The largest volume ratio of a convex body

Mariano Merzbacher

A convex body in $\mathbb{R}^n$ is a compact convex subset with non-empty interior (to start and to give us an idea, we can think that it is simply the unitary ball with respect to some norm in $\mathbb{R}^n$).

Given two convex bodies $K, L \subset \mathbb{R}^n$, the *volume ratio* between K and L is defined as

$$\mathrm{vr}(K,L) := \inf \left(\frac{|K|}{|T(L)|} \right)^{\frac{1}{n}},$$

where the infimum is taken over all affine transformations T such that $TL \subset K$ (here $|\cdot|$ stands for the n-dimensional Lebesgue measeure). Roughly speaking, this quantifies a measure relationship between the affine classes of the bodies involved (subject to the inclusion condition).

A natural question in convex geometry is the following: *Given a convex body K, how large is $vr(K,L)$ for arbitrary convex bodies $L \subset \mathbb{R}^n$?*

Thus, it is important to compute the *largest volume ratio of* K, given by

$$\mathrm{lvr}(K) := \sup_{L \subset \mathbb{R}^n} \mathrm{vr}(K,L).$$

We prove the following sharp lower bound

$$c\sqrt{n} \leq \mathrm{lvr}(K),$$

for *every* body K (where $c > 0$ is an absolute constant). This result improves the former bound, of order $\sqrt{\frac{n}{\log\log(n)}}$.

We will also review the general known upper bounds (in terms of dimension) of this quantity and show that, for many interesting classes, one can obtain correct asymptotic estimates. We will see how random techniques play a preponderant role in these bounds.

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