{\phi_2} & A' \end{array}$$

with ϕ_1 or ϕ_2 surjective. Then we have a "Mayer-Vietoris exact sequence" of K-groups

$$K_1(A) \longrightarrow K_1(A_1) \oplus K_1(A_2) \longrightarrow K_1(A') \longrightarrow K(A) \longrightarrow K(A_1) \oplus K(A_2) \longrightarrow K(A')$$

This theorem implies "excision" for the K_0-group ; in other words, if I is an ideal in a ring A, the K-group of the functor[9] $\mathcal{P}(A) \longrightarrow \mathcal{P}(A/I)$ only depends on the ideal I, i.e. is isomorphic to $Ker(K(\overline{I}) \longrightarrow K(\mathbb{Z}))$, where $\overline{I}$ denotes the $\mathbb{Z}$-algebra I with a unit added.

In the case of a group ring $A = \mathbb{Z}[G]$, the group K_1 has applications in differential topology as it was shown in the 60's. A striking one is the s-cobordism theorem ([R] p. 86) : if M is a compact connected manifold of dimension $n \geq 5$, any $h-$cobordism built on M is homeomorphic (rel. ∂M) to a product $M \times [0,1]$.

[9] Here we should be more precise about the definition of the Grothendieck group of an additive functor $\varphi : \mathcal{C} \longrightarrow \mathcal{C}$. It is the quotient of the free group generated by isomorphism classes of triples (E, F, α), where E and F are objects of $\mathcal{C}$ and $\alpha : \varphi(E) \longrightarrow \varphi(F)$ is an isomorphism, by the following relations:
$(E, F, \alpha) + (E', F', \alpha') = (E \oplus E', F \oplus F', \alpha \oplus \alpha')$
$(E, F, \alpha) + (F, G, \beta) = (E, G, \beta.\alpha)$.

There is just one algebraic condition for this : $Wh(G) = K_1(A)/G$ should be trivial. A spectacular application is the famous theorem of Smale which solves the Poincaré conjecture in dimension ≥ 5 : any manifold of dimension $n \geq 5$ homotopically equivalent to the sphere S^n is indeed homeomorphic to S^n.

A more algebraic application is the famous “congruence subgroup problem” : if R is a commutative ring and $SL(R)$ the subgroup of $GL(R)$ consisting of matrices of determinant 1, is every normal subgroup of $SL(R)$ a congruence subgroup, i.e. the kernel of a group homomorphism $SL(R) \longrightarrow SL(R/I)$ for a certain ideal I ? An analysis of a "relative" K_1 group shows this is the case for instance if $R = \mathbb{Z}$. In general one should have $SK_1(A, I) = 0$ for all ideals I([R] p.106).

Let us mention an important result with many applications, due to Bass, Milnor and Serre : if A is the ring of integers in a number field F, then $K_1(A) \cong A^*$ by the determinant map. In other words, the map $K_1(A) \longrightarrow K_1(F) \cong F^*$ is injective.

One question we may ask is whether $K_1(A)$ is the K_0-group of another ring ; this is inspired by Topology where we saw that $K^{-1}(X)$ is essentially K^0 of the suspension of X. There is indeed a dual notion of the suspension of a space which is the “loop ring” ΩA of A : it is the subring of $A[x]$, consisting of polynomials $P(x)$ such that $P(0) = P(1) = 0$. Under certain conditions (for instance if A is regular noetherian), one can show that $K_1(A) \cong K_0(\Omega A)$. This is the starting point of the so-called "Karoubi-Villamayor" K-theory which will be sketched in the next lecture.

Exercise: Let $\Omega^+(A)$ be the $A-$algebra ΩA with a unit added. Show that $\Omega^+(A)$ may be identified with the coordinate ring of the following cubic

$$\Omega^+(A) \cong A[u, v]/(u - v)^3 - uv.$$

Exercise (Milnor). Let A be the Banach algebra of continuous functions on a compact space X with complex values. Then show that $K_1(A) \cong A^* \oplus \pi_0(SL(A))$

Remark. In contrast with the situation of the K_0-group, the definition of $K_1(I)$ when I is a ring without unit is unclear. For instance, if we define $K_1(I)$ as $Ker[K_1(I^+) \longrightarrow K_1(k)]$ if I is a $k-$algebra, this definition depends strongly on the choice of k.

For certain special rings I however, there is a “good” definition of A, for instance the ideal $\mathcal{K}$ of compact operators in a Hilbert space, or the ring of continuous functions from a compact space X to $\mathcal{K}$.

The next step in Algebraic $K-$theory was done by Milnor who gave an algebraic definition of $K_2(A)$. He first introduced the (infinite) Steinberg group $ST(A)$ which is generated by elements x_{ij}^{λ}, $i \neq j$, subject to the “universal” relations between elementary matrices, i.e.

$x_{ij}^{\lambda} x_{ij}^{\mu} = x_{ij}^{\lambda+\mu}$
$[x_{ij}^{\lambda}, x_{jk}^{\mu}] = x_{ik}^{\lambda\mu}$ if $i \neq k$
$[x_{ij}^{\lambda}, x_{kl}^{\mu}] = 1$ if $j \neq k$ and $i \neq l$

Note: for convenience, we shall sometimes write these symbols as $x_{ij}(\lambda)$.

There is an obvious epimorphism

$$ST(A) \longrightarrow E(A)$$

As we said before, the group $E(A)$ is perfect (i.e. is equal to its own commutator subgroup $[E(A), E(A)]$) and Milnor proved that the exact sequence

$$1 \longrightarrow K_2(A) \longrightarrow ST(A) \longrightarrow E(A) \longrightarrow 0$$

is in fact the universal central extension of the perfect group $E(A)$. In particular, $K_2(A)$ is abelian and may be identified with $H_2(E(A); \mathbb{Z})$, the second homology group of $E(A)$ with $\mathbb{Z}$ coefficients. Analogously, one may remark that $K_1(A)$ is the homology group $H_1(GL(A); \mathbb{Z})$.

An important part of the structure of K_2 is defined by Milnor's cup-product

$$K_1(A) \times K_1(A) \longrightarrow K_2(A)$$

when A is commutative. In particular, if u and $v \in A^* \subset K_1(A)$, the *Steinberg symbol* $\{u, v\}$ is the cup-product of u and v in $K_2(A)$. It satisfies the following relations
(a) $(u, 1-u) = 1$ if u and $1-u$ belong to A^*
(b) $(u, -u) = 1$ if $u \in A^*$

If A is a field, the second relation follows from the first.

THEOREM (Matsumoto). *Let F be a (commutative) field. The Steinberg symbol identifies $K_2(F)$ with the quotient of $F^* \otimes_{\mathbb{Z}} F^*$ by the subgroup generated by the relations $u \otimes (1-u)$ for $u \neq 0, 1$.*

This theorem is the starting point for applications of K_2 in Algebra and Number Theory, in relation with the Brauer group and Galois cohomology. Let $\overline{F}$ be the separable closure of F, G the Galois group of this extension and μ_n the multiplicative group consisting of n-roots of unity in $\overline{F}$. Then, the exact sequence

$$0 \longrightarrow \mu_n \longrightarrow \overline{F}^* \longrightarrow \overline{F}^* \longrightarrow 0$$

induces an isomorphism $H^1(G; \mu_n) \cong F^*/F^{*n}$, since $H^1(G; \overline{F}^*) = 0$ by a famous theorem of Hilbert. By a classical argument, the composition

$$s : F^* \times F^* \longrightarrow F^*/F^{*n} \times F^*/F^{*n} \cong H^1(G; \mu_n) \times H^1(G; \mu_n) \longrightarrow H^2(G; (\mu_n)^{\otimes 2})$$

where the last map is the cup-product in Galois cohomology, is a Steinberg symbol (i.e. $s(u, 1-u) = 0$ if $u \neq 0, 1$). Therefore, s induces a homomorphism from $K_2(F)/nK_2(F)$ to $H^2(G; (\mu_n)^{\otimes 2})$. The following classical theorem ([Sr] p. 149) is one of the nicest in the subject:

THEOREM (**Merkurjev-Suslin**). *The homomorphism*

$$K_2(F)/nK_2(F) \longrightarrow H^2(G; (\mu_n)^{\otimes 2})$$

defined above is an isomorphism.

As a concrete application let us consider the Brauer group of F, $Br(F)$, generated (multiplicatively) by central simple algebras A with the relation $A \sim B$ iff $M_n(A) \cong M_s(B)$ for some numbers n et s. It is well known that $Br(F)$ is canonically isomorphic to the Galois cohomology group $H^2(G, \overline{F}^*)$.

COROLLARY . *Let us assume $\mu_n \subset F$. Then $H^2(G; (\mu_n)^{\otimes 2}) \cong H^2(G; \mu_n) \cong Br(F)_n$, the n-torsion of the Brauer group. Therefore, we have an isomorphism*

$$K_2(F)/nK_2(F) \longrightarrow Br(F)_n$$

A striking example is the case when $n = 2$ and F is of characteristic $\neq 2$. Then, the central simple algebra associated to the symbol $\{a, b\}$ is the quaternion algebra generated by the symbols i and j with the relations $i^2 = a, j^2 = b$ and $ij = -ji$. This corollary implies that any element of the 2-torsion in $Br(F)$ is stably isomorphic to a product of quaternion algebras, an open problem before Merkurjev and Suslin proved their theorem.

As another example, let us consider the case when A is a Banach algebra. As we have seen in a previous lecture, there is a topologically $K_2^{top}(A)$ which is simply the fundamental group $\pi_1(GL(A))$. In order to see the relation with the group $K_2(A)$, we send a generator x_{ij}^{λ} of the Steinberg group to the path $t \longmapsto x_{ij}^{\lambda t}$. The relations

$x_{ij}^{\lambda} x_{ij}^{\mu} = x_{ij}^{\lambda+\mu}$
$[x_{ij}^{\lambda}, x_{kl}^{\mu}] = 1$ if $j \neq k$ and $i \neq l$

still hold if we replace λ and μ by λt and μt. Now if we compute the commutator $[x_{ij}^{\lambda t}, x_{jk}^{\mu t}]$ for $i \neq k$, we find the path $x_{ij}^{\lambda\mu t^2}$ which is homotopic (end points fixed) to the path $x_{ij}^{\lambda\mu t}$. This shows that the previous correspondence induces a well defined homomorphism :

$$\varphi : ST(A) \longrightarrow E(GL(A))/\mathcal{R}$$

Here EX denotes in general the path space of X, i.e. the set of continuous paths $f : [0,1] \longrightarrow X$ such that $f(0) = I$, the unit matrix, and $\mathcal{R}$ the equivalence relation defined by homotopy with fixed end points. Since $K_2(A)$ is the kernel of the homomorphism $ST(A) \longrightarrow GL(A)$, we see immediately that $\varphi(x)$ is a loop in $GL(A)$ if $x \in K_2(A)$ and that φ induces a well defined homomorphism

$$\varphi : K_2(A) \longrightarrow \pi_1(GL(A)) = K_2^{top}(A)$$

In general, φ is neither an injection nor a surjection. As a typical example, if we choose A to be the field of complex numbers $\mathbb{C}$, we see that φ is reduced to 0 since its image is contained in $\pi_1(SL(\mathbb{C})) = 0$. Therefore, the kernel of φ is the full group $K_2(\mathbb{C})$, which is known to be an uncountable group. [M].

There are however Banach algebras A such that φ is an isomorphism. A typical example is the algebra $\mathcal{K}$ of compact operators in a Hilbert space or, more generally, the algebra of continuous functions $X \longrightarrow \mathcal{K}$, where X is a compact space. A technical problem appears here since these algebras have no unit. However, as it was proved in [SW][10], these algebras satisfy excision and we may choose to add a unit the way we want, as for the group K_0. For these algebras, Bott periodicity holds : $K_2(A) \cong K_0(A)$, which is quite exceptional in Algebraic K-theory.

The previous considerations suggest to replace topological homotopy by algebraic homotopy for the definition of the group $\pi_1(GL(A))$ (A being now a discrete ring). An "algebraic loop" is just an element σ of $GL(A[t])$ such that $\sigma(0) = \sigma(1)$. Two algebraic loops σ_0 and σ_1 are called homotopic if there is an element Σ of $GL(A[t,u])$ such that

$$\Sigma(t,0) = \sigma_0(t), \Sigma(t,1) = \sigma_1(t)$$
$$\Sigma(0,u) = \Sigma(1,u) = 1$$

This is an equivalence relation because $GL(A)$ is a group and we define the algebraic fundamental group $\pi_1^{alg}(GL(A))$ as the group of homotopy classes of algebraic loops (like in Topology). As before, we have an homomorphism

$$K_2(A) \longrightarrow \pi_1^{alg}(GL(A))$$

which is an isomorphism if A is a regular Noetherian ring. This gives an alternative definition of K_2, closer to Topology for this type of rings.

From another viewpoint, we also have a cup-product in Topological K-theory (for commutative Banach algebras)

$$K_1^{top}(A) \times K_1^{top}(A) \longrightarrow K_2^{top}(A)$$

[10] See also *M.* Karoubi: Homologie des groups discrets associés à des algèbres d'opérateurs. Journal of Operator Theory. 15, p. 109-161(1986).

The obvious diagram

$$\begin{array}{ccc} K_1(A)\times K_1(A) & \longrightarrow & K_2(A) \\ \downarrow & & \downarrow \\ K_1^{top}(A)\times K_1^{top}(A) & \longrightarrow & K_2^{top}(A) \end{array}$$

is then commutative. As a typical example, it is easy to show that $K_1(\mathbb{Z}) = \mathbb{Z}/2$ and the composition $K_1(\mathbb{Z}) \longrightarrow K_1(\mathbb{R}) \longrightarrow K_1^{top}(\mathbb{R})$is injective. It follows that the cup-product of the generator of $K_1(\mathbb{Z})$ by itself is a non trivial element of $K_2(\mathbb{Z})$, because the same is true for the topological K-theory of $\mathbb{R}$. As a matter of fact, as it was proved in Milnor's book, we have $K_2(\mathbb{Z}) = \mathbb{Z}/2$.

There are many other examples where we can compute the group $K_2(A)$. For finite fields F_q for instance, Milnor showed that $K_2(F_q) = 0$. Another fundamental example is the case of a number field F and its ring of integers A. We then have a "localization" exact sequence :

$$K_2(A) \longrightarrow K_2(F) \longrightarrow \oplus_{\mathcal{P}} K_1(A/\mathcal{P}) \longrightarrow K_1(A) \longrightarrow K_1(F)$$

where $\mathcal{P}$ runs over all the prime ideals of A. As a matter of fact, by a famous result of Bass, Milnor and Serre quoted in the previous lecture, the map $K_1(A) \longrightarrow K_1(F) \cong F^*$ is injective and we therefore have a well defined surjective map $K_2(F) \longrightarrow \oplus K_1(A/\mathcal{P}) \cong \oplus(A/\mathcal{P})^*$. Each component

$$K_2(F) \longrightarrow (A/\mathcal{P})^*$$

is called the tame symbol and may be defined explicitly, using the $\mathcal{P}$-adic valuation.

Finally, there is a "fundamental theorem" due to Bass, Heller and Swan which enables us to compute the Algebraic $K-$theory K_i (for $i = 1$ and 2) of the ring of Laurent polynomials $A[t, t^{-i}]$ in terms of the groups K_{i-1} of A. More precisely, we have an exact sequence

$$0 \longrightarrow K_i A \longrightarrow K_i(A[t]) \oplus K_i(A[t^{-i}]) \longrightarrow K_i(A[t, t^{-i}]) \longrightarrow K_{i-1}(A) \longrightarrow 0$$

If A is a regular Noetherian ring there is a "homotopy invariance" (first proved by Grothendieck) i.e. $K_i(A) \cong K_i(A[t])$ for $i = 0, 1, 2$. Therefore, the previous exact sequence reduces to the isomorphism

$$K_i(A[t, t^{-i}]) \cong K_i(A) \oplus K_{i-1}(A)$$

in this case. We shall come back to this matter for the definition of the negative K-groups in the next lecture.

As for the functor K_1 there is an application of K_2 in differential topology which is due to Hatcher and Wagoner. This application is related to pseudo-isotopy classes of diffeomorphisms on manifolds. As it is well known, isotopy classes of diffeomorphisms of the sphere S^n for instance parametrize the set of differentiable structures on S^{n+1} for $n \geq 5$. However, it is difficult to tell in practice when two diffeomorphisms h_0 and h_1 of a manifold M are isotopic. There is a weaker notion of "pseudo-isotopy" which is a diffeomorphism of a cyclinder $M \times [0,1]$, whose restrictions to $M \times \{0\}$ and $M \times \{1\}$ give h_0 and h_1 respectively. Let us call $\mathcal{P}(M)$ the space of pseudo-isotopies of M, i.e. diffeomorphisms of $M \times [0,1]$ which restrict to the identity on $M \times \{0\}$.

Before stating the result about this space $\mathcal{P}(M)$, we need a preliminary definition: in the Steinberg group $St(A)$ of a group algebra $A = \mathbb{Z}[G]$, we define $w_{ij}(u) = x_{ij}(u).x_{ji}(-u^{-1}).x_{ij}(u)$ and W_G the subgroup of $St(A)$ generated by the $w_{ij}(u)$. Finally, we define

$$Wh_2(G) = K_2(A)/K_2(A) \cap W_G$$

.

THEOREM (Hatcher-Wagoner). ([R] p. 242) *Let M be a smooth compact connected manifold without boundary, of dimension $n \geq 5$ and with fundamental group G. Then there is a surjection $\pi_0(\mathcal{P}(M)) \longrightarrow Wh_2(G)$.*

In some cases, the kernel has been computed and identified with a group related to $Wh_1(G)$.

Lecture 5

Higher Algebraic K-theory Some computations

After the definition of $K_1(A)$ and $K_2(A)$ by Bass and Milnor, the problem was open at the beginning of the 70's for a "good" definition of $K_n(A)$ for $n \geq 3$. It is now widely accepted that this definition was proposed by Quillen at the International Congress of Mathematicians in Nice (1970). However, other definitions were proposed at the same time and are in fact isomorphic to Quillen's definition in favourable cases. We shall discuss briefly one of them at the end of the lecture.

Let us point out first that there is no "axiomatic" definition of these groups K_n in the spirit of what has been done for Banach algebra for instance (p. 9). More precisely, Swan has pointed out the following disturbing fact : we cannot have a natural Mayer-Vietoris exact sequence associated to a cartesian square of rings with units

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \phi_1} \\ A_2 & \xrightarrow{\phi_2} & A' \end{array}$$

with ϕ_1 and ϕ_2 surjective; in other words, in exact

$$K_{n+1}(A) \to K_{n+1}(A_1) \oplus K_{n+1}(A_2) \to K_{n+1}(A') \to K_n(A) \to K_n(A_1) \oplus K_n(A_2) \to K_n(A')$$

with K_0 isomorphic to the Grothendieck group.

This means that instead of waiting for a "good" axiomatic definition of $K_n(A)$, Quillen's proposal is justified by the nice theorems and applications proved with it (the Merkurjev-Suslin theorem in the previous lecture is for instance a good example of an application ; it could not be proved without Quillen's theory).

The definition of $K_n(A)$ uses the Algebraic Topology machinery, as we have seen in Berrick's series of lectures : more precisely, we have $K_n(A) = \pi_n(BGL(A)^+)$, for $n > 0$, where $BGL(A)^+$ is a space obtained from the classifying space of the infinite general linear group $GL(A)$ by adding cells of dimension 2 and 3. For a change, we shall adopt a more geometric viewpoint, using the concept of "virtual" flat $A-$bundles.

Firstly, we define a flat $A-$bundle on a space X as a covering $P \longrightarrow X$ such that each fiber is a f.g.p. (= finitely generated projective) A-module. Note that if $A = \mathbb{C}$, we recover the usual notion of flat complex vector bundle.

A ***virtual*** flat $A-$bundle on a space X (assumed to be a finite CW-complex) is given by
1) an acyclic map $Y \longrightarrow X$ (i.e. a map inducing an isomorphism in homology with local coefficients or, equivalently, such that its homotopy fiber is an acyclic space).
2) a flat $A-$bundle E on Y

Two virtual $A-$bundles E and E' on Y and Y' respectively are called equivalent if we can find a commutative diagram (up to homotopy)

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \Big\downarrow & & \Big\uparrow \\ Z & \longleftarrow & Y' \end{array}$$

with where $g : \ Z \longrightarrow X$ is an acyclic map, together with a flat A-bundle F on Z such that the pull-back of F through the map $Y \longrightarrow Z$ (resp. $Y' \longrightarrow Z$) is isomorphic to E (resp. E'). Two virtual flat A-bundles (for instance E and E' with the previous notations) may be added by taking the homotopy fiber product T of

Y and Y' over X and adding $\gamma^*(E)$ and $\gamma'^*(E')$, where γ and γ' are the obvious projections of T on Y and Y'. This addition is compatible with the equivalence relation.

DEFINITION/THEOREM . [K2] *Let us call $K_A(X)$ the Grothendieck group of equivalence classes of virtual flat A-bundles on X. Then, as a functor of the finite CW-complex X, $K_A(X)$ is representable and we have the following isomorphism*

$$K_A(X) \cong [X, K_0(A) \times BGL(A)^+]$$

In particular, $K_A(S^n) \cong K_0(A) \oplus K_n(A)$, where $K_n(A)$ are the K-groups of Quillen.

Remarks. This theorem should be related to the analogous result in topological K-theory (for complex vector bundles) : $K^{top}(S^n) \cong \mathbb{Z} \oplus K_n^{top}(\mathbb{C})$. Note also that this definition both includes K_0 and K_n for $n > 0$.

As in topological $K-$theory, one can define tensor products of virtual flat A-bundles and construct this way a bilinear pairing

$$K_A(X) \times K_B(Z) \longrightarrow K_{A \otimes B}(X \times Z)$$

In particular, if A is commutative $K_*(A)$ becomes a graded ring and one sees in more geometrical terms Milnor's cup-product

$$K_1(A) \times K_1(A) \longrightarrow K_2(A)$$

The λ^k and ψ^k operations also extend to the ring $K_A(X)$ (if A is commutative).

The first K-groups were calculated by Quillen of course. If A is a finite field with q elements, Quillen proved that

$$K_{2n}(F_q) = 0 \text{ for } n > 0$$

$$K_{2n+1}(F_q) \cong \mathbb{Z}/(q^n - 1)\mathbb{Z}$$

On the other hand, as we have seen in Berrick's lectures, $K_*(A) \otimes \mathbb{Q}$ is the primitive part of the homology of $GL(A)$ (with rational coefficients). If A is the ring of integers in a number field, Quillen has shown on one side that $K_i(A)$ is a finitely generated abelian group ; on the other side Borel has computed the cohomology of $SL(A)$ with complex coefficients. Comparing these results, one finds the following isomorphisms with $i > 0$ (mod. finite abelian groups)

$$K_{2i}(A) = 0$$

$$K_{4i+1}(A) = \mathbb{Z}^{r_1+r_2}$$
$$K_{4i-1}(A) = \mathbb{Z}^{r_2}$$

where r_1 and r_2 are determined by the isomorphism

$$A \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$$

Following Grothendieck and Bass, Quillen has succeeded in generalizing to higher K-groups some theorems proved for K_0 and K_1. Let us mention a few of them.

THEOREM (homotopy invariance). *Let A be a regular Noetherian ring. Then $K_n(A[t]) \cong K_n(A)$. Therefore, we also have $K_n(A[t_1, ..., t_r]) \cong K_n(A)$.*

Note that if A is a field, it was an old question of Serre whether a f.g.p. module over $A[t_1, ..., t_r]$ is free. This question was answered in the affirmative by Quillen and Suslin in the 70's.

THEOREM ("fundamental" theorem). *Let A be a regular Noetherian ring and let $A[t, t^{-1}]$ be the ring of Laurent polynomial with coefficients in A. Then we have the exact sequence*

$$0 \to K_n(A) \to K_n(A[t]) \oplus K_n(A[t^{-1}]) \to K_n(A[t, t^{-1}]) \to K_{n-1}(A) \to 0$$

THEOREM (one form of the "dévissage" and localization theorems). *Let A be a regular Noetherian ring and f a non zero divisor such that A/f is regular Noetherian. Then we have an exact sequence*

$$K_{n+1}(A) \to K_{n+1}(A_f) \to K_n(A/f) \to K_n(A) \to K_n(A_f)$$

where A_f is the ring A localised at f (i.e. making f invertible).

Another variant of this theorem is the following : let A be a Dedekind ring and F its field of fractions. Then we have an exact sequence

$$K_{n+1}(A) \to K_{n+1}(F) \to \oplus_{\mathcal{P}} K_n(A/p) \to K_n(A) \to K_n(F)$$

where $\mathcal{P}$ runs through the set of all prime ideals in A.

Remarks. All these theorems may be paraphrased with a parameter space X instead of the sphere S^n by introducing a suitable fibration. For instance, the homotopy invariance may be restated as

$$K_{A[t]}(X) \cong K_A(X)$$

for any regular Noetherian ring A. We should also notice that Quillen was able to prove his theorems by using another construction in Algebraic K-theory, called the "Q-construction". This has been detailed in Berrick's lectures (see also [FW]).

Having stated all these theorems, one may ask what the definition of $K_n(A)$ for $n < 0$ should be, so that the previous theorems can be extended to all values of $n \in \mathbb{Z}$. This definition is due independently to Bass and the author of these notes. Since we are going to use the second definition in Hermitian K-theory, let us choose the latter. The key ingredient is the use of "infinite" matrices. More precisely, let us define the "cone" of A, called CA, as the set of infinite matrices such that in each row and each column we have a finite number of non zero elements chosen among a finite number of elements in A. Clearly CA is a ring by matrix multiplication, containing the finite matrices (i.e. whose entries are 0, except for a finite number) as a 2-sided ideal. We define the suspension SA of A as the quotient ring. This definition may be iterated and $S^n(A)$ will denote the n^{th} suspension of A.

DEFINITION/THEOREM . *The group $K_{-n}(A)$ is by definition $K(S^nA)$. All the theorems stated before are true for the functors K_n , $n \in \mathbb{Z}$. Note however that $K_n(A) = 0$ for $n < 0$ if A is regular Noetherian.*

A theorem which is missing in the picture is "Bott periodicity". Of course, it does not work in general : for finite fields for instance, we are far from getting the same answer for K_n and $K_{n+\alpha}$. However, we can get a partial answer for other types of K-groups which we shall introduce now. As a matter of fact, since we know that Algebraic K-theory is represented by a spectrum $\mathcal{K}(A)$ defined through the $BGL(S^nA)^+$, there is a well known procedure in Algebraic Topology taking this spectrum *mod.* n for any integer n. A more concrete way is to consider the Puppe sequence associated to a map of degree n between spheres

$$S^r \longrightarrow S^r \longrightarrow M(n,r) = X \longrightarrow S^{r+1} \longrightarrow S^{r+1}$$

Therefore, X (so-called Moore space) is a space with 2 cells of dimension r and $r+1$ respectively, the second cell being attached to the first by a map of degree n. If $n = 2$ and $r = 1$ for instance, X is just the real projective space of dimension 2.

We now define K-theory with coefficients in $\mathbb{Z}/n$, denoted by $K_{r+1}(A;\mathbb{Z}/n)$, as the quotient group $K_A(X)/K_0(A)$. From the Puppe sequence, we get an exact sequence

$$K_{r+1}(A) \longrightarrow K_{r+1}(A) \longrightarrow K_{r+1}(A;\mathbb{Z}/n) \longrightarrow K_r(A) \longrightarrow K_r(A)$$

where the arrows between the groups $K_i(A)$ are the multiplication by n.

Here is a fundamental theorem of Suslin which is the true analog of Bott periodicity.

THEOREM . *Let F be an algebraically closed field and let n be a number prime to the characteristic of F. Then there is a canonical isomorphism of graded rings $K_{2r}(F;\mathbb{Z}/n) \cong (\mu_n)^{\otimes r}$ and $K_{2r+1}(F;\mathbb{Z}/n) = 0$.*

A sketch of the proof of this theorem may be found (among other things) in [FW], lecture 8.

With this theorem in mind, one may naturally ask if there is a way to compute $K_*(F;\mathbb{Z}/n)$ for an arbitrary field (not necessarily algebraically closed). If F is the field of real numbers, and if we work in topological K-theory instead, we know that it is not an easy task since we get an 8-periodicity which looks mysterious compared to the 2-periodicity of complex K-theory. All these questions are in fact related to the so-called homotopy fixed point set relative to a group action. Roughly speaking, we know[11] that $\mathcal{K}(F)$ is the fixed point set of $\mathcal{K}(\overline{F})$, where $\overline{F}$ denotes the separable closure of F, with respect to the action of the Galois group G. We have a fundamental map

$$\phi : \mathcal{K}(F) = \mathcal{K}(\overline{F})^G \longrightarrow \mathcal{K}(\overline{F})^{hG}$$

where $\mathcal{K}(\overline{F})^{hG}$ is the "homotopy fixed point set" of $\mathcal{K}(\overline{F})$, i.e. the set of equivariant maps $EG \longrightarrow \mathcal{K}(\overline{F})$ where EG is the "universal" principal G bundle over BG. One version of the Lichtenbaum-Quillen conjecture is that ϕ induces an isomorphism on the homotopy groups π_i for $i >$ n-cohomological dimension of $G = d_n$. This implies the existence of a spectral sequence

$$E_r^{p,q} = H^p(G;\mu_n^{\otimes q}) \to K_{2q-p}(F;\mathbb{Z}/n)$$

if the characteristic of the field does not divide n and if $2q - p > d_n$. For instance, $d_n = 1$ if n is odd and if F is a number field : then the spectral sequence degenerates and we get a direct link between Algebraic K-theory and Galois cohomology, quite interesting in Number Theory. By the recent work of Voevodsky, the conjecture is true for $n = 2$. We shall hear more about this in Morel's lectures.

We should notice that the topological analog of this conjecture is true : one should replace the classifying space of Algebraic K-theory by the classifying space of (complex) topological K-theory, where $\mathbb{Z}/2$ acts by complex conjugation. Then the fixed point set (i.e. the classifying space of <u>*real*</u> topological K-theory) has the homotopy type of the homotopy fixed point set.

Although the definition of the $K_n(A)$ is getting relatively old now, they are still

[11] We now denote by $\mathcal{K}(F)$ the classifying space of Algebraic K-theory mod. n: its homotopy groups are the K-groups with coefficients in $\mathbb{Z}/n$.

quite difficult to compute, even for simple rings as the ring of integers $\mathbb{Z}$ (although we know these groups rationally by the work of Borel on the rational cohomology of arithmetic groups). Thanks to the remarkable work of Voevodsky followed by Bökstedt, Rognes and Weibel, we can now compute the 2 primary torsion of $K_n(\mathbb{Z})$ through the following homotopy cartesian square (where $\mathbb{Z}' = \mathbb{Z}[1/2]$)

$$\begin{array}{ccc} BGL(\mathbb{Z}')^+_{\#} & \longrightarrow & BGL(\mathbb{R})_{\#} \\ \downarrow & & \downarrow \\ BGL(\mathbb{F}_3)^+_{\#} & \longrightarrow & BGL(\mathbb{C})_{\#} \end{array}$$

where the symbol # means 2-adic completion and where $BGL(\mathbb{R})$ and $BGL(\mathbb{C})$ are the classifying spaces of the <u>*topological*</u> groups $GL(\mathbb{R})$ and $GL(\mathbb{C})$ respectively. From this homotopy cartesian square, Rognes and Weibel found the following results (mod. a finite odd torsion group and with $n > 0$ for the first 2 groups and $n \geq 0$ for the others)

$K_{8n}(\mathbb{Z}) = 0$
$K_{8n+1}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$
$K_{8n+2}(\mathbb{Z}) = \mathbb{Z}/2$
$K_{8n+3}(\mathbb{Z}) = \mathbb{Z}/16$
$K_{8n+4}(\mathbb{Z}) = 0$
$K_{8n+5}(\mathbb{Z}) = \mathbb{Z}$
$K_{8n+6}(\mathbb{Z}) = 0$
$K_{8n+7}(\mathbb{Z}) = \mathbb{Z}/2^r$ where 2^r is the 2 primary component of the number $4n + 4$.

One should also mention (in reverse historical order) that Milnor gave another definition of the groups $K_n(F)$, $n > 2$, for a <u>*field*</u> F. This definition does not agree with Quillen's definition in general and we should denote it by $K_n^M(F)$: it is the quotient of the tensor product $F^* \otimes_{\mathbb{Z}} ... \otimes_{\mathbb{Z}} F^*$ (n factors) by the subgroup generated by the tensors $x_1 \otimes ... \otimes x_n$ such that $x_{i+1} = 1 - x_i$ for a certain i. By the usual cup-product in Algebraic K-theory, there is an obvious map

$$K_n^M(F) \longrightarrow K_n(F)$$

which is an isomorphism if $n = 1$ or 2 according to Matsumoto's theorem. The next interesting case is $n = 3$. Then we have the following nice theorem of Suslin :

THEOREM . *The map $K_3^M(F) \longrightarrow K_3(F)$ is injective and the quotient $K_3(F)_{ind}$ fits into an exact sequence*

$$0 \longrightarrow \mu \longrightarrow K_3(F)_{ind} \longrightarrow B(F) \longrightarrow 0$$

In this sequence, $B(F)$ is the "Bloch group" i.e. the group which fits into the exact sequence

$$0 \longrightarrow B(F) \longrightarrow P(F) \xrightarrow{r} \Lambda^2(F^*) \longrightarrow K_2(F) \longrightarrow 0$$

Here $P(F)$ is the quotient of the free group generated by elements [x] in F^* by the λ group generated by the "cross-ratio relations" :

$$[x] - [y] + [\frac{y}{x}] - [\frac{1-x^{-1}}{1-y^{-1}}] + [\frac{1-x}{1-y}] = 0$$

and $r([x] = x \wedge (1-x)$. Moreover, μ is a torsion group which is deduced from the group μ_F of roots of unity in F. If, following [FW], we write $\mu_F(2)$ as $Tor(\mu F, \mu F)$, m is the group $\mu_F(2)$ if the characteristic is 2 and the non trivial extension of $\mathbb{Z}/2$ by $\mu_F(2)$ otherwise.

It is an open problem to describe the other K_n-groups of a field in such a nice algebraic way...

One of the most striking applications of Algebraic K-theory to Differential Topology is due to the hard work of Waldhausen which we have not touched in this report. For instance, in the lecture 4 we introduced the space $\mathcal{P}(M)$ of pseudo-isotopies of M, i.e. diffeomorphisms q of $M \times [0,1]$ which restrict to the identity on $M \times 0$. If M has a boundary, we ask moreover that q is the identity on $\partial M \times [0,1]$.

THEOREM . *Let M be the ball B^n of dimension n. Then, we have an isomorphism*

$$\pi_i(\mathcal{P}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_{i+2}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for $0 \leq i \leq n/6 - 7$. Therefore, mod. torsion elements, and within this range of degrees, we have $\pi_i(\mathcal{P}(M)) = \mathbb{Z}$ for $i = 4\delta - 1$ and $\pi_i(\mathcal{P}(M)) = 0$ for $i \neq 4\delta - 1$.

The same type of results applies to the group of diffeomorphisms of B^n mod. its boundary. We get $\pi_i(Diff(B^n, \partial) \otimes_{\mathbb{Z}} \mathbb{Q} = K_{i+2}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ for the same range $0 \leq i \leq n/6 - 7$ and n odd if $i = 4k - 1$.

For Banach algebras A, there are now two competing theories : topological K-theory on one side which we denoted by $K_n^{top}(A)$ previously and Algebraic K-theory $K_n(A)$. There is a canonical homomorphism

$$\phi_n : K_n(A) \longrightarrow K_n^{top}(A)$$

If A is the field of complex numbers and if we take K-theory with finite coefficients, we know by Suslin's theorem that this map is an isomorphism. Other Banach algebras of interest in functional analysis are the so-called "stable" C^*-algebras ;

this means that $A \otimes \mathcal{K}$ is isomorphic to A (for a suitable notion of tensor product), where $\mathcal{K}$ is the ring of compact operators in an Hilbert space. A surprising result proved by the author of these notes at the end of the 80's is that ϕ_n is an isomorphism for $n \leq 2$ (n may be negative). The question arose then whether ϕ_n is also an isomorphism for $n > 2$. This question was answered positively by Suslin and Wodzicki [SW]. In fact, they proved excision for this kind of rings A (without unit) : $K_n(A)$ does not depend of the ambiant ring where the ideal A lives. The idea of the proof (assuming excision) is quite simple and is based on cup-products with Bott elements as it was explained in the second lecture.

Let us conclude this lecture by trying to answer a frustration the algebraists might feel at this point. Is there a way to define $K_n(A)$ which avoids the Algebraic Topology machinery? For *regular noetherian rings*, this definition was indeed proposed by Villamayor and myself [KV] and is based on polynomial homotopies (much related to the A^1-homotopy in Morel's lectures). The definition is the following. Let us consider the ring

$$A_n = A[x_0, x_1, ..., x_n]/(x_0 + x_1 + ... + x_n - 1)$$

and the subgroup[12] G_n of $GL(A_n)$ consisting of matrices which are equal to the identity matrix if one of the variables $x_i = 0$ (for $i < n$). We define a homomorphism $d_n : G_n \longrightarrow G_{n-1}$ by setting $x_n = 0$ (if $n < 0$, $G_n = \{1\}$ by convention). Then $Kerd_n/Imd_{n+1}$ is naturally isomorphic to $K_n(A)$ for $n > 0$.

Exercise : check this for $n = 1$.

We can put these last considerations in a more sophisticated framework by considering A_n , $n \in \mathbb{N}$, as a simplicial ring and constructing its classifying space $BGL(A_*)$ as it is usual in simplicial topology (A is now an arbitrary ring). If we view A as a "constant" simplicial ring, we have a map

$$\theta : BGL(A) \longrightarrow BGL(A_*)$$

Now $BGL(A_*)$ is a connected space with a fundamental group easily seen to be $GL(A)/U(A)$, where $U(A)$ is the group generated by unipotent matrices. Since we have the inclusions

$$[GL(A), GL(A)] = E(A) \subset U(A) \subset GL(A)$$

the quotient $GL(A)/U(A)$ is an abelian group and therefore θ leads to a map

$$BGL(A)^+ \longrightarrow BGL(A_*)$$

If we follow [FW] by putting $KV_n(A) = \pi_n(BGL(A_*))$, we therefore have a homomorphism

$$\gamma : K_n(A) \longrightarrow KV_n(A)$$

[12] We should notice that $G*$ is a simplicial group. In modern language, we just say that $K_{n+1}(A)$ is the n-homotopy group of this simplicial group.

Now the groups $KV_n(A)$ are just the groups described above by polynomial homotopies and another way of stating the previous result (due essentially to Quillen) is the following fact : γ is an isomorphism if A is a regular Noetherian ring.

Lecture 6

Hermitian K-theory

As we know, usual K-theory is deeply linked with the general linear group and there is no a priori reason why we should not consider the other classical groups on an equal footing. As we shall see, it is not only desirable, but this setting appears to be suitable for a generalization of Bott periodicity and the computation of the homology of classical groups in terms of classical Witt groups.

The starting point is a ring with antiinvolution $a \mapsto \overline{a}$ together with an element ε of the center of A such that $\varepsilon\overline{\varepsilon} = 1$. In most examples, $\varepsilon = \pm 1$. For sake of simplicity, we also assume the existence of an element λ in the center of A such that $\lambda + \overline{\lambda} = 1$ (if 2 is invertible in A, we might choose $\lambda = 1/2$). We refer to the book of A. Bak (Annals of Math Studies) for a more refined notion of hermitian forms using the so called “form-parameters” (when there is no λ satisfying this property).

If M is a *right* f.g.p. module over A, we define its dual M^* to be the group of $\mathbb{Z}$ -linear maps $f : M \longrightarrow A$ such that $f(m.a) = \overline{a}.f(m)$ for $m \in M$ and $a \in A$. It is again a right f.g.p. A-module if we put $(f.b)(m) = f(m).b$ for $b \in A$. An ε-hermitian form on M is roughly speaking a A-linear map $M \longrightarrow M^*$ satisfying some conditions of ε-symmetry. More precisely, it is given by a $\mathbb{Z}$-bilinear map

$$\phi : M \times M \longrightarrow A$$

such that

$$\phi(ma, m'b) = \overline{a}\phi(m, m')b$$

$$\phi(m', m) = \varepsilon\overline{\phi(m, m')}$$

with obvious notations. Such a (M, ϕ) is called an ε-*hermitian module* . The correspondence

$$\overline{\phi} : m' \mapsto [m \mapsto \phi(m, m')]$$

defines a morphism from M to M^* and we say that ϕ is non-degenerate if $\overline{\phi}$ is an isomorphism.

Fundamental example (the hyperbolic module). Let N be a f.g.p. module and $M = N \oplus N^*$. Then a non degenerate ε-hermitian form on M is given by the formula

$$\phi((x, f), (x', f')) = \overline{f(x')} + \varepsilon f'(x)$$

We denote this module by $H(N)$. If $N = A^n$, we may identify N with its dual via the map $y \mapsto f_y$ defined by $f_y(x) = \overline{x}y$. The hermitian form on $A^n \oplus A^n$ may then be written as

$$\phi((x,y),(x',y')) = \overline{y}x' + \varepsilon\overline{x}y'$$

There is an obvious definition of the direct sum of ε-hermitian modules and of an isomorphism between them. We write ${}_\varepsilon L(A)$ for the Grothendieck group constructed from such modules[13].

Exercise : compute ${}_\varepsilon L(A)$ for $A = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and all possible antiinvolutions and ε.

Exercise : if (M,ϕ) is an ε-hermitian module, prove that $(M,\phi) \oplus (M,-\phi)$ is isomorphic to $H(M)$ (one has to use the existence of the λ above).

Exercise : let A be the ring of continuous functions on a compact space with <u>*complex*</u> values. If A is provided with the trivial involution and if we take $\varepsilon = 1$, prove that ${}_\varepsilon L(A)$ is isomorphic to the <u>*real*</u> topological K-theory of X.

The analog of the general linear group is the ε-orthogonal group which is the group of automorphisms of $H(A^n)$. We denote this group by ${}_\varepsilon O_{2n}(A)$: it can be described concretely by matrices in n-blocks

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $M^*M = MM^* = I$ where

$$M^* = \begin{pmatrix} {}^t\overline{d} & \varepsilon {}^t\overline{b} \\ \overline{\varepsilon} {}^t\overline{c} & {}^t\overline{a} \end{pmatrix}$$

Example: if $A =$ the field of real numbers $\mathbb{R}$, ${}_1O_{2n}(A)$ is the classical group $O(n,n)$ which has the homotopy type of $O(n) \times O(n)$. On the other end, ${}_{-1}O_{2n}(A)$ is the classical group $Sp(2n,\mathbb{R})$ which has the homotopy type of the unitary group $U(n)$.

The infinite orthogonal group

$${}_\varepsilon O(A) = lim_\varepsilon O_{2n}(A)$$

has the same formal property as the infinite general linear group. In particular, its commutator subgroup is perfect. Therefore, we can perform the + construction (or, more geometrically, consider virtual flat A-bundles with ε-hermitian forms as we did in lecture 5).

[13] We use here the letter L which is quite convenient, but the reader should not be confused with the definition of surgery groups, also denoted by the letter L

DEFINITION . *The higher Hermitian K-theory of a ring A (for $n > 0$) is defined as*

$$_{\varepsilon}L_n(A) = \pi_n(B_{\varepsilon}O(A)^+)$$

Example. Let F be a field of characteristic different from 2 provided with the trivial involution. Then $_{\varepsilon}L_1(F) = 0$ if $\varepsilon = -1$ and $_{\varepsilon}L_1(F) = \mathbb{Z}/2 \times F^*/F^{*^2}$ if $\varepsilon = +1$

Notation. We write

$$\mathcal{K}(A) = K(A) \times BGL(A)^+$$

for the classifiying space of Algebraic K-theory and

$$_{\varepsilon}\mathcal{L}(A) =_{\varepsilon} L(A) \times B_{\varepsilon}O(A)^+$$

for the classifying space of Hermitian K-theory

There are two interesting functors between Hermitian K-theory and Algebraic K-theory. One of them is the forgetful functor from modules with hermitian forms to modules (with no forms) and the other one from modules to modules with forms, sending N to $H(N)$, the hyperbolic module associated to N. These functors induce two maps

$$F :_{\varepsilon} \mathcal{L}(A) \longrightarrow \mathcal{K}(A) \text{ and } \mathrm{H} : \mathcal{K}(\mathrm{A}) \longrightarrow_{\varepsilon} \mathcal{L}(\mathrm{A})$$

We define $_{\varepsilon}\mathcal{V}(A)$ as the homotopy fiber of F and $_{\varepsilon}\mathcal{U}(A)$ as the homotopy fiber of H. We define this way two "relative" theories:

$$_{\varepsilon}V_nA = \pi_n(_{\varepsilon}\mathcal{V}(A)) \text{ and } _{\varepsilon}\mathrm{U_nA} = \pi_\mathrm{n}(_{\varepsilon}\mathcal{U}(\mathrm{A}))$$

THEOREM (the fundamental theorem of Hermitian K-theory)**.** *There is a natural homotopy equivalence between $_{\varepsilon}\mathcal{V}(A)$ and the loop space of $-_{\varepsilon}\mathcal{U}(A)$. In particular,*

$$_{\varepsilon}V_n(A) \cong -_{\varepsilon}U_{n+1}(A)$$

Moreover, if we work within the framework of Banach algebras, the same statement is valid for the topological analogs (i.e. replacing $BGL(A)^+$ by $BGL(A)^{top}$ and $B_{\varepsilon}O(A)^+$ by $B_{\varepsilon}O(A)^{top}$).

In order to get a feeling for this theorem, it is worthwhile to work out the classical topological examples $A = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, with various antiinvolutions (and $\varepsilon = \pm 1$). Note in general that the connected component of $_{\varepsilon}\mathcal{V}(A)$ (resp. $_{\varepsilon}\mathcal{U}(A)$) is the connected component of the homogeneous space $GL(A)/_{\varepsilon}O(A)$ (resp. $_{-\varepsilon}O(A)/GL(A)$).

If for instance $A = \mathbb{R}$, $\varepsilon = -1$, we get the spaces $GL(\mathbb{R})/{}_{-1}O(\mathbb{R})$ (which has the homotopy type of O/U) and ${}_1O(R)/GL(R)$ (which has the homotopy type of O, the infinite orthogonal group). Therefore, the previous theorem implies that O/U has the homotopy type of the loop space ΩO, one of the eight homotopy equivalences of Bott (see the end of lecture 2, where O has the homotopy type of $GL(\mathbb{R})$ and U the homotopy type of $GL(\mathbb{C})$). It is a pleasant exercise to recover the seven other homotopy equivalences by dealing with other classical groups. Since the list of classical groups is finite, it is "reasonable" to expect some periodicity...

There are two remarkable involutions[14] on the H-spaces $\mathcal{K}(A)$ and ${}_\varepsilon\mathcal{L}(A)$ and it is better to describe them in the context of classical groups. On $GL(A)$, one takes the contragredient

$$M \mapsto \overline{M}^{-1}$$

On ${}_\varepsilon O(A)$ the involution is more delicate. The idea is to take the functor which associates to any hermitian module (M, ϕ) the "opposite" module $(M, -\phi)$. On the level of groups, this means that we have to identify the hyperbolic module $H(A^n) = A^n \oplus A^n$ with its opposite Hermitian form. In terms of the orthogonal group, this involution sends the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

Therefore, if we localize the spaces $\mathcal{K}(A)$ and ${}_\varepsilon\mathcal{L}(A)$ away from 2, we obtain spaces $\mathcal{K}(A)'$ and ${}_\varepsilon\mathcal{L}(A)'$, together with splittings according to these involutions

$$\mathcal{K}(A)' \approx \mathcal{K}(A)'_+ \times \mathcal{K}(A)'_- \text{ and } {}_\varepsilon\mathcal{L}(\mathrm{A})' \approx_\varepsilon \mathcal{L}(\mathrm{A})'_+ \times_\varepsilon \mathcal{L}(\mathrm{A})'_-$$

From to this decompositon, the hyperbolic functor and the forgetful functor are both represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

If follows that, after localisation which we denote by the symbol a', the space ${}_\varepsilon\mathcal{V}(A)'$ has the homotopy type of ${}_\varepsilon\mathcal{L}(A)'_- \times \Omega\mathcal{K}(A)'_-$, whereas the space ${}_{-\varepsilon}\mathcal{U}(A)'$ has the homotopy type of $\mathcal{K}(A)'_- \times \Omega_{-\varepsilon}\mathcal{L}(A)'_-$. Therefore, the fundamental theorem of Hermitian K-theory is equivalent to saying that ${}_\varepsilon\mathcal{L}(A)'_-$ has the homotopy type of $\Omega^2_{-\varepsilon}\mathcal{L}(A)'_-$.

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[14] One has to be careful about the precise definition of these spaces when a group is acting. For instance, the decomposition $\mathcal{K}(A) = K(A) \times BGL(A)^+$ is NON canionical, A correct definition (among others) is to choose $\Omega(BGL(SA))^+$.

The preceding considerations give us an idea of how to prove the fundamental theorem in Hermitian K-theory after localisation (a more delicate proof, taking into account the 2-torsion is presented in [K4]).We define the <u>*Witt groups*</u> ${}_\varepsilon W_n(A)$ as Coker $(K_n(A) \longrightarrow {}_\varepsilon L_n(A))$. Then the fundamental theorem amounts to saying that

$${}_\varepsilon W_n(A) \cong_{-\varepsilon} W_{n+2}(A) \text{ mod.2 torsion}$$

since these Witt groups are essentially the homotopy groups of the space ${}_\varepsilon\mathcal{L}(A)'_-$. In order to prove such a statement, one may follow the pattern used to prove complex Bott periodicity in the second lecture. The base ring which plays the role of $\mathbb{C}$ is the ring $\mathbb{Z}[x]$ with the involution $x \mapsto 1-x$. Then the scheme is the following

1) Construct non connected deloopings of Hermitian K-theory with a good notion of cup-product. More precisely if we have a pairing of rings with involution

$$A \times B \longrightarrow C$$

we should have a pairing between the representing spaces of Hermitian K-theory, taking into account the sign of symmetry :

$${}_\varepsilon\mathcal{L}(A) \times_\eta \mathcal{L}(B) \longrightarrow_{\varepsilon\eta} \mathcal{L}(C)$$

This is done as in usual K-theory, using the suspension of a ring. One can proceed in the same way for Banach algebras by completing the cone and the suspension.

2) It is a well known theorem that ${}_1W(\mathbb{Z}[x]) \cong \mathbb{Z} \approx_1 W(\mathbb{R})$. The key point is now to construct elements ${}_{-1}W_{-2}(\mathbb{Z}[x])$ and v $\in_{-1}$ $\mathrm{W}_{-2}(\mathbb{Z}[\mathrm{x}])$ such that their cup-product is 4 times a generator of ${}_1W(\mathbb{Z}[x])$. The construction of u and v is done very explicitly in [K4]. To check that their cup-product is non trivial, one has to send $\mathbb{Z}[x]$ into $\mathbb{R}$, the image of x being $1/2$ and use topological K-theory of Banach algebras. This is a nice point where Algebraic and Topological K-theory are interacting.

3) Using u and v, we define morphisms

$${}_\varepsilon W_n(A) \longrightarrow_{-\varepsilon} W_{n+2}(A) \text{ and } {}_{-\varepsilon}\mathrm{W}_{\mathrm{n}+2}(\mathrm{A}) \longrightarrow_\varepsilon \mathrm{W}_\mathrm{n}(\mathrm{A})$$

such that the two compositions are 4 times the identity. This essentially concludes the proof.

Another theorem of the same spirit is the following.

THEOREM . *There exist $u' \in_{-1} W_2(\mathbb{Z})$ and $v' \in_{-1} W_2(\mathbb{Z})$ such that their cup-product is an element of ${}_1W(\mathbb{Z})$ which image in ${}_1W(\mathbb{R}) \cong \mathbb{Z}$ is 32 times the canonical generator. Now let A be ANY ring with involution (we no longer assume the existence of λ such that $\lambda + \overrightarrow{\lambda} = 1$). Then, using u' and v', we define as before* *two morphisms between ${}_{\varepsilon}W_n(A)$ and ${}_{-\varepsilon}W_{n+2}(A)$ such that their composite is 32 times the identity. In particular,*

$$ {}_{\varepsilon}W_n(A) \otimes \mathbb{Z}[1/2] \cong_{-\varepsilon} W_{n+2}(A) \otimes \mathbb{Z}[1/2] $$

for ANY ring A.

The fundamental theorem and the slight generalization above have nice applications for the computation of the homology of the general ε-orthogonal group. The philosophy is that this homology splits into two parts, one coming from the general linear group which is unknown in general and the other coming from more accessible invariants which are ${}_{\pm\varepsilon}W_n(A)$ for $n = 0$ and 1 only (since we have periodicity). As a typical example, let us consider the homology of the symplectic group $Sp(\mathbb{Z}) = limSp_{2n}(\mathbb{Z})$ and the orthogonal group ${}_1O(\mathbb{Z}) = \lim_1 O_{2n}(\mathbb{Z})$.

THEOREM . *Let $\mathbb{Z}' = \mathbb{Z}[1/2]$. Then for the homology with coefficients in $\mathbb{Z}'$, we have the following decomposition of the homology of $H_*(BSp(\mathbb{Z}))$:*

$$ H_*(BSp(\mathbb{Z})) \cong \mathbb{Z}'[x_2, x_6, ...] \otimes M_* $$

where $M_0 = \mathbb{Z}'$ and M_i is a finite group for $i > 0$.

The theorem follows from the decomposition of $BSp(\mathbb{Z})^+$ after localisation

$$ BSp(\mathbb{Z})^+ \approx_{-1} \mathcal{L}(\mathbb{Z})'_+ \times_{-1} \mathcal{L}(\mathbb{Z})'_- $$

where $\pi_i({}_{-1}\mathcal{L}(\mathbb{Z})'_-) = \mathbb{Z}'$ for i $\equiv$ 2 mod. 4 and 0 otherwise and $\pi_i({}_{-1}\mathcal{L}(\mathbb{Z})'_+))$ is the symmetric part of $K_i(\mathbb{Z})$ after localisation which is 0 mod. torsion since the regulator map sends symmetric elements to 0.

THEOREM . *Let $\mathbb{Z}' = \mathbb{Z}[1/2]$. Then for the homology with coefficients in $\mathbb{Z}'$, we have the following decomposition of the homology of $H_*({}_1O(\mathbb{Z}))$:*

$$ H_*({}_1O(\mathbb{Z})) \cong \mathbb{Z}'[x_4, x_8, ...] \otimes M_* $$

where $M_0 = \mathbb{Z}'$ and M_i is a finite group for $i > 0$.

For both groups, it is important to notice that the elements x_{4i+2} in the symplectic case are determined by embedding $Sp(\mathbb{Z})$ in the topological group $Sp(\mathbb{R})$ which has the homotopy type of the infinite (topological) unitary group. In the same way, the elements x_{4i} in the orthogonal case are determined by embedding ${}_1O(\mathbb{Z})$ in the topological group ${}_1O(\mathbb{R})$ which has the homotopy type of the product of two copies of the classical (topological) infinite orthogonal group. The situation there is in sharp contrast with what happens for the general linear group : by Chern-Weil theory, we know that the homomorphism

$$K_n(R) \longrightarrow K_n^{top}(\mathbb{R})$$

has a finite cokernel. This is not true for the homomorphism

$${}_1L_n(\mathbb{R}) \longrightarrow_1 L_n^{top}(\mathbb{R}) \text{ (resp.} {}_{-1}\mathrm{L_n}(\mathbb{R}) \longrightarrow_{-1} \mathrm{L_n^{top}}(\mathbb{R}))$$

for $n \equiv 0$ mod. 4 (resp. $n \equiv 2$ mod. 4).

Let us now turn to some computations, starting with finite fields. It has been proved by Quillen that $BGL(\mathbb{F}_q)^+$ has the homotopy fiber of the map $\psi^q - 1$, where

$$\psi^q : BGL(\mathbb{C}) \longrightarrow BGL(\mathbb{C})$$

is the map induced by the Adams operation. In the same way, for q odd, it has been proved by Friedlander that $B_1O(F_q)$ [resp.$B_{-1}O(F_q)$] is the homotopy fiber of the map $\psi^q - 1$, where

$$\psi^q : B_1O(\mathbb{C}) \longrightarrow B_1O(\mathbb{C}) \text{ [resp. } \psi^q : \mathrm{B_{-1}O}(\mathbb{C}) \longrightarrow \mathrm{B_{-1}O}(\mathbb{C})]$$

Note that ${}_1O(\mathbb{C})$ has the homotopy type of the usual infinite orthogonal group $O = injlimO(n)$ and ${}_{-1}O(\mathbb{C})$ has the homotopy type of $Sp = injlimSp(n)$. This leads to the following computations of ${}_1L_i(F_q)$ for i > 0 and i mod.8.

$i(mod.8)$	${}_1L_i(F_q)$	${}_{-1}L_i(F_q)$
i = 0	$\mathbb{Z}/2$	0
i = 1	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	0
i = 2	$\mathbb{Z}/2$	0
i = 3	$\mathbb{Z}/(q^{(i+1)/2} - 1)$	$\mathbb{Z}/(q^{(i+1)/2} - 1)$
i = 4	0	0
i = 5	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
i = 6	0	$\mathbb{Z}/2$
i = 7	$\mathbb{Z}/(q^{(i+1)/2} - 1)$	$\mathbb{Z}/(q^{(i+1)/2} - 1)$

Friedlander has also noticed that the Witt groups ${}_1W_i(F_q)$ and ${}_{-1}W_i(F_q)$ are periodic of period 8 with respect to i. More precisely, we have ${}_1W_i(F_q) \cong_{-1} W_{i+4}(F_q)$ and these are isomorphic to the following list, starting for $i \equiv 0$ mod.8

$$\mathbb{Z}/2, \mathbb{Z}/2, 0, 0, 0, 0, \mathbb{Z}/2, \mathbb{Z}/2$$

Another case of interest is when our basic ring is $\mathbb{Z}' = \mathbb{Z}[1/2]$. Some computations have been done recently by A.J. Berrick and the author. They will be published shortly. Here are the results for $i > 0$ and mod. an odd torsion finite group

i (mod. 8)	${}_1L_i(Z')$	${}_{-1}L_i(Z')$

i = 0	$\mathbb{Z} \oplus \mathbb{Z}/2$	0
i = 1	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	0
i = 2	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}$
i = 3	$\mathbb{Z}/8$	$\mathbb{Z}/16$
i = 4	$\mathbb{Z}$	$\mathbb{Z}/2$
i = 5	0	$\mathbb{Z}/2$
i = 6	0	$\mathbb{Z}$
i = 7	$\mathbb{Z}/2^{t+1}$	$\mathbb{Z}/2^{t+1}$

where 2^t is the 2-primary component of $i+1$.

Lecture 7

Cyclic homology and K-theory

Let me say a few historical words on the subject before explaining more recent achievements. Cyclic homology (and cohomology) grew from at least 3 sources :

1) In his attempt to understand the Atiyah-Singer index theorem in noncommutative geometry, Connes was forced to find the analog of the de Rham complex for suitable dense subalgebras of C^*-algebras (typical examples are the Schatten ideals introduced in Higson's lectures). Connes explained his ideas in the K-theory seminar in Paris in 1981 and with more details in his course at the College de France the following years. He called his groups "cyclic homology" and denoted them by $HC_*(A)$. These groups are very much related to Hochschild homology $HH_*(A)$, introduced 50 years ago. As a matter of fact, there is a well defined homomorphism

$$B : HC_*(A) \longrightarrow HH_{*+1}(A)$$

which plays a central role in the theory.

2) From a quite different perspective, Tsygan was trying to compute the Lie algebra homology of infinite matrices over a ring A of characteristic 0. He proved that this homology is primitively generated (as a Hopf algebra) by precisely the cyclic homology groups of A, introduced by Connes independently. This result was proved also by Quillen and Loday in 1983, using invariant theory.

3) Finally, from my work in K-theory during the late 70's, I was looking for a kind of Chern character for general rings. The target was then a new homology theory of rings called "noncommutative de Rham homology" (see [K2] § 1] and denoted $H_*^{dR}(A)$, but the relaton with cyclic homology was unclear at the first glance. However, as it was realized in 1983 by Connes and myself, this theory $H_*^{dR}(A)$ happens to be just the kernel of the map $B : HC_*(A) \longrightarrow HH_{*+1}(A)$ mentioned above. From this clarified viewpoint, $HC_*(A)$ and $H_*^{dR}(A)$ are like looking very much as Deligne cohomology in a noncommutative context.

The purpose of this lecture is to fill the gap between K-theory and cyclic homology, via various types of "Chern characters". In order not to overlap with the lectures of Berrick and Higson, we shall take here the viewpoint of functional analysis and deal essentially with Algebraic and Topological K-theory of Fréchet algebras (which are more general than Banach algebras), comparing them with the corresponding cyclic homology groups.

More precisely, let A be a unitary, but not necessarily commutative Fréchet algebra. As we have seen in previous lectures, we can define Algebraic and Topological K-theory denoted respectively by $K_n(A)$ and $K_n^{top}(A)$. For $n > 0$, we have $K_n(A) = \pi_n(BGL(A)^+)$; here $BGL(A)^+$ denotes the + construction of Quillen applied to the classifying space $BGL(A)^\delta$, with $GL(A)^\delta = GL(A)$, viewed as a *discrete* group. On the other hand, $K_n^{top}(A)$ is defined as the homotopy group $\pi_n(BGL(A))$, where $BGL(A)$ denotes the classifying space of the *topological* group $GL(A)$ (at least if A is a Banach algebra ; the general case needs simplicial methods). The obvious map $BGL(A)^+ \longrightarrow BGL(A)$ induces a natural homomorphism

$$K_n(A) \longrightarrow K_n^{top}(A)$$

We shall use cyclic homology as a tool to construct an intermediary K-theory, called the "multiplicative K-theory of A" and denoted by $\mathcal{K}_n(A)$, which is more accessible than Algebraic K-theory. The previous homomorphism can be factored as

$$K_n(A) \longrightarrow \mathcal{K}_n(A) \longrightarrow K_n^{top}(A)$$

The homomorphism $K_n(A) \longrightarrow \mathcal{K}_n(A)$ contains various interesting invariants of the Algebraic K-theory of the Fréchet algebra A. For instance $\mathcal{K}_n(A) \cong \mathbb{C}^*$ when $A = \mathbb{C}$ and n odd. The map $K_{2n-1}(\mathbb{C}) \longrightarrow \mathbb{C}^*$ is essentially the regulator map. We also have $\mathcal{K}_2(C^\infty(S^1)) \cong \mathbb{C}^*$ and the map $K_2(C^\infty(S^1)) \longrightarrow \mathbb{C}^*$ gives rise to the well-known Kac-Moody extension of $SL(C^\infty(S^1))$ by $\mathbb{C}^*$ [CK].

In order to put all these informations together, we shall construct a commutative diagram with 16 entries (when $n \geq 1$) and exact horizontal sequences :

$$\begin{array}{ccccccc}
K_n^{rel}(A) & \longrightarrow & K_n(A) & \longrightarrow & K_n^{top}(A) & \longrightarrow & K_{n-1}^{rel}(A) \\
\downarrow & & \downarrow & & \| & & \downarrow \\
HC_{n-1}(A) & \longrightarrow & \mathcal{K}_n(A) & \longrightarrow & K_n^{top}(A) & \longrightarrow & HC_{n-2}(A) \\
\| & & \downarrow & & \downarrow & & \| \\
HC_{n-1}(A) & \longrightarrow & HC_n(A) & \longrightarrow & HC_n^{top}(A) & \longrightarrow & HC_{n-2}(A) \\
\| & & \downarrow & & \downarrow & & \| \\
HC_{n-1}(A) & \longrightarrow & HH_n(A) & \longrightarrow & HC_n(A) & \longrightarrow & HC_{n-2}(A)
\end{array}$$

Most of the work describing this diagram is included in a paper by Connes and myself [CK] and in a paper of Weibel[15]. In the first row, the group $K_n^{rel}(A)$ is de-

[15]C.Weibel. Nil K-theory maps to cyclic homology. Trans. AMS 303(1987), 541-558

fined as the n^{th} homotopy group of the homotopy fiber of the map $BGL(A)^+ \longrightarrow BGL(A)$. Therefore, the first exact sequence is tautological. The other terms and arrows of this diagram will be detailed now. At the same time we shall explain what the Chern character should be in this context.

Cyclic homology has already been defined in previous lectures. Therefore, we shall innovate slightly by taking cyclic homology of Fréchet algebras[16] instead, since it is our viewpoint here. If $\otimes_\pi$ denotes the completed tensor product of Grothendieck, we define a double complex by the following formulas

$$C_{p,q}(A) = A \otimes_\pi A \otimes_\pi ... \otimes_\pi A(q+1 \text{ times})$$

for $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. The first differential $b : C_{p,q}(A) \longrightarrow C_{p,q-1}(A)$ is the usual Hochschild boundary

$$b(a_0, ..., a_q) = \sum_{i=0}^{q-1} (-1)^i (a_0, ..., a_i a_{i+1}, ..., a_q) + (-1)^q (a_q a_0, a_1, ..., a_{q-1})$$

The second differential $\partial : C_{p,q}(A) \longrightarrow C_{p-1,q}(A)$ is defined by $1-t$ if p is even and $-N$ if p is odd. Here the operators t and $N : C_{p,q}(A) \longrightarrow C_{p,q}(A)$ are defined by

$$t(a_0, ..., a_q) = (-1)^q (a_q, a_0, ..., a_{q-1})$$

$$N = 1 + t + ... + t^q$$

We therefore get the following picture :

$$\begin{array}{cccccccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \longleftarrow & C_{-2,2} & \longleftarrow & C_{-1,2} & \longleftarrow & C_{0,2} & \longleftarrow & C_{1,2} & \longleftarrow & C_{1,2} & \longleftarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \longleftarrow & C_{-2,1} & \longleftarrow & C_{-1,1} & \longleftarrow & C_{0,1} & \longleftarrow & C_{1,1} & \longleftarrow & C_{1,1} & \longleftarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \longleftarrow & C_{-2,0} & \longleftarrow & C_{-1,0} & \longleftarrow & C_{0,0} & \longleftarrow & C_{1,0} & \longleftarrow & C_{1,0} & \longleftarrow
\end{array}$$

The "total" complex of this diagram is defined by a <u>product</u> formula :

$$C_n^{per}(A) = \prod_{p+q=n} C_{p,q}(A)$$

We denote by $HC_*^{per}(A)$ the (periodic cyclic) homology of this complex. The restriction of this double complex to the first quadrant defines a complex whose homology $HC_*(A)$ is by definition the cyclic homology of the Fréchet algebra

[16] For instance, a Banach algebra or the algebra of C^∞ functions on a manifold.

A (taking the topology into account). The restriction to the second quadrant (denoted $C_*^-(A)$) of the double complex gives the definition of the negative cyclic homology $HC_*^-(A)$. We have an exact sequence (see [L1] p. 160 for instance):

$$0 \longrightarrow \underline{\lim}_r^1 HC_{n+2r+1}(A) \longrightarrow HC_n^{per}(A) \longrightarrow \underline{\lim}_r HC_{n+2r}(A) \longrightarrow 0$$

Here the projective limits lim and $\lim^1$ are computed using the periodicity operator S of Connes [C1]. We should notice that the homology spaces obtained this way are not Hausdorff in general.

For technical reasons which will appear later on, we shall truncate the complexes $E_n = C_n^{per}$ or C_n^- by 0 if $n < 0$ and $Ker(E_0 \longrightarrow E_{-1})$ if $n = 0$, the other E_n staying unchanged. For $n \geq 0$, the groups HC_n^{per} and HC_n^- are still the same as before. Finally, we notice the following diagram with two exact sequences

$$\begin{array}{ccccccc}
HC_{n-1}(A) & \longrightarrow & HC_n^-(A) & \longrightarrow & HC_n^{per}(A) & \longrightarrow & HC_{n-2}(A) \\
\| & & \downarrow & & \downarrow & & \| \\
HC_{n-1}(A) & \longrightarrow & HH_n(A) & \longrightarrow & HC_n(A) & \longrightarrow & HC_{n-2}(A)
\end{array}$$

Another way to introduce cyclic homology is to consider the double complex (B, b) of Connes (cf. [L1] p. 56 for instance). It can be written over 3 quadrants in the following way:

$$\begin{array}{ccccccccc}
 & \longleftarrow & & \longleftarrow & D_2 & \longleftarrow & D_1 & \xleftarrow{B} & D_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow b & & \\
 & \longleftarrow & D_2 & \longleftarrow & D_1 & \longleftarrow & D_0 & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
D_2 & \longleftarrow & D_1 & \longleftarrow & D_0 & & & &
\end{array}$$

where we put in general $D_q = C_{p,q} = A^{\otimes_\pi^{q+1}}$. If we truncate as before, we can restrict it to the first two quadrants. We denote by B (resp. b) the horizontal arrows (resp. the vertical ones). The bidegrees are obvious (the D_0 of the second line has bidegree $(0,0)$).

We may normalize this double complex by replacing D_q by $\Omega_q = \Omega_q(A) = A \otimes_\pi \overline{A} \otimes_\pi ... \otimes_\pi \overline{A}$ (q copies of $\overline{A} = A/C.1$). The bicomplex

$$\begin{array}{ccccccccc}
 & \longleftarrow & & \longleftarrow & \Omega_2 & \longleftarrow & \Omega_1 & \xleftarrow{B} & \Omega_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow b & & \\
 & \longleftarrow & \Omega_2 & \longleftarrow & \Omega_1 & \longleftarrow & \Omega_0 & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\Omega_2 & \longleftarrow & \Omega_1 & \longleftarrow & \Omega_0 & & & &
\end{array}$$

also enables us to compute cyclic homology and its variants $(H\mathbb{C}_*^{per}, H\mathbb{C}_*^-, ...)$.

Finally, there is a last variant of cyclic homology, called non commutative de Rham homology, mentionned at the beginning, which we shall define now. The direct sum of the $\Omega_n(A)$ above is a differential graded algebra : we write formally an element of $A \otimes_\pi \overline{A} \otimes_\pi ... \otimes_\pi \overline{A}$ as a linear combination (infinite since we are dealing with the topological case) of elements of the type

$$a_0 da_1 ... da_n$$

and we multiply them formally using the Leibniz rule: for instance, $(x.dy).(z.dt) = x.d(yz).dt - xy.dz.dt$. The differential is defined simply by $d(a_0da_1...da_n) = 1.da_0da_1...da_n$. Let us now consider the algebra $\overline{A}$ obtained by adding a unit to A and the quotient $\overline{\Omega}_*(\overline{A})$ of $\Omega_*(\overline{A})$ by the $\mathbb{C}$-module generated by commutators $\lambda\omega - \omega\lambda, \lambda \in \overline{A}$, and their differentials. This $\mathbb{C}$-module is easily seen to be the quotient of the algebra $\overline{\Omega}_*(\overline{A})$ by the module generated by graded commutators.

THEOREM ([C1][K2]). *We have a natural isomorphism*

$$H_n^{dR}(\overline{A}) \cong Ker[B : HC_n(A) \longrightarrow HH_{n+1}(A)]$$

From the classical proof of the Poincaré lemma in de Rham cohomology, we deduce that H_n^{dR} is a homotopy invariant. More precisely, we have $H_n^{dR}(\overline{A}) \cong H_n^{dR}(\overline{A} \otimes C^\infty(\mathbb{R}))$. As a consequence, we deduce from the previous theorem that $KerB$ is also a homotopy invariant which in turn implies the following theorem:

THEOREM . *Let A_r be the algebra of C^∞ functions on the canonical simplex*[17] *Δ_r with values in A, interpreted as the completed tensor product $C^\infty(\Delta_r) \otimes_\pi A$. Then, the obvious map $A \longrightarrow A_r$ induces an isomorphism $HC_n^{per}(A) \approx HC_n^{per}(A_r)$.*

In order to go further, we need a "differential form" version of the Dold-Kan correspondance. If we denote by $\mathcal{C}$ the category of chain complexes (positively graded and assumed to be $\mathbb{C}$-vector spaces, which is our framework) and by $\mathcal{G}$ the category of <u>*simplicial*</u> $\mathbb{C}$-vector spaces, they are equivalent. In particular, we can associate to a chain complex (C_n) a space such that its homotopy groups are the homology groups of the chain complex. There are many choices of spaces[18] which

[17] We may avoid "angle problem" by defining Δ_r as the <u>$affine\ space$</u> of points with coordinates $(t_0, t_1, ..., t_r)$ in $\mathbb{R}^{r+1}$ such that $t_0 + t_1, +..., +t_r = 1$ (we do not assume that t_i are ≥ 0.) The face and degeneracy operators enable us to define a simplicial ring from the $C^\infty(\Delta_r)$. If X is a manifold and A a Banach algebra, it is a well know fact that $C^\infty(X) \otimes_\pi A$ may be identified with the algebra of C^∞ functions on X with values in A.

[18] M. karoubi. Formes différentielles et théotème de Dold-Kan. Journal of Algebra 198, p. 618-626 (1997)

are equivalent up to homotopy. The one which is the most suitable for our purpose is the following. Let $\Omega^*(\Delta_r)$ be the DGA of usual C^∞ differential forms on the r-simplex. We associate now to (C_n) a double simplicial complex defined on the second quadrant by

$$D_{p,q}(\Delta_r) = \Omega^{-p}(\Delta_r) \otimes_{\mathbb{C}} C_q$$

The two differentials (which decrease the degrees p and q by 1) are induced by the differentials on Ω^{-*} and C_*. To this double complex we associate the <u>*simplicial*</u> complex defined by

$$C_n(\Delta_r) = \bigoplus_{p+q=n} D_{p,q}(\Delta_r) = \bigoplus_{q \geq n} \Omega^{q-n}(\Delta_r) \otimes_{\mathbb{C}} C_q$$

where $n \in \mathbb{Z}$. There is a remarkable simplicial subgroup of $C_0(\Delta_*)$: it consists of <u>*closed*</u> chains of total degree 0 : it is a subgroup of the direct sum $\Omega^q(\Delta_*) \otimes_{\mathbb{C}} C_q$. The face and degeneracy operators are induced by those on $\Omega^q(\Delta_*)$. We call it $DK(C_*)$.

THEOREM . *The functor $DK : C_* \mapsto DK(C_*)$ from $\mathcal{C}$ to $\mathcal{G}$ is inverse up to homotopy to the well-known category equivalence $F : \mathcal{G} \longrightarrow \mathcal{C}$. We shall simply write $|C_*|$ for $DK(C_*)$ and we call it the "simplicial realization" of the chain complex C_*.*

Let X_* be a simplicial abelian group and let $C_* = F(X_*)$ be the associated chain complex. According to the previous theorem, there should be an explicit homotopy equivalence

$$\phi : X_* \longrightarrow |C_*|$$

This may be applied to the particular case when X_* is the $\mathbb{C}$-vector space with basis the simplicial set BG or EG, whose G is a discrete group. In order to be more explicit, let $C_* = C_*(EG)$ be the "homogeneous" Eilenberg-Mac Lane chain complex defining the homology of $BG = G \setminus EG$. We define a simplicial map

$$\Phi : BG \longrightarrow |C_*|$$

by the following formula for $g = (g_0, g_1, ..., g_r)$ of degree r in $BG = G \setminus EG$:

$$\Phi(g) = \sum_i x_i \otimes g_j + \sum_{(i,j)} x_i dx_j \otimes \{g_i, g_j\} + \sum_{(i,j,k)} x_i dx_j \wedge dx_k \otimes \{g_i, g_j, g_k\} + ...$$

where$\{g_i, g_j\}$, $\{g_i, g_j, g_k\}$, etc... are the corresponding classes in $C_*(BG)$. Here $x_i dx_j$, $x_i dx_j \wedge dx_k$, etc... are the usual differential forms on the standard r-simplex Δ_r with the x_i as barycentric coordinates. It is easy to check that ϕ is the expected map.

With these definitions, let us now describe the Chern character in Algebraic K-theory. For this purpose we consider the double complex

$$\overline{C}_{**}(G)$$

defined in [K2] and $C_{**} = G/\overline{C}_{**}(G)$ the quotient complex by the left action of G. The definition of this double complex follows the same pattern as C_{**}, using the obvious cyclic module associated to the bar construction of EG. The double indices in $C_{p,q}(G)$ or $\overline{C}_{p,q}(G)$ are in the same range as before : $p \in \mathbb{Z}, q \in \mathbb{N}$. In the same way, we put

$$C_n^{per}(G) = \prod_{p+q=n} C_{p,q}(G) \ for \ n > 0 \text{ and}$$

$$C_0^{per}(G) = Ker[\prod_{p+q=0} C_{p,q}(G) \longrightarrow \prod_{p+q=-1} C_{p,q}(G)],$$

and analogous definitions for $C_n^-(G)$ if $n \geq 0$. In particular, we define $HC_*(G)$ as the homology of the total complex associated to the previous double complex restricted to the first quadrant. With the method described in [K2], it is easy to show, using classical homological algebra (see [K2] or the book of Weibel) that

$$HC_n(G) = H_n(G) \oplus H_{n-2}(G) \oplus ...$$

$$HC_n^-(G) = \prod_{k\geq 0} H_{n+2k}(G)$$

$$HC_n^{per}(G) = \prod_{k} H_{n+2k}(G)$$

According to the previous considerations, we therefore have well defined maps of simplicial complexes

$$BG \longrightarrow |C_*(G)| \longrightarrow |C_*^-(G)|$$

In particular, if $G = GL(A)$, we get a morphism $|C_*(G)| \longrightarrow |C_*^-(\overline{A})|$. If we apply the + construction, we obtain a well defined map up to homotopy $BGL(A)^+ \longrightarrow |C_*^-(\overline{A})| \longrightarrow |C_*^-(A)|$ and therefore a homomorphism

$$K_n(A) \longrightarrow HC_n^-(A)$$

defined by Hood and Jones[19], generalizing the results obtained in [K2], with the same method.

In order to define the same type of invariants in topological K-theory, we just apply the same ideas, but with $G = GL(A)$ replaced by the simplicial group $G_* = GL(A_*)$. We write the composition

$$BG_* \longrightarrow |C_*(G_*)| \longrightarrow |C_*^-(G_*)| \longrightarrow |C_*^-(A_*)| \longrightarrow |C_*^{per}(A_*)|$$

[19] C. HOOD and J.D.S. JONES. Some algebraic properties of cyclic homology groups. K-theory 1(1987),361-384

Now comes the crucial fact : since periodic cyclic homology is homotopy invariant, the space $|C_*^{per}(A_*)|$ is homotopically equivalent to the space $|C_*^{per}(A)|$ and finally we get the Chern character in topological K-theory defined as a map

$$BGL(A) \longrightarrow |C_*^{per}(A)|$$

By taking homotopy groups, one finally gets a homomorphism

$$K_n^{top}(A) \longrightarrow HC_n^{per}(A)$$

DEFINITION/THEOREM . *Let G_* be the simplicial group $GL(A_*)$ and let $\mathcal{K}(A)$ be the homotopy fiber of the map*

$$BG_* \longrightarrow |C_*^{per}(A)| \longrightarrow |C_*^{per}(A)|/|C_*^-(A)|$$

For $n > 0$, the multiplicative K-theory groups $\mathcal{K}_n(A)$ are then defined as the homotopy groups of the space $\mathcal{K}(A)$. We have an exact sequence

$$K_{n+1}^{top}(A) \longrightarrow HC_{n-1}(A) \longrightarrow \mathcal{K}_n(A) \longrightarrow K_n^{top}(A) \longrightarrow HC_{n-2}(A)$$

It is easy to see that the composite map

$$BGL(A)^\delta \longrightarrow BGL(A_*) \longrightarrow |C_*^{per}(A_*)| \approx | C_*^{per}(A)|$$

can be factored through $|C_*^-(A)|$. We deduce from this observation a <u>canonical</u> map from $BGL(A)^\delta$ to $\mathcal{K}(A)$. In the same way, if we denote by $\mathcal{G}$ the homotopy fiber of the map

$$BGL(A) \longrightarrow BGL(A_*)$$

we have a canonical map from $\mathcal{G}$ to $|C_{*-1}(A)|$. Thanks to the + construction, we can write the following diagram of homotopy fibrations (with $\mathcal{F} = \mathcal{G}^+$) :

$$\begin{array}{ccccccc}
\Omega BGL(A_*) & \longrightarrow & \mathcal{F} & \longrightarrow & BGL(A)^+ & \longrightarrow & BGL(A_*) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega BGL(A_*) & \longrightarrow & \Omega|C_{*-2}(A)| & \longrightarrow & \mathcal{K}(A) & \longrightarrow & BGL(A_*)
\end{array}$$

Applying the functor π_n, we find the commutative diagram

$$\begin{array}{ccccccc}
K_n^{rel}(A) & \longrightarrow & K_n(A) & \longrightarrow & K_n^{top}(A) & \longrightarrow & K_{n-1}^{rel}(A) \\
\downarrow & & \downarrow & & \| & & \downarrow \\
HC_{n-1}(A) & \longrightarrow & \mathcal{K}_n(A) & \longrightarrow & K_n^{top}(A) & \longrightarrow & HC_{n-2}(A)
\end{array}$$

The commutativity of the diagram

$$\begin{array}{ccccccc}
HC_{n-1}(A) & \longrightarrow & \mathcal{K}_n(A) & \longrightarrow & K_n^{top}(A) & \longrightarrow & HC_{n-2}(A) \\
\| & & \downarrow & & \downarrow & & \| \\
HC_{n-1}(A) & \longrightarrow & HC_n^-(A) & \longrightarrow & HC_n^{per}(A) & \longrightarrow & HC_{n-2}(A)
\end{array}$$

is a consequence of the homotopy fibrations :

$$\begin{array}{ccccc}
\mathcal{K}_n(A) & \longrightarrow & BGL(A_*) & \longrightarrow & |C_*^{per}(A)|/|C_*^-(A)| \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
|C_*^-(A)| & \longrightarrow & |C_*^{per}(A)| & \longrightarrow & |C_*^{per}(A)|/|C_*^-(A)|
\end{array}$$

Finally, the commutativity of the diagram

$$\begin{array}{ccccccc}
HC_{n-1}(A) & \longrightarrow & HC_n^-(A) & \longrightarrow & HC_n^{per}(A) & \longrightarrow & HC_{n-2}(A) \\
\| & & \downarrow & & \downarrow & & \| \\
HC_{n-1}(A) & \longrightarrow & HH_n(A) & \longrightarrow & HC_n(A) & \longrightarrow & HC_{n-2}(A)
\end{array}$$

follows from the classical definitions (cf. [L], p. 158 for instance).

References

[At1] TIYAH M.-F. : *K-theory*. Benjamin, New-York (1967).

[Ba] ASS H.: *Algebraic K-theory*. New York. Benjamin (1968).

[Ber] ERRICK J. : *An approach to Algebraic K-theory*. Pitman (1982).

[Bla] LACKLADAR : *K-theory of operator algebras*. Springer (1986).

[C1] ONNES A. : *Noncommutative Geometry*. Academic Press, San Diego (1994).

[FW] RIEDLANDER E. and WEIBEL C. : *School on Algebraic K-theory and applications*. An overview of the subject. ICTP Trieste (1997).

[K1] AROUBI M. : *K-theory, an introduction*. Grundlehren der Math. Wissen. 226, Springer, Berlin (1978).

[K2] AROUBI M. : *Homologie cyclique et K-théorie*. Astérisque 149. Société Mathématique de France (1987).

[L1] ODAY J.-L. : *Cyclic homology*. Grundelehren der Math. Wissen. 301, Springer, Berlin (1992).

[M] ILNOR J. : *Introduction to Algebraic K-theory*. Ann. of Math. Studies 197. Princeton, NJ. Princeton University Press (1974).

[R] OSENBERG J. : *Algebraic K-theory and its applications*. Graduate Texts *No* 147, Springer, Berlin (1994).

[Sr] RINIVAS V. : *Algebraic K-theory*. Birkhauser, Boston (1996).

[SLN] pringer Lecture Notes *Nr* 341, 342 and 342 (1973).

Papers

[Ad] DAMS J.-F. : *Vector fields on the sphere*. Ann. of Math. 75, 603-632 (1962).

[At2] TIYAH M.-F. : *Global theory of elliptic operators*. Proc. Internat. Conf. on Functional Analysis and related topics, p. 21-30. Univ. of Tokyo Press (1970).

[AH1] TIYAH M.-F. and HIRZEBRUCH F. : *Vector bundles and homogeneous spaces*. Proc. Symposium in Pure Maths, Amer. Math. Soc. 3, 7-38 (1961).

[AH2] TIYAH M.-F. and HIRZEBRUCH F. : *Riemann-Roch theorems for differentiable manifolds*. Bull. Amer. Math. Soc. 65, 276-281 (1959).

[AS1] TIYAH M.-F. and SINGER I.M. : *The index of elliptic operators on compact manifolds*. Bull. Amer. Math. Soc. 69, 422-433 (1963).

[AS2] TIYAH M.-F. and SINGER I.M. : *The index of elliptic operators*. Ann. of Maths. 87, 484-530, 564-604 (1968), 93, 119-138 (1971).

[Be] EILINSON A.A. : *Higher regulators and values of L-functions*. J. Soviet. Math. 30, 2036-2070 (1985).

[BHM] OKSTEDT M., HSIANG W.-C. and MADSEN I. : *The cyclotomic trace and algebraic K-theory of spaces*. Invent. Math. 111, 465-540 (1993).

[Bor] OREL A. : *Stable real cohomology of arithmetic groups*. Ann. Sc. Ec. Norm. Sup. 7, 235-272 (1974).

[BS] OREL A. et SERRE J.-P. : *Le théorème de Riemann-Roch (d'après Grothendieck)*. Bull. Soc. Math. France 86, 97-136 (1958).

[Bott] OTT R. : *The stable homotopy of the classical groups*. Ann. of Math. 70, 313-337 (1959).

[Br] ROWDER W. : *Surgery on simply connected manifolds*. Ergebnisse der Mathematik, Springer (1970).

[Bu] URGHELEA D. : *Cyclic homology and the algebraic K-theory of spaces I*. Contemporary Mathematics, 55, 89-115 (1986).

[BF] URGHELEA D. and FIEDOROWICZ Z. : *Cyclic homology and the algebraic K-theory of spaces II*. Topology 25, 303-317 (1986).

[C2] ONNES A. : *Non commutative differential geometry*. Publ. Institut Hautes Etudes Scientifiques N 62, 257-360 (1985).

[CK] ONNES A. et KAROUBI M. : *Caractère multiplicatif d'un module de Fredholm*. K-theory 2, 431-463 (1988).

[Cu] UNTZ J. : *A new look at KK-theory*. K-theory 1, 31-51 (1987).

[HH] AUSMANN J.-C. and HUSEMOLLER D. : *Acyclic maps*. L'Enseignement mathématique, XXV, 53-75 (1979).

[H] IGSON N. *A characterization of KK-theory*. Pacific J. Math. 126, 253-276 (1987).

[HJ] OOD C., JONES J.D.S. : *Some algebraic properties of cyclic homology groups*. K-theory 1, 361-384 (1987).

[Gra] RAYSON D. : *Higher Algebraic K-theory II (after Daniel Quillen)*. Lecture Notes in Math. 551 (1976).

[K3] AROUBI M. : *Quatre Notes sur l'homologie cyclique*. C.R. Acad. Sci. Paris 297, 381-384, 447-450, 513-516 et 557-560 (1983).

[K4] AROUBI M. : *Le théorème fondamental de la K-théorie hermitienne.* Annals of Math. 112, 259-282 (1980).

[KV] AROUBI M. et VILLAMAYOR O. : *Foncteurs K^n en algèbre et en topologie, Comptes Rendus Acad.* Sci. Paris, t. 269, p. 416 (1969).

[KL] AROUBI M. et LAMBRE T. : *Quelques classes caractéristiques en théorie des nombres (avec T. Lambre).* J. Reine angew. Math. 543, p. 169-186 (2002)

[Kas] ASPAROV G. : *The operator K-functor and extension of C^*-algebras.* Izv. Akad. Nauk. SSSR, Ser. Math. 44, 571-636 (1980).

[L2] ODAY J.-L. : *Homotopie des espaces de concordance.* Séminaire Bourbaki N 516 (1977/78).

[LQ] ODAY J.-L. and QUILLEN D. : *Cyclic homology and the Lie algebra homology of matrices.* Comm. Math. Helvetici 59, 565-591 (1984).

[MS] ERKURJEV A.S. and SUSLIN A.A. : *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism.* Math. USSR Izv. 21, 307-440 (1983).

[NT] EST R. and TSYGAN B. : *Algebraic Index Theorem.* Comm. Math. Phys. 172, 223-262 (1995).

[Q] UILLEN D. : *Higher Algebraic K-theory.* Lecture Notes in Maths 341, 85-147. Berlin-Heidelberg-New York : Springer (1973).

[So] OULE C. : *K-théorie des anneaux d'entiers de corps de nombres.* Inventiones math. 55, 251-295 (1979).

[Su] USLIN A. : *On the K-theory of local fields.* J. Pure and Applied Algebra 34, 301-318 (1984).

[SW] USLIN A. and WODZICKI M. : *Excision in Algebraic K-theory.* Ann. of Maths 136, 51-122 (1992).

[Th] HOMASON R. : *Algebraic K-theory and etale cohomology.* Ann. Sci. Ec. Norm. Supér. 13, 437-552 (1985).

[Ts] SYGAN B.L. : *The homology of matrix Lie algebras over rings and the Hochschild homology.* Russian Math. Survey 38 N 2, 198-199 (1983).

[Vo] OEVODSKY V. (to appear).

[Wald] ALDHAUSEN W. : *Algebraic K-theory of topological spaces I.* Proc. Symp. Pure Math. 32 AMS, 35-60 (1978), II Springer Lecture Notes in Math. 1051, 173-196 (1984).

[Wall] ALL C.T.C. : *Surgery on compact manifolds.* Academic Press (1967).

[Wo] OOD R. : *Banach algebras and Bott periodicity.* Topology 4, 371-389 (1966).

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