<u>Quasi-homomorphisms</u>

Let A, B be rings. A <u>quasi-homomorphism</u> from A to B consists of:

(1) An ideal embedding $B \lhd \mathcal{E}$.

(2) Two homomorphisms $\varphi_0, \varphi_1 : A \longrightarrow \mathcal{E}$ such that

$$(\forall a \in A) \quad \varphi_0(a) - \varphi_1(a) \in B.$$

Notation: $(\varphi_0, \varphi_1) : A \rightrightarrows \mathcal{E} \rhd B$.

<u>obs</u>: $A = \mathbb{Z}\{x_1, \dots, x_n\}$; $\varphi_0(x_i) - \varphi_1(x_i) \in B \Longrightarrow$ true $\forall a \in A$. Follows from $(\varphi_0 - \varphi_1)(XY) = \varphi_0(X)(\varphi_0 - \varphi_1)(Y) + (\varphi_0 - \varphi_1)(X)\varphi_1(Y)$.

<u>Ex</u>: $A \underset{\pi}{\longleftarrow} \mathcal{E} \rhd B$ with $\varphi_1, \varphi_0 : A \to \mathcal{E}$, $\pi \varphi_i = \mathrm{id}_A$. } <u>Special</u> quasi-homo.

$\Longrightarrow (\varphi_0 - \varphi_1)(a) \in B$.

Construction:

$$A \overset{\varphi_0}{\underset{\varphi_1}{\rightrightarrows}} \mathcal{E} \rhd B \quad \text{a quasi-homo.}$$

Associated Special quasi-homo.

$$A \overset{\varphi_0'}{\underset{\varphi_1'}{\rightrightarrows}} \mathcal{E}' = A \oplus B, \quad (a, b)(X, Y) = (aX, \varphi_1(a)Y + b\varphi_1(X) + bY)$$

$$= A \ltimes_{\varphi_1} B. \quad \cancel{= aX, Y}$$

$$a \xrightarrow{\varphi_1'} (a, 0)$$

$$a \xrightarrow{\varphi_0'} (a, \varphi_0(a) - \varphi_1(a)).$$

$F : \text{Rings} \longrightarrow \text{Ab}$. split-exact.

$A \underset{\varphi_1}{\overset{\varphi_0}{\rightrightarrows}} \mathcal{E} \rhd B$ special quasi-homo.

$$\begin{tikzcd}
0 & F(A) \arrow[l] \arrow[r, bend left, "F(\varphi_0)"] \arrow[r, bend right, "F(\varphi_1)"'] \arrow[dr, bend right, "F(\varphi_0)"] \arrow[dr, bend right=50, "F(\varphi_1)"'] & F(\mathcal{E}) \arrow[l, "F(p)"'] \arrow[d] & F(B) \arrow[l, "F(\iota)"'] & 0 \arrow[l] \\
 & & F(A) \oplus F(B) & &
\end{tikzcd}$$

$(x, 0)$

$F(\varphi_0) - F(\varphi_1) : F(A) \longrightarrow F(B)$
$\shortparallel$
$F(\varphi_0, \varphi_1)$.

If the qh is not special, apply to associated special qh.

(0) $\begin{matrix} A \\ \uparrow f \\ A' \end{matrix} \underset{\varphi_1}{\overset{\varphi_0}{\rightrightarrows}} \mathcal{E} \rhd B \Rightarrow F(\varphi_0 f, \varphi_1 f) = F(\varphi_0, \varphi_1) F(f)$.

Properties:

(1). $$\begin{tikzcd}
A \arrow[r, shift left, "\varphi_0"] \arrow[r, shift right, "\varphi_1"'] \arrow[d, "qh"'] \arrow[dr, bend left=10, "\Psi\varphi_0"] \arrow[dr, bend right=10, "\Psi\varphi_1"'] & \mathcal{E}_1 \rhd B_1 \ qh \arrow[d, "\Psi"] \\
 & \mathcal{E}_2 \rhd B_2
\end{tikzcd}$$
$\Psi|_{B_1}$

$\Rightarrow (\Psi\varphi_0, \Psi\varphi_1)$ is qh and

$\Rightarrow F(\Psi|_{B_1}) \circ F(\varphi_0, \varphi_1) = F(\Psi\varphi_0, \Psi\varphi_1)$.

(2) (φ_0, φ_1) qh $\Rightarrow (\varphi_1, \varphi_0)$ qh & $F(\varphi_1, \varphi_0) = -F(\varphi_0, \varphi_1)$.

(3) $(\varphi_0, \varphi_1), (\varphi_1, \varphi_2) : A \rightrightarrows \mathcal{E} \rhd B$. qh $\Rightarrow (\varphi_0, \varphi_2)$ qh.

& $F(\varphi_0, \varphi_2) = F(\varphi_0, \varphi_1) + F(\varphi_1, \varphi_2)$

(4) $f_0, f_1 : A \to B$ homos $\Rightarrow$ $F(f_0) - F(f_1) = F(f_0, f_1)$

$\Rightarrow (f_0, f_1) : A \rightrightarrows B \triangleright B$
qh

~~(5)~~

Def: $f, g : A \to B$ homos, $f \perp g$ if $f(a_1)g(a_2) = g(a_1)f(a_2) = 0$
$(\forall a_1, a_2 \in A)$

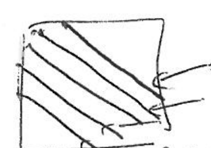
if $f \perp g \Rightarrow (f+g)(a_1)(f+g)(a_2) = f(a_1)f(a_2) + g(a_1)g(a_2)$.
$= (f+g)(a_1 a_2)$.

(5) If $(\varphi_0, \varphi_1), (\psi_0, \psi_1) : A \rightrightarrows E \triangleright B$ are qh with

$\varphi_0 \perp \psi_0$, $\varphi_1 \perp \psi_1 \Rightarrow F(\varphi_0 + \psi_0, \varphi_1 + \psi_1) = F(\varphi_0, \varphi_1) +$
$+ F(\psi_0, \psi_1)$

— o —

Back to FT.

$j(\Gamma) = M_\infty +$  constant along these.

$$\langle S, S^* \rangle = \Gamma$$

$$\begin{tikzcd} M_\infty \triangleleft \Gamma \arrow[r, "p"] & \mathbb{Z}[t, t^{-1}] \arrow[d, "\varepsilon"] \\ \mathbb{Z} \arrow[u, "j"] \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

$j(m) = \begin{pmatrix} m & & & \\ & m & & \\ & & m & \\ & & & \ddots \end{pmatrix}$

Consider $\xi : \Gamma \to \Gamma$ $\quad S \mapsto S^2 S^*$
$\quad S^* \mapsto S S^{*2}$.

intuitively $\xi \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$s - \xi(s) = e_{2,1}$
$s^* - \xi(s^*) = e_{2,1}^* = e_{1,2}$. $\quad \in M_\infty \Rightarrow (\mathrm{id},\xi) : \mathcal{T} \rightrightarrows \mathcal{T} \rhd M_\infty$ is q.h.

~~In gene~~

observe $(\mathrm{id},\xi)\circ j = (j, \xi j) : \mathbb{Z} \rightrightarrows \mathcal{T} \rhd M_\infty$.

$$\xi j(n) = \begin{pmatrix} 0 & & & \\ & n & & \\ & & n & \\ & & & \ddots \end{pmatrix}, \quad j(n) = \begin{pmatrix} n & & & \\ & n & & \\ & & \ddots & \\ & & & n \end{pmatrix} = i + \xi j$$

Here $i : \mathbb{Z} \longrightarrow M_\infty \lhd \mathcal{T}$
$\quad n \longmapsto n e_{11}$

Thus
$$F(\mathrm{id},\xi)\,F(j) = F(j,\xi j) = F(\xi j + i, \xi j)$$
$$= F(\xi j, \xi j) + F(i,0).$$
$$= F(i)$$
iso, since F is M_∞-stable

$\Rightarrow F(i)^{-1} F(\mathrm{id},\xi)\, F(j) = \mathrm{id}$.

Must show $F(j)\, F(i)^{-1} F(\mathrm{id},\xi) = \mathrm{id}$.

consider
$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "i"] \arrow[d, "j"] & M_\infty \lhd \mathcal{T} \arrow[d, "M_\infty \otimes j"] & \mathcal{T} \arrow[l, shift right, "\mathrm{id}"'] \arrow[l, shift left, "\xi"] \arrow[d, "\mathrm{id}\otimes j = \mathcal{T}j"] \\
\mathcal{T} \arrow[r, "i \mathcal{T}"'] & M_\infty \mathcal{T} \lhd \mathcal{T}(\mathcal{T}) &
\end{tikzcd}$$

$$\Longrightarrow F(i_\tau \circ j) = F(M_\infty j \circ i)$$

$$\|$$

$$F(i_\tau)F(j) = F(M_\infty j)F(i).$$

$$\Downarrow$$

$$F(j) = F(i_\tau)^{-1} F(M_\infty j) F(i).$$

$$F(j)F(i)^{-1}F(id,\xi) = \; = F(i_\tau)^{-1} \underbrace{F(M_\infty j)F(id,\xi)}_{= F(\psi_0,\psi_1)}$$

Let $\psi_0 = \tau(j)$, $\psi_1 = \tau(j)\xi \Rightarrow (\psi_0,\psi_1) = \tau(j)\circ(id,\xi)$.

Have $\tau(j)\big|_{M_\infty} = M_\infty j$. $\Longrightarrow F(\psi_0,\psi_1) = F(M_\infty j)\, F(id,\xi)$.

~~Claim~~ To prove: $F(\psi_0,\psi_1) = F(i_\tau)$.

consider

$$\psi_{1/2} : \tau \longrightarrow \tau \otimes \tau = \tau(\tau).$$

$$s \longmapsto (1-ss^*)\otimes s + s^2 s^* \otimes 1.$$

$$s^* \longrightarrow (1-ss^*)\otimes s^* + ss^{*2}\otimes 1.$$

$$\psi_{1/2}(s) = \left(\begin{array}{c|cccc} s & 0 & 0 & - & \\ \hline 0 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ & & 1 & & \\ \cdot & & & 1 & \ddots \end{array}\right), \quad \psi_{1/2}(s^*) = \left(\begin{array}{c|cccc} s^* & 0 & 0 & & \\ \hline 0 & 0 & 1 & 0 & \\ 0 & & & 1 & \\ & & & & 1 \ddots \end{array}\right)$$

$$\psi_1(s) = \begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & 1 & & \\ & & 1 & \ddots \end{pmatrix} \Rightarrow \psi_1 \perp i_\tau, \quad \psi_{1/2} = i_\tau + \psi_1.$$

$$\Rightarrow \quad F(\varphi_0, \varphi_1) = F(\varphi_0, \varphi_{1/2}) + F(\varphi_{1/2}, \varphi_1).$$

$$= F(\varphi_0, \varphi_{1/2}) + F(i_{\mathcal{T}}).$$

Suffices: $\quad F(\varphi_0, \varphi_{1/2}) = 0$

$$\varphi_0(s) = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} = \underbrace{\left(\begin{array}{cc|c} 1-ss^* & -s & 0 \\ s^* & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)}_{u_1} \left(\begin{array}{cc|c} s & 0 & \\ 0 & 0 & \\ \hline 0 & 1 & \\ 0 & 0 & 1 \\ & & & \ddots \end{array}\right)$$

$$u_t = \left(\begin{array}{cc|c} 1-t^2 ss^* & -ts & 0 \\ ts^* & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \in \mathcal{T}(\mathcal{T}[t]).$$

$$v_t = \left(\begin{array}{cc|c} 1 & ts & 0 \\ -ts^* & 1-t^2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \qquad v_t u_t = 1.$$

$$H : \mathcal{T} \longrightarrow \mathcal{T}(\mathcal{T}(t)) \qquad \begin{aligned} H(s) &= u_t \, \varphi_{1/2}(s) \\ H(s^*) &= \varphi_{1/2}(s^*) v_t. \end{aligned}$$

$$ev_0 H = \varphi_{1/2}, \quad ev_1 H = \varphi_0.$$

Besides, $(H - \varphi_0)(\mathcal{T}) \subset M_\infty \mathcal{T}[t]$.

$\Rightarrow (H, \varphi_0) : \mathcal{T} \rightrightarrows \mathcal{T}^2[t] \triangleright M_\infty \mathcal{T}(t)$ is qh.

We have

$$ev_0 \circ (H, \Psi_0) = (\Psi_{1/2}, \Psi_0) \quad \Rightarrow F(\Psi_{1/2}, \Psi_0)$$

$$ev_1 \circ (H, \Psi_0) = (\Psi_0, \Psi_0). \quad \Rightarrow F(\Psi_{1/2}, \Psi_0) = F(\Psi_{1/2}, \Psi_0) = F(\Psi_0, \Psi_0) = 0.$$