

Fundamental Theorem & Bott periodicity

①

$$\mathbb{A}[t, t^{-1}] = \left\{ \sum_{i=-m}^m a_i t^i : m, m' \in \mathbb{Z}, a_i \in A \right\}$$

↑  
makes sense even if  $1 \notin A$   
( $\Rightarrow t^m \notin A$ )

↑  
makes sense

$$= A \otimes \mathbb{Z}[t, t^{-1}]$$

$$\sigma : \mathbb{Z}[t, t^{-1}] \xrightarrow{\cong} \mathbb{Z}$$

$$t^{\pm 1} \mapsto 1$$

$$\sum a_i t^i \mapsto \sum a_i$$

$$\sigma A = \sigma \otimes A$$

Fundamental theorem:  $KH_n(\sigma A) = KH_{n-1} A$

$$\Rightarrow KH_n(A[t, t^{-1}]) = KH_n A \oplus KH_{n-1} A$$

Idea Proof (Cuntz)

$$\mathcal{Z} = \frac{\mathbb{Z} \langle s, s^* \rangle}{\langle s^* s - 1 \rangle} \xrightarrow{P} \mathbb{Z}[t, t^{-1}] = \mathcal{Z} / \langle [2, 2] \rangle$$

$$s \mapsto t$$

$$s^* \mapsto t^{-1}$$

Linear  
(but not  
multiplicative)

splitting

$$\sum_{i \geq 0} a_i s^i + \sum_{i < 0} a_i s^{*-i} \longleftarrow \sum a_i t^i$$

$$p: \mathbb{Z} \longrightarrow \mathbb{Z}$$

$$s \longmapsto \sum_{i=1}^{\infty} e_{i+1,i} = \begin{bmatrix} 0 & 0 & & & \\ 1 & 0 & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \end{bmatrix}$$

$$s^* \longmapsto \sum_{i=1}^{\infty} e_{i,i+1} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix}$$

Lemma:  $p$  is injective and maps

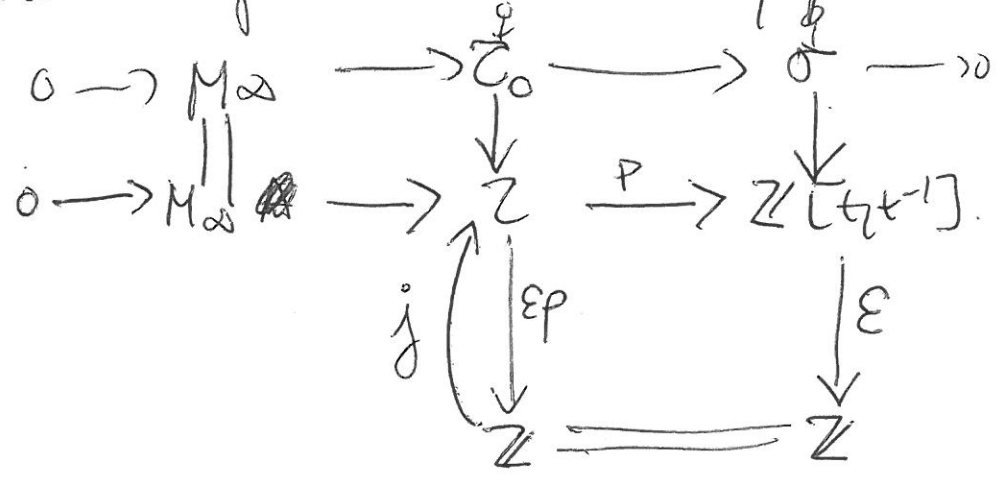
$$\text{Ker } p \xrightarrow{\sim} M_{\infty}$$

$$1 - ss^* \longmapsto e_{11}$$

$$s^{i-1}(1 - ss^*)s^{j-1} \longmapsto e_{ij} \quad i, j \geq 1$$

We shall identify  $\mathbb{Z} = p(\mathbb{Z})$ .

Have diagram with linearly split exact rows.



Reasoning with  $A$  we still get exact diagram.  $\mathbb{Z}A, \mathbb{Z}_0A$ , etc. (3)

To prove FT, it suffices to prove.

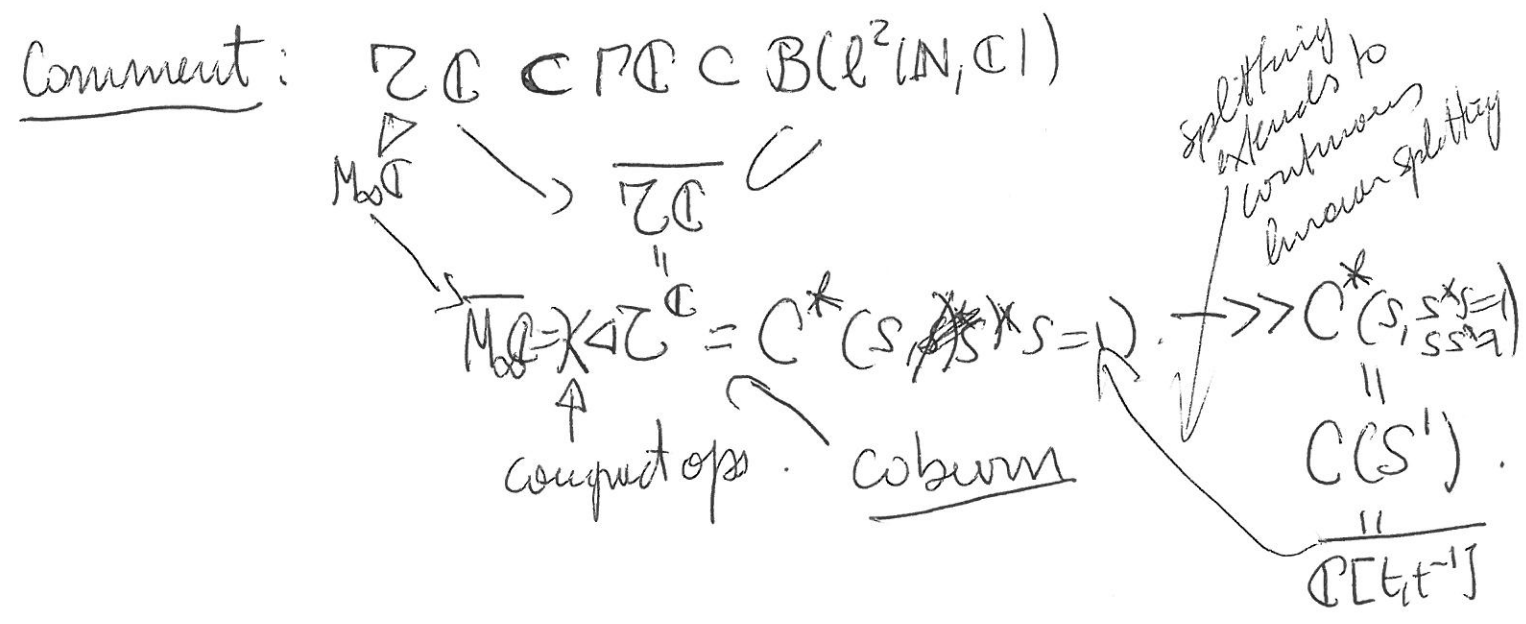
$$KH_* (\mathbb{Z}_0 A) = 0.$$

Because vertical sequence in the middle is split-exact.  
 It suffices to show

$$(\forall A) \quad KH_* \left( \begin{array}{c} \mathbb{Z}A \\ \downarrow \\ \mathbb{Z}_0A \end{array} \right) \text{ is an inv.} \quad \partial A = \text{id}_A$$

Thm: Let  $F: \text{Rings} \rightarrow \text{abs}$  be  $\left\{ \begin{array}{l} \text{split-exact} \\ M_\infty\text{-stable} \\ (\text{Polynomially htpy inv}) \end{array} \right.$   
 $\Rightarrow F(\partial A)$  inv.

obs:  $F$  as in thm  $\Rightarrow G(B) = F(B \otimes A)$   
 is still as in thm. So it suffices to prove the  
 thm for  $A = \mathbb{Z}$ .



$$C_0(S') = \text{Ker}(C(S') \xrightarrow{\omega_1} \mathbb{C}) = \overline{\sigma C}$$

Thm (Cuntz)  $F: C^* \text{-alg} \rightarrow ab$   
 $P: C^* \text{-alg} \rightarrow ab$   $a \rightarrow a \otimes 1$   
 $F$  is  $\kappa$ -stable of  $F(A \rightarrow A \otimes K = M_n A)$  iso  
 $C^* \text{-algebra} \otimes$

Thm (Cuntz)  $F: C^* \text{-alg} \rightarrow ab$   $\begin{cases} \kappa\text{-stable} \\ \text{split-exact} \\ \text{HPV-inv} \end{cases}$

$\Rightarrow F(\mathbb{C} \xrightarrow{f} \mathbb{Z}^n)$  is iso.

Corollary:  $K_{m+1}^{top}(A) = K_m^{top}(A \otimes K)$

Thm (Kasubiri)  $B$  a Banach algebra,  $A \subset B$  a subalg.

- closed under holomorphic calculus
  - dense.
- $\Rightarrow K_{\otimes}^{top} A \xrightarrow{\sim} K_{\otimes}^{top} B$

If  $B$  is itself Banach  $\Rightarrow A(0,1)$  satisfies the same and  $K_1^{top} A \xrightarrow{\sim} K_1^{top} B$  iso.

Kasubiri  
 Apply to  $A = M_{\infty} A(0,1) \xrightarrow{in} K_{\otimes}^{top} A(0,1)^m \xrightarrow{K_{M^m}} K_{\otimes}^{top} A(0,1)^{m+1} = K_0(A(0,1)^{m+1}) = K_0(A(0,1) \otimes K)$   
 $K_{m+1}^{top} A = K_0(A(0,1)^{m+1})$   
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