§ Spanier Whitehead duality for $KK_X(-,-)$.

$$KK_X(-,-) \doteq \{C(X)\text{-linear homs}\}/\sim_h.$$

We replace $\mathbb{C}$ with $C(X)$ and

unit & counit live in

$$\mu \in KK_X(C(X), \mathcal{A} \otimes_{C(X)} D(\mathcal{A}))$$
$$\nu \in KK_X(D(\mathcal{A}) \otimes_{C(X)} \mathcal{A}, C(X)).$$

Let X be S^1 for simplicity.

A & $D(A)$: stable Kirchberg alg.

We identify $C(S^1)$ as $\{f \in C[0,1] \mid f(0) = f(1)\}$.

and every loc trivial conti field of A is given by

$$\mathcal{A} = M_\alpha(A) := \{a \in C[0,1] \otimes A \mid a(0) = \alpha(a(1))\}$$

for some $\alpha \in \mathrm{Aut}(A)$

GOAL

We will find $\alpha_D \in \mathrm{Aut}(D(A))$ providing

$$D(A) = M_{\alpha_D}(D(A))$$

and the unit & counit.

Lem $\exists\ \mu_0 : O_\infty \otimes \mathbb{K} \longrightarrow A \otimes D(A)$

$\exists\ \nu_0 : D(A) \otimes A \longrightarrow O_\infty \otimes \mathbb{K}$ $*$-homs.

such that

$$(\mathrm{id}_A \otimes \nu_0) \circ (\mu_0 \otimes \mathrm{id}_A)$$

$$\sim_h \quad \theta : (O_\infty \otimes \mathbb{K}) \otimes A \longrightarrow A \otimes (O_\infty \otimes \mathbb{K})$$

$$x \otimes a \longmapsto a \otimes x$$

One can also observe that

$$(\nu_0 \otimes \mathrm{id}_{D(A)}) \circ (\mathrm{id}_{D(A)} \otimes \mu_0)$$

$$\sim_h \quad \sigma : D(A) \otimes (O_\infty \otimes \mathbb{K}) \longrightarrow (O_\infty \otimes \mathbb{K}) \otimes D(A)$$

For the proof, we use a KK-equivalence

$$\gamma \in KK(\mathcal{O}_\infty \otimes \mathbb{K}, \mathbb{C})^{-1}$$

and the equation

$$\underset{\substack{\cap \\ KK(\mathcal{O}_\infty \otimes \mathbb{K} \otimes A,\, A)}}{\gamma \otimes \mathrm{id}_A} = \underset{\substack{\cap \\ KK(\mathcal{O}_\infty \otimes \mathbb{K} \otimes A,\, A \otimes \mathcal{O}_\infty \otimes \mathbb{K})}}{\theta} \hat{\otimes} \underset{\substack{\cap \\ KK(A \otimes \mathcal{O}_\infty \otimes \mathbb{K},\, A)}}{(\mathrm{id}_A \otimes \gamma)}.$$

Lem $\exists\, \alpha_D \in \mathrm{Aut}(D(A))$ such that

$$\nu_0 \circ (\alpha_D \otimes \alpha) \sim_h \nu_0$$

Proof

It is enough to show that.

there exists $\alpha_D \in KK(D(A), D(A))^{-1}$ satisfying

$$(\alpha_D \otimes \alpha) \hat{\otimes} \nu_0 = \nu_0 \in KK(D(A) \otimes A, \mathcal{O}_\infty \otimes \mathbb{K}).$$

which follows from the basic properties of unit & counit. $\square$

$$\boxed{\underline{\text{Cor}} \quad \mu_0 \sim_h \mu_0 \circ (\alpha \otimes \alpha_D)^{-1} \quad \text{in } \mathrm{Hom}(\mathcal{O}_\infty \otimes \mathbb{K}, A \otimes D(A))}$$

<u>Proof</u> By the above two lemmas, one gets

$$(\mathrm{id}_{\mathcal{O}_\infty \otimes \mathbb{K}}) \otimes (\alpha \otimes \alpha_D \circ \mu_0)$$

$$\sim_h ((\mathrm{id}_{\mathcal{O}_\infty \otimes \mathbb{K}}) \otimes \mu_0) \circ \xi .$$

Here

$$\begin{array}{rcl} \xi : (\mathcal{O}_\infty \otimes \mathbb{K})^{\otimes 2} & \longrightarrow & (\mathcal{O}_\infty \otimes \mathbb{K})^{\otimes 2} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x \otimes y & \longmapsto & y \otimes x \end{array}$$

is the flip automorphism, and $\xi \sim_h \mathrm{id}_{(\mathcal{O}_\infty \otimes \mathbb{K})^{\otimes 2}}$

The map

$$\begin{array}{rcl} KK(\mathcal{O}_\infty \otimes \mathbb{K}, A \otimes D(A)) & \longrightarrow & KK((\mathcal{O}_\infty \otimes \mathbb{K})^{\otimes 2}, (\mathcal{O}_\infty \otimes \mathbb{K}) \otimes A \otimes D(A)) . \\ \rotatebox{90}{$\in$} & & \\ \varphi & \longmapsto & (\mathrm{id}_{\mathcal{O}_\infty \otimes \mathbb{K}}) \otimes \varphi \end{array}$$

is bijective, which implies

$$\alpha \otimes \alpha_D \circ \mu_0 \sim_h \mu_0 \qquad \square$$

Now $M_{\alpha_D}(D(A))$ gives a dual alg

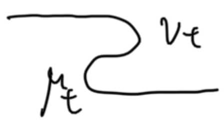
and one can obtain unit & counit.

$$\mu_t : C(S^1) \longrightarrow M_\alpha(A) \otimes_{C(S^1)} M_{\alpha_D}(D(A))$$
$$\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad M_{\alpha\otimes\alpha_D}(A \otimes D(A))$$

$$\nu_t : M_{\alpha_D}(D(A)) \otimes_{C(S^1)} M_\alpha(A) \longrightarrow C(S^1)$$
$$\| $$
$$M_{\alpha_D\otimes\alpha}(D(A) \otimes A)$$

<u>Rem</u> μ_t & ν_t satisfy only

$\in KK_X(M_\alpha(A), M_\alpha(A))^{-1}$



$\in KK_X(M_{\alpha_D}(D(A)), M_{\alpha_D}(D(A)))^{-1}$.

But the above is enough to show that.

$M_\alpha(A)$ & $M_{\alpha_D}(D(A))$ are dual