C^*-algebras associated with real reductive symmetric spaces

based on j. work with A. Afgoustidis, N. Higson, P. Hochs, and Y. Song

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Recall $C_r^*(G)$, one of our favorite C^*-algebras defined by the regular representation

$$\lambda_G\colon G \to U(L^2(G)),\ \ (\lambda_G(g)f)(s) \coloneqq f(g^{-1}s).$$

If G is of type 1 (unimodular, separable), the abstract Plancherel theorem (Dixmier) gives

$$L^2(G) \cong \int_{\widehat{G}}^{\oplus} V_\pi \otimes V_{\pi^*}\, d\mu(\pi), \qquad f(e) = \int_{\widehat{G}} \mathrm{Tr}(\pi(f))d\mu(\pi).$$

The isomorphism on the left is iso of unitary representations of $G \times G$.

Here π^* is the dual (contragradient) of π, defined on V_π^*, by $\pi^*(g) \coloneqq \pi(g^{-1})^*$; $V_\pi \otimes V_{\pi^*} \cong V_\pi \otimes V_\pi^* \cong \mathcal{B}_2(V_\pi)$.

The support of μ is the reduced dual $\widehat{G}_r$, the spectrum of $C_r^*(G)$.

I will first discuss what happens if we consider G as a symmetric space $X = G \times G/\mathrm{diag}(G)$ and take the image/quotient of $C^*(G \times G)$.

Let G be type 1, unimodular. Let $X = G \times G/\mathrm{diag}(G) \cong G$. Let

$$A = C^*_{\lambda_X}(G \times G) = \lambda_X(C^*(G \times G)) \subset B(L^2(X)).$$

By the abstract Plancherel theorem, we have

$$\hat{A} = \overline{\{\pi \otimes \pi^* \mid \pi \in \widehat{G}_r\}} \subset \widehat{G}_r \times \widehat{G}_r.$$

Note, $\hat{A}$ contains all the limit points of $\{\pi \otimes \pi^*\}$ for $\pi \in \widehat{G}_r$.

Thus, we may think A as the quotient of $C^*_r(G) \otimes C^*_r(G)$ with spectrum $\overline{\{\pi \otimes \pi^*\}}$.

The map $f \mapsto \overline{f}$ on $C^*(G)$ induces an (anti-linear) automorphism on $C^*(G)$, mapping the primitive ideal I_π for π to I_{π^*}. Thus, the skew diagonal $\{\pi \otimes \pi^*\}$ and the diagonal $\{\pi \otimes \pi\}$ are homeomorphic (same for their closures).

So, $\{\pi \otimes \pi^* \mid \pi \in \widehat{G}_r\}$ is closed in $\widehat{G}_r \times \widehat{G}_r$ iff $\{\pi \otimes \pi \mid \pi \in \widehat{G}_r\}$ is closed, iff $\widehat{G}_r$ is Hausdorff.

Let us consider for any type 1-C^*-algebra B, the quotient A of $B \otimes B$ whose spectrum $\hat{A}$ is $\overline{\{\pi \otimes \pi\}}$, the closure of the diagonal inside $\hat{B} \times \hat{B}$.

Example: $B = C_0(\mathbb{R}) \rtimes C_2$ (by the flip action):

We have $B \subset C_0([0, \infty), M_2(\mathbb{C}))$ (diagonal matrices at 0).
The spectrum of B is the half-line with double points at the origin.

Then, we have $A \subset C_0([0, \infty), M_4(\mathbb{C}))$ (diagonal matrices at 0).
$\hat{A}$ is the half-line with four points at the origin.

In this case, there is an ideal I of A whose spectrum $\hat{I} = \{\pi \otimes \pi\}$.
Moreover, I is Morita-equivalent to B.

Following Knapp, we say G is a linear connected real reductive group if it is a closed connected subgroup of $\mathrm{GL}(n, \mathbb{R})$ stable under transpose. For example, $\mathrm{SL}(n, \mathbb{R}), \mathrm{SO}(n, m)_0$.

Theorem: Let G be a linear connected real reductive group. Let $X = G \times G / \mathrm{diag}(G) \cong G$. Consider the quotient

$$A = C^*_{\lambda_X}(G \times G) \subset B(L^2(X)).$$

Then, there is an ideal I of A whose spectrum is $\{\pi \otimes \pi^* \mid \pi \in \hat{G}_r\}$. In other words, the skew diagonal $\{\pi \otimes \pi^* \mid \pi \in \hat{G}_r\}$ is open in its closure. Moreover, the ideal I is Morita equivalent to $C^*_r(G)$, hence they have isomorphic K-theory.

Theorem: The same assumption on G. The diagonal $\{\pi \otimes \pi \mid \pi \in \hat{G}_r\}$ is open in its closure. The corresponding ideal-quotient I of $C^*(G \times G)$ is Morita-equivalent to $C^*_r(G)$.

I will illustrate how these properties are uncommon for other groups G, and for type 1-C^*-algebras.

Proposition: Let X be a topological space. Then, the diagonal Δ_X in $X \times X$ is open in its closure (i.e. Δ_X is locally closed) if and only if X is locally Hausdorff.

Proof: Δ_X is open in $\overline{\Delta_X}$ if and only if for any x in X, there is $U_x \ni x$ open so that $(U_x \times U_x) \cap \overline{\Delta_X} = \Delta_{U_x}$, i.e. U_x is Hausforff.

Example: The half-line with two points at the origin is locally Hausdorff.

Non-Example: If $\widehat{A}$ has a non-closed point, then $\hat{A}$ is not locally Hausdorff. Hence, for any such A, there is no ideal-quotient I of $A \otimes A$ whose spectrum is $\{\pi \otimes \pi \mid \pi \in \hat{A}\}$.

Non-Example: Let G be the Heisenbeg group $\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ over the real.

Then, $\hat{G} = \widehat{G_r}$ is not locally Hausdorff: $\hat{G}$ is the real line "with plane at the origin".

Even if $\hat{B}$ is Hausdorff, the quotient A of $B \otimes B$ with spectrum $\{\pi \otimes \pi \mid \pi \in \hat{B}\}$ are often not Morita equivalent to B:

Example: Take $B \subset C_0([0, \infty), M_2(\mathbb{C}))$ (scalar matrices at 0). $K_1 \cong Z/2$. Then, $A \subset C_0([0, \infty), M_4(\mathbb{C}))$ (scalar matrices at 0). $K_1 \cong Z/4$.

Example: Take B a continuous trace C^*-algebra with (locally compact, Hausdorff) spectrum X. According to the Dixmier-Douady theory, $A = B \otimes_X B$ is Morita-equivalent to B if and only if their DD-invariants coincide: $\delta_B + \delta_B = \delta_B$ in $H^3(X, \mathbb{Z})$, i.e. $\delta_B = 0$. Same conclusion if we instead use "the skew-diagonal" $B \otimes_X B^{\mathrm{op}}$. If $\delta_B = 0$, we have $B = \mathcal{K}(C_0(X, \mathcal{H}_X))$, and A is Morita-equivalent to B via continuous field $\mathcal{H}_X \otimes_X \mathcal{K}(\mathcal{H}_X)$.

The main ingredients of the proof of Theorem (briefly):

For G real reductive (in Harish-Chandra's class), one of the major results of Harish-Chandra is the Plancherel decomposition of the Schwartz space $\mathcal{C}(G)$, which is a subspace of $L^2(G)$, but also a dense subalgebra of $C_r^*(G)$. The corresponding decomposition of $C_r^*(G)$ was obtained by Clare, Crisp, and Higson, utilising Hilbert-module techniques:

$$C_r^*(G) \cong \bigoplus_{[P]\in\mathcal{P}} C_r^*(G)_P,$$

$$C_r^*(G)_P \cong \left(\bigoplus_{\xi\in\widehat{M}_{\mathrm{ds}}} C_0(\mathfrak{a},\mathfrak{K}) \right)^{W_P} \cong \bigoplus_{[\xi]\in\widehat{M}_{\mathrm{ds}}/W_P} C_0(\mathfrak{a},\mathfrak{K})^{W_{P,\xi}}$$

I omit more explanations, but this implies

$$\widehat{G}_r = \bigsqcup_{[P,\xi]} C_0\widehat{(\mathfrak{a},\mathfrak{K})^{W_{P,\xi}}}$$

If G is connected and linear, there is an extra feature of each summand $C_0(\mathfrak{a}, \mathfrak{K})^{W_{P,\xi}}$, due to Kanpp–Stein, and separately by Vogan.

Let K be a maximal compact subgroup of G.

For each P, ξ, there is a (canonically defined) finite subset $S_{P,\xi} \subset \widehat{K}$ with the following properties:

For a (any) rank one projection p_τ in $C^*(K)$ of type $\tau \in S_{P,\xi}$,

$$p_\tau C_0(\mathfrak{a}, \mathfrak{K})^{W_{P,\xi}} p_\tau \cong C_0(\mathfrak{a}/W_{P,\xi})$$

and

$$\sum_{\tau \in S_{P,\xi}} C_0(\mathfrak{a}, \mathfrak{K})^{W_{P,\xi}} p_\tau C_0(\mathfrak{a}, \mathfrak{K})^{W_{P,\xi}} = C_0(\mathfrak{a}, \mathfrak{K})^{W_{P,\xi}}.$$

These imply that $C_0(\mathfrak{a}, \mathfrak{K})^{W_{P,\xi}}$ has locally Hausdorff spectrum. Therefore, $C_r^*(G)$ has a locally Hausdorff spectrum.

Remark: There is a linear, disconnected, real reductive G such that $\widehat{G}_r$ is not locally Hausdorff.

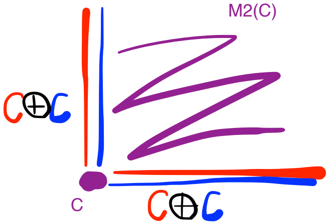
Example (Vogan): $G = (\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})) \rtimes H_8$ (quoternionic group).

$C_r^*(G)$ contains a direct summand that is Morita equivalent to the following C^*-algebra

$$A \subset C_0([0,\infty) \times [0,\infty), M_2(\mathbb{C})),$$

diagonal matrices on $\{0\} \times \mathbb{R}$ and on $\mathbb{R} \times \{0\}$ and scalar matrices at the origin $(0,0)$.

$\hat{A}$ is not locally Hausdorff at the origin.



A homogeneous space G/H is called a real reductive symmetric space if G is real reductive and H is a symmetric subgroup of G: H is an open subgroup of G^σ where $\sigma \in \mathrm{Aut}(G)$ is an involution ($\sigma^{-1} = \sigma$).

Examples: $G = G_1 x G_1/\mathrm{diag}(G_1)$, $\mathrm{GL}_{m+n}(\mathbb{R})/\mathrm{GL}_m(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{C})/\mathrm{SL}_n(\mathbb{R})$, ... and a lot more.

From what we just saw regarding the group case $G = G_1 x G_1/\mathrm{diag}(G_1)$, it is expected that the image/quotient of the regular representation

$$\lambda_X \colon C^*(G) \to B(L^2(G/H)),$$

may contain a proper ideal that deserves some attention in terms of K-theory: $C^*_{\lambda_{G_1}}(G_1 \times G_1)$ has an ideal Morita equivalent to $C^*_r(G_1)$ if G_1 is linear connected real reductive.

We also know that this is quite delicate: the aforementioned ideal of $C^*_{\lambda_{G_1}}(G_1 \times G_1)$ exists iff the skew-diagonal $\{\pi \otimes \pi^*\}$ is locally closed iff $\widehat{G_1}$ is locally Hausdorff.

For symmetric spaces G/H, the situation is analogous but much more complicated. I will focus on some aspects that are easy to digest.

The Harish-Chandra's Plancherel decomposition has been extended to this setting by van den Ban, Delorme, and Schlichtkrull in the 1990s, utilizing the work of Flensted-Jensen, Oshima-Matuski, et al on the discrete series representations (irreducible direct summand) of $L^2(G/H)$:

$$\mathcal{C}(G/H) \cong \bigoplus_{[P]\in\mathcal{P}} \mathcal{C}(G/H)_P,$$

$$\mathcal{C}(G/H)_P \cong \left(\bigoplus_{v\in\mathcal{W}} \bigoplus_{\xi\in\widehat{M}_{H,v,\mathrm{ds}}} \mathcal{S}(\mathfrak{a}_\mathfrak{q}, \mathcal{H}\otimes\mathcal{V}) \right)^{W_P} \cong \bigoplus_{[\xi]\in\widehat{M}_{H,\mathrm{ds}}/W_P} \mathcal{S}(\mathfrak{a}_\mathfrak{q}, \mathcal{H}\otimes\mathcal{V})^{W_{P,\xi}}$$

To be precise, we need to consider the dense subspace of K-finite functions, and I also abused some notations.

I omit more explanations, but the crucial difference from the group case is that $\mathcal{C}(G/H)$ is not an algebra, and it is not straightforward to transfer this decomposition to a decomposition of $C^*_{\lambda_X}(G)$, the image/quotient of the regular representation

$$\lambda_X\colon C^*(G) \to B(L^2(G/H)).$$

But in some cases, the Plancherel theorem is more than enough to fully determine $C^*_{\lambda_X}(G)$.

Tempered symmetric spaces

A symmetric space (more generally a homogeneous space) G/H is called a tempered symmetric/homogeneous space if $\operatorname{supp}(\lambda_{G/H}) \subset \widehat{G}_{\mathrm{temp}} = \widehat{G}_r$.

Benoist and Kobayashi classified all reductive pairs (G, H) such that the homogeneous space G/H is not tempered. In their work, they showed that if $G^{\mathbb{C}}/H^{\mathbb{C}}$ is tempered, then G/H is tempered.

For a symmetric space, the condition for $G^{\mathbb{C}}/H^{\mathbb{C}}$ being tempered is equivalent to the following:

We say a symmetric space G/H is well-tempered if for a (any) Cartan subspace $\mathfrak{b}$ of $\mathfrak{q}$, the centralizer $Z_{\mathfrak{g}}(\mathfrak{b})$ is abelian (thus $Z_{\mathfrak{g}}(\mathfrak{b})$ is a Cartan subspace of $\mathfrak{g}$). Here, $\mathfrak{q}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$.

Any well-tempered symmetric space is tempered. If G/H is a complex symmetric space, the converse holds. These can also be deduced from the Plancherel theorem and the theory of discrete series.

Examples of well-tempered homogenous spaces: $G_1 \times G_1/\mathrm{diag}(G_1)$, $\mathrm{SL}(m+n,\mathbb{R})/\mathrm{SO}_0(m,n)$, $\mathrm{SU}(m,n)/\mathrm{SO}_0(m,n)$, G/K_ϵ (G quasi-split), $\mathrm{SL}(2n,\mathbb{R})/S(\mathrm{GL}(n,\mathbb{R})x\mathrm{GL}(n,\mathbb{R}))$, $\mathrm{SO}_0(2n,2)/\mathrm{SO}(n) \times \mathrm{SO}_0(n,2)$, ...

Symmetric spaces of type $G_{\mathbb{C}}/G_{\mathbb{R}}$

Let G be a connected complex reductive Lie group, and H be an open subgroup of its real points. We call G/H a symmetric space of type $G_{\mathbb{C}}/G_{\mathbb{R}}$.

Examples: $\mathrm{SL}(n,\mathbb{C})/\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SO}_0(\mathrm{n},\mathrm{m})^{\mathbb{C}}/\mathrm{SO}_0(n,m)$, $K^{\mathbb{C}}/K$, ...

The Plancherel formula for $L^2(G/H)$ was obtained by Pascale Harinck (1990).

Any G/H of type $G_{\mathbb{C}}/G_{\mathbb{R}}$ is well-tempered: a subspace $\mathfrak{b}_{\mathfrak{q}} \subset \mathfrak{q}$ is a Cartan subspace of $\mathfrak{q}$ iff $\mathfrak{b}_{\mathfrak{h}} = i\mathfrak{b}_{\mathfrak{q}} \subset \mathfrak{h}$ is a Cartan subspace of $\mathfrak{h}$ iff $\mathfrak{b} = \mathfrak{b}_{\mathfrak{h}} + \mathfrak{b}_{\mathfrak{q}}$ is a Cartan subspace of $\mathfrak{g}$. In particular, $Z_{\mathfrak{g}}(\mathfrak{b}_{\mathfrak{q}}) = \mathfrak{b}$.

C^*-algebra on symmetric spaces of type $G_{\mathbb{C}}/G_{\mathbb{R}}$

Theorem Let G/H be a symmetric space of type $G_{\mathbb{C}}/G_{\mathbb{R}}$. Then, the following are equivalent:

1. $K_*(C^*_{\lambda_{G/H}}(G)) \not\cong 0$;
2. For a (any) Cartan subspace $\mathfrak{a}_{\mathfrak{h}}$ of $\mathfrak{p} \cap \mathfrak{h}$, $Z_{\mathfrak{h}}(\mathfrak{a}_{\mathfrak{h}})$ is abelian, i.e. H is quasi-split;
3. For a (any) Cartan subspace $\mathfrak{t}_{\mathfrak{q}}$ of $\mathfrak{k} \cap \mathfrak{q}$, $Z_{\mathfrak{k}}(\mathfrak{t}_{\mathfrak{q}})$ is abelian;
4. $K/K \cap H$ is well-tempered, i.e. $K_{\mathbb{C}}/(K \cap H)_{\mathbb{C}}$ is tempered,

If these conditions are satisfied, we have

$$K_*(C^*_{\lambda_{G/H}}(G)) \cong \begin{cases} \bigoplus_{\widehat{K}_{K\cap H,\mathrm{reg}}} \mathbb{Z} & * = \dim(\mathfrak{t}_{\mathfrak{h}}) \\ 0 & * \neq \dim(\mathfrak{t}_{\mathfrak{h}}) \end{cases}$$

where $\mathfrak{t}_{\mathfrak{h}} \oplus \mathfrak{t}_{\mathfrak{q}}$ is the most-σ-split Cartan of $\mathfrak{k}$. By Helgason's work, $\widehat{K}_{K\cap H}$ is parametrized by their highest weights in $(\widehat{T/T \cap H})_{\mathrm{dom}}$ where $T = \exp(\mathfrak{t}_{\mathfrak{h}} \oplus \mathfrak{t}_{\mathfrak{q}})$. The set $\widehat{K}_{K\cap H,\mathrm{reg}}$ corresponds to strictly dominant characters.

C^*-algebra on symmetric spaces of type $G_{\mathbb{C}}/G_{\mathbb{R}}$

For any symmetric pair (G, H) defined by involution σ, it has a companion (G, H') defined by involution $\theta\sigma$.

If (G, H) is of type $G_{\mathbb{C}}/G_{\mathbb{R}}$, its companion is $(K_{\mathbb{C}}, (K \cap H)_{\mathbb{C}})$. "$G = K_{\mathbb{C}}$".

The condition of the previous theorem is that the latter is tempered. In this case, the associated C^*-algebras have similar decompositions:

$$K_*(C^*_{\lambda_{G_{\mathbb{C}}/G_{\mathbb{R}}}}(G)) \cong \begin{cases} \bigoplus_{\widehat{K}_{K\cap H,\mathrm{reg}}} \mathbb{Z} & * = \dim(\mathfrak{t}_{\mathfrak{h}}) \\ 0 & * \neq \dim(\mathfrak{t}_{\mathfrak{h}}) \end{cases}$$

$$K_*(C^*_{\lambda_{K_{\mathbb{C}}/(K\cap H)_{\mathbb{C}}}}(G)) \cong \begin{cases} \bigoplus_{\widehat{K}_{K\cap H,\mathrm{reg}}} \mathbb{Z} & * = \dim(\mathfrak{t}_{\mathfrak{q}}) \\ 0 & * \neq \dim(\mathfrak{t}_{\mathfrak{q}}) \end{cases}$$

where $\mathfrak{t}_{\mathfrak{h}} \oplus \mathfrak{t}_{\mathfrak{q}}$ is the most-σ-split Cartan of $\mathfrak{k}$.

This "duality" does not hold without the tempered assumption.
Example: $(G, H) = (K_{\mathbb{C}}, K)$ where $\sigma = \theta$. Then, the companion is $(K_{\mathbb{C}}, K_{\mathbb{C}})$ with trivial involution. They have different K-theory: $\mathbb{Z}$ for $(K_{\mathbb{C}}, K_{\mathbb{C}})$ and zero for $(K_{\mathbb{C}}, K)$ (unless K is abelian).

Thank you for your time listening!