$\boxed{\text{Fundamental Thm \& Bott periodicity}}$

$$A[t, t^{-1}] = \left\{ \sum_{i=m}^{M} a_i t^i \;:\; \cancel{m, M, a_i \in \mathbb{Z}}\; M, m \in \mathbb{Z},\ a_i \in A \right\}.$$

makes?

Makes sense even if $1 \notin A$ ($\Rightarrow t^n \notin A$).

$$= A \otimes \mathbb{Z}[t, t^{-1}].$$

$$\sigma \quad \cancel{\ker}\, \mathbb{Z}[t, t^{-1}] \xrightarrow{\varepsilon} \mathbb{Z})$$

$$t^{+/-} \longrightarrow 1.$$

$$\sum a_i t^i \longmapsto \sum a_i$$

$$\sigma A = \sigma \otimes A.$$

<u>Fundamental theorem:</u> $KH_n(\sigma A) = KH_{n-1} A.$

$(\Rightarrow KH_n(A[t, t^{-1}]) = KH_n A \oplus KH_{n-1} A)$

<u>Idea ~~Cun~~ (Cuntz)</u>

$$\mathcal{T} = \frac{\mathbb{Z}\{s, s^*\}}{\langle s^* s - 1 \rangle} \xrightarrow{p} \mathbb{Z}[t, t^{-1}] = \mathcal{T} / \langle [\mathcal{T}, \mathcal{T}] \rangle$$

$$s \longmapsto t$$

$$s^* \longrightarrow t^{-1}.$$

Linear (but not multiplicative) splitting. $\sum_{i \geq 0} a_i s^i + \sum_{i < 0} a_i s^{*-i} \longleftarrow\!\!| \sum a_i t^i$

$$\rho : \mathcal{T} \longrightarrow \Gamma$$

$$s \longmapsto \sum_{i=1}^{\infty} e_{i+1\,i} = \begin{bmatrix} 0 & 0 & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & 1 & \ddots & \\ & & & \ddots & \end{bmatrix}$$

$$s^{*} \longrightarrow \sum_{i=1}^{\infty} e_{i,i+1} = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & \end{bmatrix}$$

<u>Lemma</u>: ρ is injective and maps.

$$\operatorname{Ker} p \xrightarrow{\sim} M_\infty$$

$$1 - ss^{*} \longmapsto e_{11}.$$

$$s^{i-1}(1-ss^{*})s^{*\,j-1} \longmapsto e_{ij} \qquad i,j \geq 1.$$

We shall identify $\mathcal{T} = \rho(\mathcal{T})$.

Have diagram with linearly split exact rows.

$$\begin{tikzcd}
0 \arrow[r] & M_\infty \arrow[r] \arrow[d, equal] & \mathcal{T}_0 \arrow[r] \arrow[d] & \sigma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_\infty \arrow[r] & \mathcal{T} \arrow[r, "p"] \arrow[d, "\varepsilon p"] & \mathbb{Z}[t,t^{-1}] \arrow[d, "\varepsilon"] & \\
 & & \mathbb{Z} \arrow[r, equal] \arrow[u, bend left, "j"] & \mathbb{Z} &
\end{tikzcd}$$

Tensoring with A we still get exact diagram. $\mathcal{I}A$, $\mathcal{I}_0A$, etc.

To prove FT, it suffices to prove.

$$KH_*(\mathcal{I}_0 A) = 0.$$

Because vertical sequence in the middle is <u>split-exact</u> it suffices to show

$(\forall A)$ $KH_*(j_A)$ is an iso. $j_A = j \otimes id_A$

<u>Thm</u>: Let $F : \text{Rings} \to \text{Ab}$ be $\begin{cases} \text{split-exact} \\ M_\infty\text{-stable} \\ (\text{Polynomially htpy inv} \end{cases}$

$\Rightarrow$ $F(j_A)$ iso.

<u>Obs</u>: F as in thm $\Rightarrow$ $G(B) = F(B \otimes A)$ is still as in thm. So it suffices to prove the thm for $A = \mathbb{Z}$.

<u>Comment</u>: $\mathcal{I}\mathbb{C} \subset \Gamma\mathbb{C} \subset B(\ell^2(\mathbb{N}, \mathbb{C}))$

$M_\infty\mathbb{C} \triangleleft \mathcal{I}\mathbb{C}$, $\quad \mathcal{I}\mathbb{C} \to \overline{\mathcal{I}\mathbb{C}} \subset$

$$\overline{M_\infty\mathbb{C}} = \mathcal{K} \triangleleft \mathcal{I}^{\mathbb{C}} = C^*(s, s^*s = 1) \twoheadrightarrow C^*(s, s^*s=1, ss^*=1) = C(S^1) = \mathbb{C}[t, t^{-1}]$$

compact ops.   <u>coburn</u>

splitting extends to continuous linear splitting

$$C_0(S^1) = \mathrm{Ker}\,(C(S^1) \xrightarrow{ev_1} \mathbb{C}) = \Sigma\mathbb{C}$$

~~Thm (Cuntz) $F: C^*\text{-Alg} \longrightarrow ab$~~

$F: C^*\text{-Alg} \longrightarrow Ab$

F is $\mathcal{K}$-stable if $F(a \longrightarrow a \otimes \mathcal{K} = M_\infty a)$ iso, $a \mapsto a \otimes e$

$\uparrow$ C^*-algebra $\otimes$

Thm (Cuntz) $F: C^*\text{-Alg} \longrightarrow ab$ — $\mathcal{K}$-stable, split-exact, HTPY-inv

$\Longrightarrow F(\mathbb{C} \xrightarrow{j} Z^{\mathbb{C}})$ is iso.

Coro: $K^{top}_{n+1}(a) = K^{top}_{n-1}(a \hat{\otimes} \mathcal{K})$.

Thm (Karoubi) B a Banach algebra, $A \subset B$ a subalg.

- closed under holomorphic calculus
- dense.

$\Big\} \Longrightarrow K^{top}_0 A \xrightarrow{\sim} K^{top}_0 B$.

~~If A is itself Banach (with~~ $\Longrightarrow A(0,1)$ satisfies the same and $K^{top}_1 A \xrightarrow{\sim} K^{top}_1 B$ iso.

Apply Karoubi to $A = M_\infty a(0,1)^n$ $\qquad K_0(a(0,1)^{n+1}) = K_0(a(0,1)^{n+1} \hat{\otimes} \mathcal{K})$

$\cap$ $\qquad K^{top}_{n+1} a =$

$\mathcal{K} \hat{\otimes} a(0,1)^n$ $\qquad$ Karoubi $\rightarrow$ $\Vert$

$K^{top}_{n-1} a = K_0(a(0,1)^{n-1})$