Math 4430-Spring 2013
First Midterm Exam

- This is a closed book/closed notes exam. You may have one 3-by-5 note-card with formulas only on one side.
- You must show all your work. There will be no credit for answers, even correct ones, given without supporting work.
- No electronic equipment is allowed.

Name: Solutions

Problem	Max. points	Points
1	25	
2	25	
3	15	
4	35	
Total	100	

1. Find the general solution of each of the following equations.

(a) (10pts.) $y' = (y^2+4)t^3$.

$$t^3 = y'/_{y^2+4} \Rightarrow \frac{t^4}{4} + c = \int \frac{y'dt}{y^2+4} = \int \frac{dy}{y^2+4}$$

$$\| $$

For another c

$$\tan\left(\frac{t^4}{2}+c\right) = y/2 \Leftarrow \frac{1}{2}\arctan(y/2) = \frac{1}{2}\int \frac{d(y/2)}{(y/2)^2+1} = \frac{1}{4}\int \frac{dy}{(y/2)^2+1}$$

$$\Downarrow$$

$$\boxed{y = 2\tan(t^4/2 + c)}.$$

(b) (15 pts.) $(t+1)y' - y = (t+1)^2\sin(t)$. $\Rightarrow y' \underbrace{-\frac{1}{t+1}}_{=p} y = \underbrace{(t+1)\sin(t)}_{=q}$

Integ. factor $\mu = e^{\int p\,dt} = e^{-\int \frac{dt}{t+1}} = 1/_{t+1}$.

$$\left(y/_{t+1}\right)' = y'/_{t+1} - \frac{1}{(t+1)^2}y = \sin(t).$$

$$\Downarrow$$

$$\frac{y}{(t+1)} = -\cos(t) + c \Rightarrow \boxed{y = (t+1)(c - \cos(t))}$$

2. Find the general solution of each of the following equations.

(a) (10pts.) $t^2y' = y^2 + ty + t^2$

$$\Updownarrow$$

$$y' = (y/t)^2 + (y/t) + 1 \quad (*)$$

Set $u = y/t$. Then $(*)$ becomes:

$$u + tu' = (tu)' = u^2 + u + 1$$

$$\Updownarrow$$

$$tu' = u^2 + 1 \Longleftrightarrow \frac{u'}{u^2+1} = 1/t$$

$$\Downarrow$$

$$\arctan(u) = \int \frac{du}{u^2+1} = \int \frac{u'dt}{u^2+1} = \int \frac{dt}{t} + C = \ln|t| + C.$$

$$\Downarrow$$

$$u = \tan(\ln|t| + C)$$

$$\Downarrow$$

$$\boxed{y = t\tan(\ln|t| + C)}$$

(b) (15pts) $\underbrace{2ty-1}_{=\,p}+y'(\underbrace{2(1+t^2)-t/y}_{=\,q})=0$

$$\frac{\partial p}{\partial y}-\frac{\partial q}{\partial t}=2t-4t+1/y=-2t+1/y=-\frac{(2ty-1)}{y}=-\frac{p}{y}$$

$$\Rightarrow \frac{\frac{\partial p}{\partial y}-\frac{\partial q}{\partial t}}{p}=-1/y \text{ depends ONLY on } y \Rightarrow$$ Integrating factor $\mu=e^{\int\frac{dy}{y}}=y.$

$$\Rightarrow \underbrace{(2ty-1)y}_{=\,P}+y'\,\underbrace{(2y(1+t^2)-t)}_{=\,Q}=0 \text{ is EXACT.}$$

$$\Phi(t,y)=\int Q\,dy+\psi(t)=y^2(1+t^2)-ty+\psi(t).$$

$$\frac{\partial\Phi}{\partial t}=\underbrace{2y^2t-y}+\psi'(t)=P\Rightarrow \text{take } \psi=0$$

$$\Downarrow$$

$$\Phi(t,y)=y^2(1+t^2)-ty$$

$$\boxed{y^2(1+t^2)-ty=C}$$

3. (a) (5pts.) Verify that e^t is a solution of the equation

$$ty" - (t+2)y' + 2y = 0.$$

$(e^t)' = e^t$, $(e^t)'' = e^t \Rightarrow t(e^t)'' - (t+2)(e^t)' + 2e^t$

$$\| $$

$e^t((t+2)-(t+2)) = e^t(t-(t+2)+2)$

$$\| $$

$e^t \cdot 0 = 0.$

(b) (15pts.) Find the general solution of the equation

$$ty" - (t+2)y' + 2y = 0.$$

$$\Updownarrow$$

$$y'' \underbrace{-(1+2/t)}_{= p} y' + \underbrace{(2/t)}_{q} y = 0$$

$y_2 = v y_1$, $v' = \dfrac{e^{-\int p\,dt}}{y_1^2} = \dfrac{e^{\int (1+2/t)dt}}{e^{2t}}$
$= ve^t$.

$$\dfrac{e^{\int (1+2/t)dt}}{e^{2t}} = \dfrac{e^t t^2}{e^{2t}} = t^2 e^{-t}.$$

$2\int te^{-t}dt - (t^2e^{-t}) = \int t^2 e^{-t} = v \Leftarrow \dfrac{e^t t^2}{e^{2t}}$

$$\| $$

$-(t^2e^{-t}) - 2te^{-t} + 2\int e^{-t}dt = -e^{-t}(t^2+2t+2)$

$$\Downarrow$$

General Solution $\boxed{y = c_1 e^t + c_2(t^2+2t+2)}$ $\quad Y_2 = -(t^2+2t+2)$.

4. (a) (10pts.) Solve the initial value problem

$$y" - y' - 2y = 0, \quad y(1) = 0, \quad y'(1) = 1.$$

$$p(r) = r^2 - r - 2 = (r+1)(r-2)$$

General solution

$$y = c_1 e^{-(t-1)} + c_2 e^{2(t-1)}$$

$$0 = y(1) = c_1 + c_2 .$$

$$1 = y'(1) = -c_1 + 2c_2 .$$

$$\Downarrow$$

$$1 = 3c_2 \Rightarrow c_2 = 1/3 \Rightarrow c_1 = -1/3 .$$

$$\boxed{y = \frac{1}{3}\left(e^{2(t-1)} - e^{-(t-1)}\right)}$$

(b) (10pts.) Solve the initial value problem

$$y'' - 14y' + 49y = 0, \quad y(0) = 1, \quad y'(0) = 8.$$

$$p(r) = r^2 - 14r + 49.$$

$$= (r-7)^2.$$

General Solution $y = c_1 e^{7t} + c_2 t e^{7t}.$

$$\left.\begin{aligned} 1 = y(0) &= c_1 \\ 8 = y'(0) &= 7c_1 + c_2 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 1 \\ c_2 &= 1 \end{aligned}$$

$$\boxed{y = e^{7t} + t e^{7t}}$$

(c) (15pts.) Find a particular solution of the equation

$$y'' - y' - 6y = \cos(t). \quad (*)$$

$\cos(t) = \mathrm{Re}(e^{it})$. so if $z'' - z' - 6z = e^{it}$.

then $y = \mathrm{Re}(z)$ will be a solution to $(*)$.

Set $p(r) = r^2 - r^1 - 6$, $z = ue^{it}$.

Then

$$e^{it} = z'' - z' - 6z = \left(u'' + p'(i)u' + p(i)u\right)e^{it}.$$

$$\cancel{e^{it}} = \left(u'' + (2i-1)u' - (7+i)u\right)\cancel{e^{it}}$$

$$\Downarrow$$

$$1 = u'' + (2i-1)u' - (7+i)u$$

CHOOSE $u = -\frac{1}{7+i} = -\frac{(7-i)}{7^2+(1)^2} = \frac{i}{50} - \frac{7}{50}$

$$\Downarrow$$

$$z = \left(\frac{i}{50} - \frac{7}{50}\right)e^{it}$$

$$\|$$

$$\mathrm{Re}(z) = -\frac{1}{50}(7\cos t + \sin t) \Leftarrow \left(\frac{i}{50} - \frac{7}{50}\right)(\cos t + i\sin t)$$

$$\boxed{y = -\frac{1}{50}(7\cos t + \sin t)}$$