A NOTE ON GENERALIZED FUJII-WILSON CONDITIONS AND BMO SPACES

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Abstract. In this note we generalize the definition of Fujii-Wilson condition providing quantitative characterizations of some interesting classes of weights, such as A_∞, A_∞^{weak} and C_p, in terms of BMO type spaces suited to them. We will provide as well some self improvement properties for some of those generalized BMO spaces and some quantitative estimates for Bloom's BMO type spaces.

1. Introduction and main results

Given a weight v, namely, non-negative locally integrable function in $\mathbb{R}^n$, and a functional $Y : \mathcal{Q} \to (0,\infty)$ defined over the family of all cubes in $\mathbb{R}^n$ with sides parallel to the axes, we define the class of functions $\mathrm{BMO}_{v,Y}$ by

$$\mathrm{BMO}_{v,Y} = \{f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{\mathrm{BMO}_{v,Y}} < \infty\}$$

where

$$\|f\|_{\mathrm{BMO}_{v,Y}} := \sup_Q \frac{1}{Y(Q)} \int_Q |f - f_Q| v < \infty,$$

and, as usual, $f_Q = \frac{1}{|Q|}\int_Q f$ denotes the average of f over Q. In the case that $Y(Q) = v(Q)$ for every cube Q, a classical result due to Muckenhoupt and Wheeden in [MW76] asserts that

$$\mathrm{BMO} = \mathrm{BMO}_{v,v}$$

holds whenever $v \in A_\infty$. Also the case of $Y(Q) = w(Q)$ for some weight w and $v = 1$ was considered in [MW76] and independently by J. García-Cuerva [GC79] in the context of Hardy spaces[1]. Later on, S. Bloom's [Blo85] also considered this special case in the context of commutators and used the notation BMO_w to denote the space $\mathrm{BMO}_{1,w}$. We remit the reader to [GCHST91, HLW17, LORR17, LORR18, Hyt, AMPRR] for the latest advances and related results in that direction. The unweighted case $v = 1$ and $Y(Q) = |Q|$ corresponds, obviously, to the classical BMO space of John-Nirenberg [JN61] and in that case we shall drop the subscripts.

More recently it was established that $I : \mathrm{BMO} \hookrightarrow \mathrm{BMO}_{w,w}$ is bounded with norm at most $c_n[w]_{A_\infty}$, namely

$$\|f\|_{\mathrm{BMO}_{w,w}} \le c_n\, [w]_{A_\infty}\, \|f\|_{\mathrm{BMO}} \tag{1.1}$$

1991 *Mathematics Subject Classification.* Primary: 42B25. Secondary: 43A85.

Key words and phrases. Muckenhoupt weights, BMO.

C. P. This work was supported by the Spanish Ministry of Economy and Competitiveness, MTM2017-82160-C2-2-P and SEV-2017-0718.

E. R. is partially supported by grants UBACyT 20020170200057BA and PICT-2015-3675.

S. O. and I. P. R.-R. are supported by grant PIP (CONICET) 11220130100329CO.

[1]Results involving that space appear in an abstract of that author in Notices of the AMS Feb 1974 p.A-309.

where $[w]_{A_\infty}$ denotes the Fujii-Wilson constant defined by

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q), \tag{1.2}$$

and the dependence on $[w]_{A_\infty}$ is sharp. Definition (1.2) was introduced in [HP13] where estimate (1.1) was obtained combining the classical John-Nirenberg theorem and the optimal reverse Hölder inequality obtained as well in [HP13], namely, if $w \in A_\infty$ and if

$$r(w) := 1 + \frac{1}{\tau_n\, [w]_{A_\infty}}$$

then,

$$\left(\fint_Q w^{r(w)}\right)^{\frac{1}{r(w)}} \le 2 \fint_Q w. \tag{1.3}$$

with τ_n a dimensional constant that we may take to be $\tau_n \approx 2^n$.

Up until now it was not known whether (1.1) would still be true for general weights, in other words, whether the emdedding $I : \mathrm{BMO} \hookrightarrow \mathrm{BMO}_{w,w}$ would hold for weights beyond the A_∞ class. Our first result gives a negative answer to that question providing a new quantitative characterization of A_∞ in terms of $[w]_{A_\infty}$ and the BMO class. We recommend [DMRO16] for a detailed account on characterizations of A_∞. Actually, we are going to present a more general result that encloses the aforementioned results as particular cases. For that purpose we introduce here the following variation of the Fujii-Wilson A_∞ constant (1.2).

Definition 1.1. Let v be a weight and $Y : \mathcal{Q} \to (0,\infty)$ a functional defined over the family of all cubes in $\mathbb{R}^n$. We define

$$[v]_{A_{\infty,Y}} = \sup_Q \frac{1}{Y(Q)} \int_Q M(v\chi_Q). \tag{1.4}$$

When the supremum above is finite, we say that $v \in A_{\infty,Y}$.

We include here some examples that motivate this definition and that will be used later on.

(1) $Y(Q) := w(Q)$. This corresponds to the A_∞ class of weights. See Section 2.1 for a more detailed discussion on this subject.

(2) $Y(Q) := w(2Q)$. Associated to this functional is the so called weak A_∞ class of weights and will be treated on Section 2.2.

(3) $Y(Q) := \displaystyle\int_{\mathbb{R}^n} M(\chi_Q)^p w$. Associated to the C_p class of weights, see Section 2.3.

(4) $Y(Q) = w_r(Q) := |Q| \left(\dfrac{1}{|Q|}\displaystyle\int_Q w^r dx\right)^{1/r}$, $1 < r < \infty$. Also related to the A_∞ class which produces more precise estimates as shown in Corollary 2.1.

Theorem 1.2. *Let v be a weight. There exist some dimensional constants c_n, C_n independent of v such that*

$$c_n\, [v]_{A_{\infty,Y}} \le \sup_{b:\|b\|_{\mathrm{BMO}}=1} \|b\|_{\mathrm{BMO}_{v,Y}} \le C_n\, [v]_{A_{\infty,Y}}. \tag{1.5}$$

In other words, we have that

$$[v]_{A_{\infty,Y}} \approx \sup_{b:\|b\|_{\mathrm{BMO}}=1} \sup_Q \frac{1}{Y(Q)} \int_Q |b(x) - b_Q|\, v dx$$

We will present some interesting corollaries of this result in Section 2. We observe that no condition is assumed on the functional $Y : \mathcal{Q} \to (0,\infty)$ nor on the weight in the theorem above. However, for the next theorem we need to restrict ourselves to a special class of functionals Y. Before introducing this special class of functionals we recall the notion of L-smallness that was introduced in [PR] within the context of generalized Poincaré inequalities.

Definition 1.3. We say that a family of pairwise disjoint subcubes $\{Q_i\}$ contained in a cube Q is L-small if

$$\sum_i |Q_i| \leq \frac{|Q|}{L}. \tag{1.6}$$

In that case we say that $\{Q_i\} \in S_Q(L)$ or if the cube Q is clear by the context that $\{Q_i\} \in S(L)$.

This condition arises tipically when considering the Calderón-Zygmund decomposition of the level sets of a non-negative function f in a given cube Q at level $L > 1$ under the assumption $\fint_Q f = 1$.

We define now the following class of functionals.

Definition 1.4. We say that a functional $Y \in \mathcal{Y}_q$ for $q > 1$ if there exists $c > 0$ such that for any cube Q and any family Λ of pairwise disjoint subcubes of Q with $\Lambda \in S(L)$, the following inequality holds

$$\sum_{P\in\Lambda} Y(P) \leq c\, Y(Q) \left(\frac{1}{L}\right)^{\frac{1}{q}}, \tag{1.7}$$

and we will denote by β_Y the smallest of the constants c.

Remark 1.5. Our model example in this class is given by

$$Y(Q) = w_r(Q) := \left(\frac{1}{|Q|}\int_Q w^r\,dx\right)^{1/r} |Q| = \left(\int_Q w^r\,dx\right)^{1/r} |Q|^{1/r'} \tag{1.8}$$

where $r > 1$. Hölder's inequality yields (1.7) with constant $\beta_Y \leq 1$ and exponent $q = r'$.

We can now state the following theorem.

Theorem 1.6. *Let w be a weight and let $Y \in \mathcal{Y}_q$, $q > 1$, namely a functional satisfying (1.7). Suppose further that there is a constant c such that for any cube Q*

$$w(Q) \leq Y(Q).$$

Then if $f \in \mathrm{BMO}$, there is dimensional constant c such that for each cube Q,

$$\left(\frac{1}{Y(Q)}\int_Q |f - f_Q|^p\, w dx\right)^{1/p} \leq c\, p\, q\, \beta_Y \|f\|_{\mathrm{BMO}}$$

We will show some consequences of this result in Section 2. Our next result can be seen as an update version of the work [MW76] which is related to the BMO_w classes following our notation. Being more precise we obtain quantitative versions of the results in [MW76].

Theorem 1.7. *Let $b \in \mathrm{BMO}_{1,w}$, namely $\sup_Q \frac{1}{w(Q)}\int_Q |f - f_Q| < \infty$.*

(1) *If $w \in A_1$ we have that for every $q > 1$,*

$$\left(\frac{1}{w(Q)}\int_Q \left|\frac{b(x) - b_Q}{w}\right|^q w(x)dx\right)^{\frac{1}{q}} \leq c_n \|b\|_{\mathrm{BMO}_{1,w}} q [w]_{A_1}^{\frac{1}{q'}} [w]_{A_\infty}^{\frac{1}{q}},$$

and hence for any cube Q

$$\left\| \frac{f-f_Q}{w} \right\|_{\exp L(Q,w)} \leq c\, [w]_{A_1} \, \|f\|_{\mathrm{BMO}_{1,w}}. \tag{1.9}$$

(2) *If $w \in A_p$ then,*

$$\left(\frac{1}{w(Q)} \int_Q \left| \frac{b(x)-b_Q}{w} \right|^{p'} w(x)dx \right)^{\frac{1}{p'}} \leq c_n p' \|b\|_{\mathrm{BMO}_{1,w}} [w]_{A_p}^{\frac{1}{p}} \, [w]_{A_\infty}^{\frac{1}{p'}}.$$

The remainder of the paper is organized as follows. Section 2 is devoted to provide consequences of some of the main results and in Section 3 we provide the proofs of the main results.

2. Some applications and consequences of the main results

2.1. The A_∞ class. Our first Corollary provides interesting information related to A_∞ weights. In particular we will provide a new characterization of the class via Fujii-Wilson constant.

Corollary 2.1. *Let w be a weight. Then*

$$[w]_{A_\infty} \approx \sup_{b:\|b\|_{\mathrm{BMO}}=1} \sup_Q \frac{1}{w(Q)} \int_Q |b(x)-b_Q| \, w(x)dx.$$

Our second corollary allows us to reprove known John-Nirenberg type estimates.

Corollary 2.2. *Let $w \in A_\infty$ and $f \in \mathrm{BMO}$. There exists a dimensional constant c_n independent of f and w such that for each cube Q,*

$$\left(\frac{1}{w(Q)} \int_Q |f-f_Q|^p \, wdx \right)^{1/p} \leq c_n \, p \, [w]_{A_\infty} \|f\|_{\mathrm{BMO}} \tag{2.1}$$

and hence

$$\|f-f_Q\|_{\exp L(Q,\frac{wdx}{w(Q)})} \leq C_n \, [w]_{A_\infty} \|f\|_{\mathrm{BMO}}. \tag{2.2}$$

For the proof of the latter we consider the functional $Y(Q) = w_r(Q) = \left(\frac{1}{|Q|} \int_Q w^r dx \right)^{1/r} |Q|$ which satisfies Definition 1.4 with constants $\beta_Y \leq 1$ and exponent $q = r'$ by Remark 1.5. Hence by Theorem 1.6

$$\left(\frac{1}{w_r(Q)} \int_Q |f-f_Q|^p \, wdx \right)^{1/p} \leq c_n \, p \, r' \, \|f\|_{\mathrm{BMO}}$$

Finally, if $w \in A_\infty$, the reverse Hölder inequality (1.3) with $r = r(w)$ so that $r' \approx [w]_{A_\infty}$, yields (2.1). Inequality (2.2) will be obtained from the following well known measure theory argument. Consider a probability space (X,μ) and a function g such that for some $p_0 \geq 1$, $c > 0$, and $\alpha > 0$ we have that

$$\|g\|_{L^p(X,\mu)} \leq c\, p^\alpha \qquad p \geq p_0.$$

Then for a universal multiple of c,

$$\|g\|_{\exp L^{\frac{1}{\alpha}}(X,\mu)} \leq c.$$

We conclude the section by mentioning that above argument cannot be so precise if we use the more natural functional $Y(Q) = w(Q)$ instead of the functional $Y(Q) = w_r(Q)$. Indeed, if we check the details we would get an exponential growth $e^{c[w]_{A_\infty}}$ both in (2.1) and in (2.2) instead of linear $c[w]_{A_\infty}$.

2.2. **The weak A_∞ class.** Besides the example of functional displayed in (1.8) another case of interest is the functional defined by the *weak* condition:

$$Y(Q) = w(2Q).$$

This is related to the condition introduced by E. Sawyer in [Saw82] by defining the "weak" A_∞ class as those weights satisfying the estimate

$$w(E) \le c \left(\frac{|E|}{|Q|}\right)^{\delta} w(2Q).$$

This class of weights is very interesting since appears in many contexts like the theory of quasiregular mappings or regularity fot solutions of elliptic PDE's (see, for example, [BI87]).

In [AHT17], the weak A_∞ class was characterized by means of a suitable Fujii-Wilson type A_∞ constant, namely

$$[w]_{A_\infty}^{weak} := \sup_Q \frac{1}{w(2Q)} \int_Q M(w\chi_Q).$$

Notice that the constant 2 in the average could be replaced by any parameter $\sigma > 1$ as shown in [AHT17]. It is also shown there that the following reverse Holder's inequality holds

$$\left(\fint_Q w^{r(w)}\right)^{\frac{1}{r(w)}} \le 2\fint_{2Q} w. \tag{2.3}$$

with $r(w) := 1 + \frac{1}{\tau_n [w]_{A_\infty}^{weak}}$ where τ_n is a dimensional constant that we may take to be $\tau_n \approx 2^n$.

Relying upon Theorems 1.2 and 1.6 and arguing as in the preceding section we obtain the following corollaries

Corollary 2.3. *Let w be weight. Then*

$$[w]_{A_\infty}^{weak} \approx \sup_{f:\|f\|_{\mathrm{BMO}}=1} \sup_Q \frac{1}{w(2Q)} \int_Q |f(x) - f_Q|\, w dx.$$

Corollary 2.4. *Let $w \in A_\infty^{weak}$, and $f \in \mathrm{BMO}$. There exists a dimensional constant c_n independent of f and w such that for each cube Q,*

$$\left(\frac{1}{w(2Q)} \int_Q |f - f_Q|^p\, w dx\right)^{1/p} \le c_n\, p[w]_{A_\infty}^{weak} \|f\|_{\mathrm{BMO}}$$

and hence

$$\|f - f_Q\|_{\exp L(Q, \frac{wdx}{w(2Q)})} \le C_n\, [w]_{A_\infty}^{weak} \|f\|_{\mathrm{BMO}}.$$

2.3. **The C_p class.** The C_p class is a class of weights containing the weak A_∞ class considered above and hence larger than the A_∞ class. It is a very interesting class of weights which is related intimately to the theory of singular integrals although very recently in [ABES] appeared some application to PDE. Indeed, it is a well known fact that $w \in A_\infty$ is a sufficient condition for the so called Coifman-Fefferman estimate, namely

$$\int_{\mathbb{R}^n} T^* f(x)^p w(x) dx \le c_{n,p,T,w} \int_{\mathbb{R}^n} Mf(x)^p w(x) dx \qquad 0 < p < \infty \tag{2.4}$$

where T^* is the maximal singular integral operator associated to the Calderón-Zygmund singular integral operator T.

Muckenhoupt [Muc81] proved that $w \in A_\infty$ is not necessary for (2.4) to hold. He showed in the case of the Hilbert transform that for $1 < p < \infty$, the correct necessary condition is that $w \in C_p$, namely, that for every cube Q and every measurable subset of $E \subset Q$,

$$w(E) \leq c\left(\frac{|E|}{|Q|}\right)^{\delta} \int_{\mathbb{R}^n} M(\chi_Q)^p w \tag{2.5}$$

Later on Sawyer [Saw83] showed that the C_p condition is "almost" sufficient in the sense that $w \in C_{p+\varepsilon}$ implies (2.4) for $1 < p < \infty$. A natural counterpart for the classical Fefferman-Stein estimate relating M and $M^\#$, the classical sharp maximal function, was provided by Yabuta [Yab90] and slightly improved by Lerner in [Ler10]. Recently, in [CLPR] Sawyer's result was extended to the full range $0 < p < \infty$ imposing as a sufficient condition that $w \in C_{\max\{1,p\}+\varepsilon}$ and including estimates for other operators as well. Similar estimates were settled too for the weak norm relying upon sparse domination. At this point, note that whether C_p is sufficient for (2.4) to hold remains an open question.

A fact that makes the C_p classes interesting is that weights in those classes are allowed to have "holes", namely to be zero in sets of no null measure, in some reasonable sense. Examples of those kind of weights can be found in [Muc81, Buc90].

Very recently Canto [Can18] settled a suitable quantitative reverse Hölder inequality for C_p classes in terms of the following Fujii-Wilson type constant

$$[w]_{C_p} := \sup_Q \frac{1}{\int_{\mathbb{R}^n} M(\chi_Q)^p w} \int_Q M(w\chi_Q).$$

Relying upon that reverse Hölder inequality it was also established in [Can18] that if $w \in C_q$ for $1 < p < q < \infty$ then

$$\|T^* f\|_{L^p(w)} \lesssim [w]_{C_q} \log\left(e + [w]_{C_q}\right) \|Mf\|_{L^p(w)}$$

Our next Corollary, which is again a direct consequence of Theorem 1.2, provides a new characterization of the C_p class.

Corollary 2.5. *Let w a weight and $p > 1$. We have that*

$$[w]_{C_p} \approx \sup_{f:\|f\|_{\mathrm{BMO}}=1} \sup_Q \frac{1}{\int_{\mathbb{R}^n} M(\chi_Q)^p w} \int_Q |f(x) - f_Q|\, w dx$$

3. Proofs of the main results

3.1. **Proof of Theorem 1.2.** We recall that a family of cubes $\mathcal{S}$ is η-sparse if for every $Q \in \mathcal{S}$ there exists a measurable subset $E_Q \subset Q$ such that

(1) $\eta|Q| \leq |E_Q|$.
(2) The sets E_Q are pairwise disjoint.

We will start by showing the second inequality from (1.5). We will rely upon a simplified version of [LORR17, Lemma 5.1] (see also [Hyt]).

Lemma 3.1. *Let Q a cube. There exists a sparse family $\mathcal{S} \subset \mathcal{D}(Q)$, where $\mathcal{D}(Q)$ stands for the dyadic grid relative to Q, such that*

$$|b(x) - b_Q|\chi_Q(x) \leq c_n \sum_{P\in\mathcal{S}} \frac{1}{|P|} \int_P |b(y) - b_P|\, dy \chi_P(x).$$

Armed with that lemma we can argue as follows:

$$\begin{aligned}\frac{1}{Y(Q)}\int_Q |b-b_Q|v &\leq c_n\frac{1}{Y(Q)}\int_Q\sum_P\left(\frac{1}{|P|}\int_P|b(y)-b_P|dy\right)\chi_P(x)v(x)dx\\ &\leq c_n\frac{1}{Y(Q)}\sum_P\left(\frac{1}{|P|}\int_P|b(y)-b_P|dy\right)v(P)\end{aligned}$$

Now, since $b\in BMO$, we have that

$$\begin{aligned}\frac{1}{Y(Q)}\int_Q |b-b_Q|v &\leq C_n\frac{1}{Y(Q)}\sum_P\left(\frac{1}{|P|}\int_P|b(y)-b_P|dy\right)v(P)\\ &\leq C_n\|b\|_{\mathrm{BMO}}\frac{1}{Y(Q)}\sum_P v(P)\\ &\lesssim \|b\|_{\mathrm{BMO}}\frac{1}{Y(Q)}\sum_{P\in\mathcal{S}}\frac{v(P)}{|P|}|E_P|\\ &\leq \|b\|_{\mathrm{BMO}}\frac{1}{Y(Q)}\sum_{P\in\mathcal{S}}\int_{E_P}M(v\chi_Q)\\ &\leq \|b\|_{\mathrm{BMO}}\frac{1}{Y(Q)}\int_Q M(v\chi_Q)\leq \|b\|_{\mathrm{BMO}}[v]_{A_{\infty,Y}}.\end{aligned}$$

This proves that

$$\|b\|_{\mathrm{BMO}_{v,Y}}\lesssim \|b\|_{\mathrm{BMO}}\,[v]_{A_{\infty,Y}}$$

For the second part of the proof, let us denote

$$X:=\sup_{\|b\|_{\mathrm{BMO}}=1}\|b\|_{\mathrm{BMO}_{v,Y}}.$$

Then it suffices to show that $[v]_{A_{\infty,Y}}\lesssim_n X$ and we shall assume that $X<\infty$ since otherwise there's nothing to prove. We note as well that from the definition of X it follows that

$$\|b\|_{\mathrm{BMO}_{v,Y}}\leq X\,\|b\|_{\mathrm{BMO}}. \tag{3.1}$$

Taking this into account, we claim first that for every cube Q

$$v(Q)\leq 4X\,Y(Q). \tag{3.2}$$

We may assume first that $v(Q)>0$, otherwise is trivial. Now, for a cube Q we let $\tilde{Q}\subset Q$ be another cube such that $\frac{1}{2}|Q|=|\tilde{Q}|$. Then $(\chi_{\tilde{Q}})_Q=\frac{1}{2}$ and we have that

$$\begin{aligned}\frac{v(Q)}{Y(Q)}&=\frac{2}{Y(Q)}\int_Q|\chi_{\tilde{Q}}-(\chi_{\tilde{Q}})_Q|v\leq 2\|\chi_{\tilde{Q}}\|_{\mathrm{BMO}_{v,Y}}\\ &\leq 2X\|\chi_{\tilde{Q}}\|_{\mathrm{BMO}}\leq 4X,\end{aligned}$$

from which the claim follows.

We now observe that to prove the theorem it suffices to show that there exists some finite constant $\alpha_n\geq 1$ such that for every cube Q, the inequality

$$\frac{1}{|Q|}\int_Q\log^+\left(\frac{v(x)}{\alpha_n^2 v_Q}\right)v(x)dx\ \lesssim_n X\,\frac{Y(Q)}{|Q|} \tag{3.3}$$

holds. Indeed, provided (3.3) holds, and taking into account (3.2),

$$\begin{aligned}\frac{1}{|Q|}\int_Q M(\chi_Q v)dx &\simeq \frac{1}{|Q|}\int_Q\left(1+\log^+\left(\frac{v(x)}{v_Q}\right)\right)v(x)dx\\ &= \frac{v(Q)}{|Q|}+\frac{1}{|Q|}\int_Q \log^+\left(\frac{\alpha_n^2 v(x)}{\alpha_n^2 v_Q}\right)v(x)dx\\ &\leq 3\frac{v(Q)}{|Q|}\log^+(\alpha_n)+\frac{1}{|Q|}\int_Q\log^+\left(\frac{v(x)}{\alpha_n^2 v_Q}\right)v(x)dx\\ &\leq 3\log^+(\alpha_n)\frac{v(Q)}{|Q|}+\nu_n X\frac{Y(Q)}{|Q|}\\ &\lesssim_n \frac{Y(Q)}{|Q|}X,\end{aligned}$$

and this yields

$$\frac{1}{Y(Q)}\int_Q M(\chi_Q v)dx \lesssim_n X.$$

Now, to prove (3.3) we note first that

$$\frac{1}{|Q|}\int_Q \log^+\left(\frac{v(x)}{\alpha_n^2 v_Q}\right)v(x)dx=\frac{1}{|Q|}\int_{L_Q}\log^+\left(\frac{v(x)}{\alpha_n^2 v_Q}\right)v(x)dx$$

where $L_Q=\{x\in Q : v(x)\geq \alpha_n^2 v_Q\}$ and α_n is to be chosen. Hence, settling (3.3) is equivalent to prove that

$$\frac{1}{|Q|}\int_{L_Q}\log^+\left(\frac{v(x)}{\alpha_n^2 v_Q}\right)v(x)dx \leq \gamma_n X\frac{Y(Q)}{|Q|} \tag{3.4}$$

for appropriate constants $\alpha_n,\gamma_n\geq 1$.

Now, recall that if $w\in A_1$ then $\log(w)\in \mathrm{BMO}$. Furthermore, tracking the dependence on the A_1 constant in [GCRdF85, Theorem 3.3 p. 157]

$$\|\log(w)\|_{\mathrm{BMO}}\leq 2\log([w]_{A_1}).$$

Hence, in particular, if we choose

$$w=M\left(\frac{v\chi_Q}{v_Q}\right)^{1/2},$$

it is well known (see for instance [GCRdF85, Theorem 3.4 p. 158]) that $w\in A_1$ with

$$[w]_{A_1}\leq d_n.$$

Hence, if we let

$$b=\log w$$

there exists a constant a dimensional constant ρ_n, i.e. independent of Q, such that

$$\|b\|_{\mathrm{BMO}}\leq \rho_n.$$

Observe that although b depends on Q its BMO constant is just dimensional. Combining this estimate with (3.1)

$$\frac{1}{Y(Q)}\int_Q |b-b_Q|\,v(x)dx\leq X\,\|b\|_{\mathrm{BMO}}\leq X\rho_n,$$

namely

$$\fint_Q |b-b_Q|\, v(x)dx \leq X\rho_n \frac{Y(Q)}{|Q|}.$$

In view of the preceding estimate, to settle (3.4), and hence ending the proof, it suffices to check that for every $x \in L_Q$ for $\alpha_n > 1$ to be chosen we have that

$$|b(x)-b_Q| \geq \frac{1}{2}\log^+\left(\frac{v(x)}{\alpha_n^2 v_Q}\right) \tag{3.5}$$

To verify this we observe first that combining Jensen's inequality and Kolmogorov's inequality with dimensional constant $c_n = 2\|M\|^{\frac{1}{2}}_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)}$ (see [Gra14, Ex. 2.1.5 p. 100]), we have

$$b_Q = \fint_Q \log w = \fint_Q \log\left(\frac{M(v\chi_Q)}{v_Q}\right)^{1/2} \leq \log\left[\fint_Q \left(\frac{M(v\chi_Q)}{v_Q}\right)^{1/2}\right] \leq \log c_n.$$

If we further assume that $x \in L_Q$, namely that $v(x) \geq \alpha_n^2 v_Q$, where α_n is yet to be chosen, we have

$$b_Q \leq \log c_n \leq \log\left(c_n \frac{v(x)^{1/2}}{\alpha_n (v_Q)^{1/2}}\right) \leq \log\left(\frac{M(v\chi_Q)(x)^{1/2}}{(v_Q)^{1/2}}\right) \leq \log w = b$$

choosing $\alpha_n = c_n$. Hence, for these $x \in L_Q$

$$\begin{aligned}|b(x)-b_Q| &= b(x)-b_Q \geq b(x) - \log c_n = \log\left(\frac{w(x)}{c_n}\right)\\ &= \frac{1}{2}\log\left[\frac{1}{c_n^2}\left(\frac{M(v\chi_Q)}{v_Q}\right)\right] \geq \frac{1}{2}\log\left[\frac{1}{c_n^2}\left(\frac{v(x)}{v_Q}\right)\right].\end{aligned}$$

This ends the proof of (3.5) and hence the proof of (3.4) with $\alpha_n = c_n$ and $\gamma_n = 2\rho_n$.

3.2. **Proof of Theorem 1.6.** In this section we will present two different approaches for Theorem 1.6. One of them is based in some new ideas from [PR] involving self improving properties for smallness preserving functionals related to generalized Poincaré inequalities. This was in fact inspired by the proof of John-Nirenberg's lemma given in [Jou83]. Relying upon this approach we will establish the following inequality

$$\left(\frac{1}{Y(Q)}\int_Q |f-f_Q|^p\, w dx\right)^{\frac{1}{p}} \leq c_n\, pq \max\left\{1, \beta_Y^q\right\} \|f\|_{\mathrm{BMO}}, \tag{3.6}$$

Even though that the estimate above doesn't provide the best dependence on β_Y we have included its proof for the sake of the interest of the approach used.

The other proof that we will present here can be seen as a very interesting application of the so called *sparse approach* for studying singular integrals. We remark that in this BMO type estimate this approach provides a linear bound in terms of β_Y, namely, we will prove that

$$\left(\frac{1}{Y(Q)}\int_Q |f-f_Q|^p\, w dx\right)^{\frac{1}{p}} \leq c_n\, pq\, \beta_Y \|f\|_{\mathrm{BMO}}.$$

Unfortunately, this method does not work so precisely in the general scenario of generalized Poincaré inequalities

3.2.1. *Proof based on the smallness property.* As we announced above, in this section we will settle (3.6). By homogeneity we may assume that $\|f\|_{\mathrm{BMO}} = 1$,

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq 1. \tag{3.7}$$

We may assume that f is bounded. Fixed one cube Q. We can consider the local Calderón-Zygmund decomposition of $|f - f_Q|$ relative to Q at level L on Q for a large universal constant $L > 1$ to be chosen. Let $\mathcal{D}(Q)$ be the family of dyadic subcubes of Q. The Calderón-Zygmund (C-Z) decomposition yields a collection $\{Q_j\}$ of cubes such that $Q_j \in \mathcal{D}(Q)$, maximal with respect to inclusion, satisfying

$$L < \frac{1}{|Q_j|}\int_{Q_j} |f - f_Q|\,dy. \tag{3.8}$$

Then, if P is dyadic with $P \supset Q_j$

$$\frac{1}{|P|}\int_P |f - f_Q|\,dy \leq L \tag{3.9}$$

and hence

$$L < \frac{1}{|Q_j|}\int_{Q_j} |f - f_Q|\,dy \leq L\,2^n \tag{3.10}$$

for each integer j. Also note that

$$\left\{x \in Q : M_Q^d\left(|f - f_Q|\chi_Q\right)(x) > L\right\} = \bigcup_j Q_j =: \Omega_L$$

where M_Q^d stands for the dyadic maximal function adapted to the cube Q. That is,

$$M_Q^d(f)(x) := \sup_{P \ni x} \fint |f(y)|\,dy \qquad x \in Q, P \in \mathcal{D}(Q)$$

Then, by the Lebesgue differentiation theorem it follows that

$$|f - f_Q| \leq L \qquad a.e.\ x \notin \Omega_L$$

Also, observe that by (3.8) (or the weak type $(1,1)$ property of M) and recalling our starting assumption (3.7), we have that $\{Q_i\} \in S(L)$, namely

$$|\Omega_L| = \left|\bigcup_j Q_j\right| < \frac{|Q|}{L}.$$

Now, given the C-Z decomposition of the cube Q, we perform the classical C-Z of the function $f - f_Q$ as

$$f - f_Q = g_Q + b_Q, \tag{3.11}$$

where the functions g_Q and b_Q are defined as usual. We have that

$$g_Q(x) = \begin{cases} f - f_Q, & x \notin \Omega_L \\ \fint_{Q_i}(f - f_Q), & x \in \Omega_L, x \in Q_i \end{cases} \tag{3.12}$$

Note that this definition makes sense since the cubes $\{Q_i\}$ are disjoint, so any $x \in \Omega_L$ belongs to only one Q_i. Also note that condition (3.10) implies that

$$g_Q(x) \leq 2^n L \tag{3.13}$$

for almost all $x \in Q$. The function b_Q is determined by this choice of g_Q as the difference

$$b_Q = f - f_Q - g_Q,$$

but we also have a representation as

$$b_Q(x) = \sum_i (f(x) - f_{Q_i})\, \chi_{Q_i}(x) = \sum_i b_{Q_i}, \tag{3.14}$$

where $b_{Q_i} = (f(x) - f_{Q_i})\chi_{Q_i}(x)$.

Now we start with the estimation of the desired L^p norm from (3.6). Consider on Q the measure μ defined by $d\mu = \frac{w}{Y(Q)}\chi_Q$. Then, by the triangle inequality, we have

$$\begin{aligned}\left(\frac{1}{Y(Q)}\int_Q |f - f_Q|^p\, w dx\right)^{\frac{1}{p}} &\leq \|g_Q\|_{L^p(\mu)} + \|b_Q\|_{L^p(\mu)} \\ &\leq 2^n L + \left(\frac{1}{Y(Q)}\int_{\Omega_L} \sum_j |b_{Q_j}|^p\, w dx\right)^{1/p}\end{aligned}$$

since we assume $w(Q) \leq Y(Q)$.

Let us observe that the last integral of the sum, by the localization properties of the functions b_{Q_i}, can be controlled:

$$\begin{aligned}\int_{\Omega_L} |\sum_j b_{Q_j}|^p\, w dx &\leq \sum_j \int_{Q_j} \left|b_{Q_j}\right|^p\, w dx \\ &= \sum_j \frac{Y(Q_j)}{Y(Q_j)}\int_{Q_j} \left|f - f_{Q_j}\right|^p\, w dx \\ &\leq X^p \sum_j Y(Q_j),\end{aligned}$$

where X is the quantity defined by

$$X = \sup_Q \left(\frac{1}{Y(Q)}\int_Q |f - f_Q|^p\, w dx\right)^{1/p},$$

which is finite since we are assuming that f is bounded. Then we obtain that

$$\begin{aligned}\left(\frac{1}{Y(Q)}\int_Q |f - f_Q|^p\, w dx\right)^{\frac{1}{p}} &\leq 2^n L + X\left(\frac{\sum_i Y(Q_i)}{Y(Q)}\right)^{1/p} \\ &\leq 2^n L + X\, (\frac{\beta_Y}{L^{1/q}})^{1/p}\end{aligned}$$

by the smallness preserving hypothesis. This holds for every cube Q, so taking the supremum we obtain

$$X \leq 2^n L + (\frac{\beta_Y}{L^{1/q}})^{1/p} X.$$

Now we choose $L = 2e \max\left\{\beta_Y^q, 1\right\}$ so the above inequality yields

$$X \leq 2^n 2e \max\left\{\beta_Y^q, 1\right\} \left((2e)^{1/pq}\right)' \leq e 2^{n+2}\, pq\, \max\left\{\beta_Y^q, 1\right\},$$

using that $\left((2e)^{1/s}\right)' \leq 2s$, $s > 1$. This is the desired inequality (3.6):

$$\left(\frac{1}{Y(Q)}\int_Q |f - f_Q|^p\, w dx\right)^{\frac{1}{p}} \leq c_n pq \max\left\{\beta_Y^q, 1\right\}.$$

3.2.2. *Proof based on the sparse approach.* First we need to prove that the smallness condition implies suitable sparse conditions for functionals.

Lemma 3.2. *Let $q>1$ and $Y\in\mathcal{Y}_q$ be a functional. Let $\mathcal{F}\subset\mathcal{D}(Q)$ be a family of cubes. If there exists $L>1$ such that for every $P\in\mathcal{F}$*

$$\sum_{R\in\mathcal{F},R\subsetneq P,R \text{ maximal}}|R|\leq\frac{1}{L}|P|$$

then

$$\sum_{P\in\mathcal{F}}Y(P)\leq\kappa Y(Q)$$

where $\kappa=\beta_Y\sum_{k=0}^{\infty}\frac{1}{L^{\frac{k}{q}}}$

Proof. We observe that

$$\sum_{P\in\mathcal{F}}Y(P)=\sum_{k=0}^{\infty}\sum_{P\in\mathcal{F}_k}Y(P)$$

where $\mathcal{F}_0=\{Q\}$, $\mathcal{F}_k=\{P\subsetneq R\,:\,P\in\mathcal{F},R\in\mathcal{F}_{k-1},P\text{ maximal }\}$. We note that, taking into account the properties of $\mathcal{F}$,

$$\sum_{P_j^1\in\mathcal{F}_1}|P_j^1|\leq\frac{1}{L}|Q|$$

$$\sum_{P_j^2\in\mathcal{F}_2}|P_j^2|\leq\frac{1}{L}\sum_{P_j^1\in\mathcal{F}_1}|P_j^1|\leq\frac{1}{L^2}|Q|$$

and in general

$$\sum_{P_j^k\in\mathcal{F}_k}|P_j^k|\leq\frac{1}{L^k}|Q|.$$

Then, since $Y\in\mathcal{Y}_q$,

$$\sum_{k=0}^{\infty}\sum_{P\in\mathcal{F}_k}Y(P)\leq\beta_Y\sum_{k=0}^{\infty}\left(\frac{1}{L^k}\right)^{\frac{1}{q}}Y(Q)$$

and the desired estimate holds with $\kappa=\beta_Y\sum_{k=0}^{\infty}\frac{1}{L^{\frac{k}{q}}}$. □

We may assume, without loss of the generality that the sparse family in Lemma 3.1 satisfies a sparseness property as the one in the preceding Lemma with $L=2$. We remit the reader to [LN17, Section 6] for the equivalence between Carleson families, as the ones in the lemma, and sparse families and futher details. Taking that into account we can proceed as follows. Let k be the only non-negative integer such that $k<p\leq k+1$. Then, since $\frac{w(Q)}{Y(Q)}\leq 1$

$$\begin{aligned}
&\left(\frac{1}{Y(Q)}\int_Q|f-f_Q|^pw\right)^{\frac{1}{p}}\leq\left(\frac{1}{Y(Q)}\int_Q|f-f_Q|^{k+1}w\right)^{\frac{1}{k+1}}\\
&\leq c_n\left(\frac{1}{Y(Q)}\int_Q\left(\sum_{P\in\mathcal{F}}\frac{1}{|P|}\int_P|f-f_P|\chi_P(x)\right)^{k+1}w\right)^{\frac{1}{k+1}}\\
&\leq c_n\|f\|_{\mathrm{BMO}}\left(\frac{1}{Y(Q)}\int_Q\left(\sum_{P\in\mathcal{F}}\chi_P(x)\right)^{k+1}w(x)dx\right)^{\frac{1}{k+1}}
\end{aligned}$$

$$\leq c_n\|f\|_{\mathrm{BMO}}\left((k+1)!\right)^{\frac{1}{p}}\left(\frac{1}{Y(Q)}\sum_{\substack{P_{k+1}\subseteq P_k\subseteq\cdots\subseteq P_1\subseteq Q\\ P_i\in\mathcal{F}}}\int_Q \chi_{P_1}\chi_{P_2}\cdots\chi_{P_k}\chi_{P_{k+1}}w\right)^{\frac{1}{k+1}}$$

$$\leq c_n\|f\|_{\mathrm{BMO}}\left((k+1)!\right)^{\frac{1}{k+1}}\left(\frac{1}{Y(Q)}\sum_{\substack{P_{k+1}\subseteq P_k\subseteq\cdots\subseteq P_1\subseteq Q\\ P_i\in\mathcal{F}}} w(P_{k+1})\right)^{\frac{1}{k+1}}$$

$$\leq c_n\|f\|_{\mathrm{BMO}}\left((k+1)!\right)^{\frac{1}{k+1}}\left(\frac{1}{Y(Q)}\sum_{\substack{P_{k+1}\subseteq P_k\subseteq\cdots\subseteq P_1\subseteq Q\\ P_i\in\mathcal{F}}} Y(P_{k+1})\right)^{\frac{1}{k+1}}$$

$$\leq c_n\|f\|_{\mathrm{BMO}}\left((k+1)!\right)^{\frac{1}{k+1}}\left(\frac{1}{Y(Q)}\kappa_q\sum_{\substack{P_k\subseteq P_{k-1}\subseteq\cdots\subseteq P_1\subseteq Q\\ P_i\in\mathcal{F}}} Y(P_k)\right)^{\frac{1}{k+1}}$$

$$\ldots$$

$$\leq c_n\|f\|_{\mathrm{BMO}}\left((k+1)!\right)^{\frac{1}{k+1}}\left(\frac{1}{Y(Q)}\kappa_q^{k+1}Y(Q)\right)^{\frac{1}{k+1}}$$

$$\leq c_n\|f\|_{\mathrm{BMO}}\left((k+1)!\right)^{\frac{1}{k+1}}\kappa_q$$

where $\kappa_q = \beta_Y\sum_{k=0}^{\infty}\frac{1}{2^{k/q}}$. Finally, a simple computation, using elementary Stirling estimates for $(k+1)!$, shows that this final inequality yields the desired estimate:

$$\left(\frac{1}{Y(Q)}\int_Q |f-f_Q|^p w\right)^{\frac{1}{p}} \leq c_n p\, q\, \beta_Y\, \|f\|_{\mathrm{BMO}}.$$

4. Proof of Theorem 1.7

We rely upon Lemma 3.1 to provide the proof of Theorem 1.7.

Proof. We start with the second estimate

$$\left(\frac{1}{w(Q)}\int_Q\left|\frac{b(x)-b_Q}{w}\right|^{p'} w(x)dx\right)^{\frac{1}{p'}}$$

$$\leq 2^{n+2}\left(\frac{1}{w(Q)}\int_Q\left(\sum_{P\in\mathcal{S},P\subseteq Q}\frac{w(P)}{|P|}\frac{1}{w(P)}\int_P|b-b_P|\chi_P(x)\right)^{p'}\sigma(x)dx\right)^{\frac{1}{p'}}$$

$$\leq 2^{n+2}\|b\|_{\mathrm{BMO}_{1,w}}\left(\frac{1}{w(Q)}\int_Q\left(\sum_{P\in\mathcal{S},P\subseteq Q}\frac{w(P)}{|P|}\chi_P(x)\right)^{p'}\sigma(x)dx\right)^{\frac{1}{p'}}$$

$$\leq c_n\|b\|_{\mathrm{BMO}_{1,w}}\left(\frac{(k+1)!}{w(Q)}\sum_{\substack{P_i\in\mathcal{S}\\ P_{k+1}\subseteq P_k\subseteq\cdots\subseteq P_1\subseteq Q}}\left(\prod_{j=1}^{k}\langle w\rangle_{P_j}\right)\left(\langle w\rangle_{P_{k+1}}\right)^{p'-k}\sigma(P_{k+1})\right)^{\frac{1}{p'}}$$

taking into account [Hyt14, Lemma 5.1] and choosing k to be the unique integer such that $k \leq p' < k+1$. A careful tracking of the constants involved in the argument in [Hyt14, p. 102] allows us to obtain the desired result.

For the first estimate it suffices to notice that if $q > 1$ then, $[w]_{A_{q'}} \leq [w]_{A_1}$. Then using the computations above we are done. □

References

[ABES] P. Auscher, S. Bortz, M.Egert, and O. Saari, *Non-local ghering lemmas*, Preprint, arXiv:1707.02080v1 (2018).

[AHT17] T. C. Anderson, T. Hytönen, and O. Tapiola, *Weak A_∞ weights and weak reverse Hölder property in a space of homogeneous type*, J. Geom. Anal. **27** (2017), no. 1, 95–119.

[AMPRR] N. Accomazzo, J. Martínez Perales, and I. P. Rivera Ríos, *On Bloom type estimates for iterated commutators of fractional integrals*, To appear in Indiana Univ. Math. Journal.

[BI87] B. Bojarski and T. Iwaniec, *p-harmonic equation and quasiregular mappings*, Partial differential equations (Warsaw, 1984), Banach Center Publ., vol. 19, PWN, Warsaw, 1987, pp. 25–38.

[Blo85] S. Bloom, *A commutator theorem and weighted BMO*, Trans. Amer. Math. Soc. **292** (1985), no. 1, 103–122.

[Buc90] S. M. Buckley, *Harmonic analysis on weighted spaces*, ProQuest LLC, Ann Arbor, MI, 1990, Thesis (Ph.D.)–The University of Chicago. MR 2611907

[Can18] J. Canto, *Quantitative C_p estimates for Calderón-Zygmund operators*, arXiv e-prints (2018), arXiv:1811.05209.

[CLPR] M. E. Cejas, K. Li, C. Perez, and I. P. Rivera-Rios, *Vector-valued operators, optimal weighted estimates and the C_p condition*, to appear in Sci. China Math.

[DMRO16] J. Duoandikoetxea, F. J. Martín-Reyes, and S. Ombrosi, *On the A_∞ conditions for general bases*, Math. Z. **282** (2016), no. 3-4, 955–972. MR 3473651

[GC79] J. García-Cuerva, *Weighted H^p spaces*, Dissertationes Math. (Rozprawy Mat.) **162** (1979), 63. MR 549091

[GCHST91] J. García-Cuerva, E. Harboure, C. Segovia, and J. L. Torrea, *Weighted norm inequalities for commutators of strongly singular integrals*, Indiana Univ. Math. J. **40** (1991), no. 4, 1397–1420.

[GCRdF85] J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985.

[Gra14] L. Grafakos, *Classical Fourier analysis*, third ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2014.

[HLW17] I. Holmes, M. T. Lacey, and .B D. Wick, *Commutators in the two-weight setting*, Math. Ann. **367** (2017), no. 1-2, 51–80.

[HP13] T. Hytönen and C. Pérez, *Sharp weighted bounds involving A_∞*, Anal. PDE **6** (2013), no. 4, 777–818.

[Hyt] T. Hytönen, *The L^p-to-L^q boundedness of commutators with applications to the Jacobian operator.*

[Hyt14] T. Hytönen, *The A_2 theorem: remarks and complements*, Harmonic analysis and partial differential equations, Contemp. Math., vol. 612, Amer. Math. Soc., Providence, RI, 2014, pp. 91–106. MR 3204859

[JN61] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

[Jou83] J. L. Journé, *Calderón-Zygmund operators, pseudodifferential operators and the Cauchy integral of Calderón*, Lecture Notes in Mathematics, vol. 994, Springer-Verlag, Berlin, 1983.

[Ler10] A. K. Lerner, *Some remarks on the Fefferman-Stein inequality*, J. Anal. Math. **112** (2010), 329–349. MR 2763004

[LN17] A. K. Lerner and F. Nazarov, *Intuitive dyadic calculus: the basics*, To appear in Expo. Math. (2017).

[LORR17] A. K. Lerner, S. Ombrosi, and I. P. Rivera-Ríos, *On pointwise and weighted estimates for commutators of Calderón-Zygmund operators*, Adv. Math. **319** (2017), 153–181.

[LORR18] A. K. Lerner, S. Ombrosi, and I. P. Rivera-Ríos, *Commutators of singular integrals revisited*, Bulletin of the London Mathematical Society (2018).

[Muc81] B. Muckenhoupt, *Norm inequalities relating the Hilbert transform to the Hardy-Littlewood maximal function*, Functional analysis and approximation (Oberwolfach, 1980), Internat. Ser. Numer. Math., vol. 60, Birkhäuser, Basel-Boston, Mass., 1981, pp. 219–231. MR 650277

[MW76] B.Muckenhoupt and R. L. Wheeden, *Weighted bounded mean oscillation and the Hilbert transform*, Studia Math. **54** (1975/76), no. 3, 221–237.

[PR] C. Pérez and E. Rela, *Degenerate poincaré-sobolev inequalities*, To appear Trans. A.M.S., https://arxiv.org/abs/1805.10388.

[Saw82] E. T. Sawyer, *Two weight norm inequalities for certain maximal and integral operators*, Harmonic analysis (Minneapolis, Minn., 1981), Lecture Notes in Math., vol. 908, Springer, Berlin-New York, 1982, pp. 102–127. MR 654182

[Saw83] E. T. Sawyer, *Norm inequalities relating singular integrals and the maximal function*, Studia Math. **75** (1983), no. 3, 253–263. MR 722250

[Yab90] K. Yabuta, *Sharp maximal function and C_p condition*, Arch. Math. (Basel) **55** (1990), no. 2, 151–155. MR 1064382

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