ON THE CONVERGENCE OF RANDOM POLYNOMIALS AND MULTILINEAR FORMS

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Abstract. We consider different kinds of convergence of random homogeneous polynomials and multilinear forms. We show that for a variety of complex random variables, the almost sure convergence of the random polynomial is equivalent to that of the multilinear form, and to the square summability of the coefficients. Also, we present polynomial Khintchine inequalities for complex gaussian and Steinhaus variables. All these results have no analogous in the real case. Moreover, we study the $L_p$ convergence of random polynomials and derive certain decoupling inequalities without the usual tetrahedral hypothesis. We also consider convergence on “full subspaces” in the sense of Sjögren, both for real and complex random variables, and relate it to domination properties of the polynomial or the multilinear form, establishing a link with the theory of homogeneous polynomials on Banach spaces.

Introduction

In this article we study the convergence of random polynomials and multilinear forms. Given a $k$-homogeneous random polynomial \( \sum_{i_1,\ldots,i_k=1}^n a_{i_1,\ldots,i_k} X_{i_1} \cdots X_{i_k} \), where \( \{X_i\}_i \) is a sequence of complex or real random variables, we relate properties of the coefficients \( (a_{i_1,\ldots,i_k})_{i_1,\ldots,i_k} \) (or mappings related to these coefficients) with different kinds of convergence of the polynomial (almost sure, convergence in $L_p$, convergence in full subspaces, etc.).

For real random variables, the almost sure convergence of a multilinear form
\[
\sum_{i_1,\ldots,i_k=1}^n a_{i_1,\ldots,i_k} X_{i_1} \cdots X_{i_k}
\]
is not equivalent to the convergence of the random polynomial
\[
\sum_{i_1,\ldots,i_k=1}^n a_{i_1,\ldots,i_k} X_{i_1} \cdots X_{i_k}
\]
(see, for example, [21] for the bilinear/quadratic case). Conditions on the coefficients $a_{i_1,\ldots,i_k}$ with repeated subindexes must be considered and also, in many cases, one has to impose all of them to be null in order to relate the convergence of the random polynomial with the multilinear form. With the same spirit, for real random variables (or real function spaces), there are multilinear Khintchine inequalities but...
no polynomial ones, and coefficients with repeated indexes are again the problem: for Rademacher or Gaussian variables, the $L_p$-convergence of $\sum_i a_{i,i}X_i^2$ (the diagonal quadratic form) is not related to the square summability of the coefficients $a_{i,i}$ (but to the convergence of $\sum_i a_{i,i}$).

We show that for rotation-invariant complex random variables, coefficients with repeated indexes is not a problem, and almost sure convergence of the random polynomial is equivalent to that of the random multilinear form and, also, to the square summability of the coefficients. Moreover, for complex Gaussian and Steinhaus random variables, we present a polynomial Khintchine inequality (which has no analogous for real random variables), and allows us to relate the square summability of the coefficients also to the $L_p$ convergence of the polynomial and the multilinear form. Another consequence of our polynomial Khintchine inequalities is a particular case of decoupling inequality, which again holds without conditions on the coefficients with repeated indexes.

In [21], Sjögren considered the convergence of gaussian quadratic and bilinear forms on full subspaces (see the definitions in Section 2). He shows that this convergence is equivalent to the coefficients defining a nuclear operator on $\ell_2$, but that this is no longer true for degree three. In order to study higher degrees we introduce the standard full subspaces and show, for example, that convergence on these subspaces is equivalent to the coefficients defining a 2-dominated polynomial (or multilinear form) on $\ell_2$. We also consider non-gaussian random variables. Finally, we use our polynomial Khintchine inequality to extend a result on dominated polynomials due to Melendez and Tonge [15].

Some of our results are proved using recent techniques on integral representation of holomorphic functions on Banach spaces introduced in [18, 20]. We devote Section 3 to summarize some aspects of this theory, as well as to prove some new results needed in this work. The proofs of most of the results of the first two sections are then postponed to Section 4.

1. POLYNOMIAL KHINTCHINE TYPE INEQUALITIES AND ALMOST SURE CONVERGENCE

Let us fix some terminology. The (multi-indexed) sequence of complex numbers $\{a_{i_1,\ldots,i_k}\}_{i_j \geq 1}$ is said to be symmetric if

$$a_{i_1,\ldots,i_k} = a_{j_1,\ldots,j_k}$$

whenever $\{j_1,\ldots,j_k\}$ is a permutation of $\{i_1,\ldots,i_k\}$.

A complex random variable $X : (\Omega, \mathcal{A}, P) \to \mathbb{C}$ is said to be rotation-invariant if $X$ and $e^{i\theta}X$ have the same distribution law for all $\theta \in [0, 2\pi]$. Note that for such a random variable we must have $E(X) = 0$, since in particular

$$E(X) = E(e^{i\pi}X) = e^{i\pi}E(X) = -E(X).$$

Moreover, the same argument shows that $E(X^k) = 0$ for any $k \in \mathbb{N}$.

In the sequel, given $k \in \mathbb{N}$, we will need to work with sequences of independent complex random variables $\{X_j\}_{j \in \mathbb{N}}$, which satisfy the following hypothesis:

\[
(*) \quad \inf_{j \in \mathbb{N}} E(|X_j|) > 0 \quad \text{and} \quad \sup_{j \in \mathbb{N}} E(|X_j|^{2k}) < \infty.
\]
We will call it the \((\ast)\)-condition (for \(k\)). Note that, for a sequence of identically distributed non-zero random variables, this condition is merely to have a finite \(2k\)-th moment.

The following result makes it apparent the difference between real and complex variables in terms of polynomial convergence:

**Proposition 1.1.** Let \(k \in \mathbb{N}\) and \(\{X_j\}_{j \in \mathbb{N}}\) be a sequence of independent and rotation-invariant complex random variables satisfying the \((\ast)\)-condition. Then, there exist \(A_k, B_k \in \mathbb{R}_{>0}\) such that for any symmetric sequence of complex numbers \(\{a_{j_1,\ldots,j_k}\}_{j_s \geq 1}\) and any \(n \in \mathbb{N}\), we have

\[
A_k^{-1} \left[ \sum_{j_1,\ldots,j_k=1}^{n} |a_{j_1,\ldots,j_k}|^2 \right]^{\frac{1}{2}} \leq \left[ \mathbb{E} \left( |F_n|^2 \right) \right]^{\frac{1}{2}} \leq B_k \left[ \sum_{j_1,\ldots,j_k=1}^{n} |a_{j_1,\ldots,j_k}|^2 \right]^{\frac{1}{2}},
\]

where \(F_n = \sum_{j_1,\ldots,j_k=1}^{n} a_{j_1,\ldots,j_k} X_{j_1} \cdots X_{j_k}\).

If the \(\{X_i\}_{i \in \mathbb{N}}\) are independent standard complex gaussian variables, we actually have

\[
\left[ \mathbb{E} \left( |F_n|^2 \right) \right]^{1/2} = \sqrt{k!} \left( \sum_{i_1,\ldots,i_k=1}^{n} |a_{i_1,\ldots,i_k}|^2 \right)^{1/2}.
\]

As we can see in the proof of the previous proposition (in Section 4), the set of random monomials \(\{X_{j_1} \cdots X_{j_k}\}_{j_1 \leq \cdots \leq j_k}\) is an orthogonal system. Note that we are including monomials with repeated indexes. The implication \((i) \Rightarrow (ii)\) in next theorem will be a consequence of this proposition, together with the martingale properties of the sequence \(\sum_{j_1,\ldots,j_k=1}^{n} a_{j_1,\ldots,j_k} X_{j_1} \cdots X_{j_k}\), to be shown in Section 4.

Let us note that in [21], convergence of bilinear and quadratic forms in real valued random variables are studied. Under certain assumptions, it is shown that the almost sure convergence of the bilinear form \(\sum_{i,j=1}^{\infty} a_{i,j} X_i Y_j\) is equivalent to the coefficients \((a_{i,j})_{i,j}\) being square summable. For quadratic forms, extra conditions on the diagonal \((a_{i,i})_i\) are necessary for the equivalence. We see that for complex-valued random variables the situation is different:

**Theorem 1.2.** Given \(k \in \mathbb{N}\) and a symmetric sequence \(\{a_{j_1,\ldots,j_k}\}_{j_s \geq 1}\) of complex numbers, the following are equivalent:

\begin{enumerate}
    \item[(i)] \(\sum_{j_1,\ldots,j_k \geq 1} |a_{j_1,\ldots,j_k}|^2 < \infty\).
    \item[(ii)] For every sequence \(\{X_j\}_{j \in \mathbb{N}}\) of independent and rotation-invariant complex random variables which satisfies the \((\ast)\)-condition, the series

    \[
    \sum_{j_1,\ldots,j_k=1}^{n} a_{j_1,\ldots,j_k} X_{j_1} \cdots X_{j_k}
    \]

    converges almost surely.
    \item[(iii)] For every choice of \(k\) independent sequences \(\{Y_{j_1}^1\}_{j \in \mathbb{N}}, \ldots, \{Y_{j_k}^k\}_{j \in \mathbb{N}}\) of independent and rotation-invariant complex random variables which satisfy the
\[(*)\)-condition, the series
\[
\sum_{j_1, \ldots, j_k = 1}^n a_{j_1, \ldots, j_k} Y_{j_1}^1 \cdots Y_{j_k}^k
\]
converges almost surely.

If the sequence \(\{a_{j_1, \ldots, j_k}\}_{j_k \geq 1}\) is not symmetric, the equivalence between (i) and (iii) remains true.

Now we restrict ourselves to complex gaussian variables. As we have mentioned, the proof of Proposition 1.1 shows the orthogonality of the whole family of functions \(X_{i_1} \cdots X_{i_k}\) in \(L_2\), including those with repeated indexes. For the other \(L_p\)'s, we have the following polynomial Khintchine inequality:

**Theorem 1.3.** If \(\{X_i\}_{i \in \mathbb{N}}\) is a sequence of independent standard complex gaussian variables, then for \(1 \leq p < \infty\) there are constants \(A_{k,p}\) and \(B_{k,p} \geq 1\) such that for every symmetric sequence of complex numbers \(\{a_{i_1, \ldots, i_k}\}_{i_1 \geq 1}\), we have:

\[
A_{k,p}^{-1} \left[ \sum_{i_1, \ldots, i_k = 1}^n |a_{i_1, \ldots, i_k}|^2 \right]^{\frac{1}{2}} \leq \left[ \mathbb{E} \left( |F_n|^p \right) \right]^{\frac{1}{2}} \leq B_{k,p} \left[ \sum_{i_1, \ldots, i_k = 1}^n |a_{i_1, \ldots, i_k}|^2 \right]^{\frac{1}{2}},
\]

for all \(n \in \mathbb{N}\), where \(F_n = \sum_{i_1, \ldots, i_k = 1}^n a_{i_1, \ldots, i_k} X_{i_1} \cdots X_{i_k}\).

Although this is probably known, we can derive the multilinear Khintchine inequality for complex Gaussian variables from the polynomial one to obtain:

**Corollary 1.4.** Let \(\{Z_{i_1}^1\}_{i_1 \in \mathbb{N}}, \ldots, \{Z_{i_1}^k\}_{i_1 \in \mathbb{N}}\) a finite set of independent sequences of independent standard complex gaussian variables, then for \(1 \leq p < \infty\) there are constants \(A_{k,p}\) and \(B_{k,p} \geq 1\) such that for every sequence of complex numbers \(\{b_{i_1, \ldots, i_k}\}_{i_1 \geq 0}\), we have:

\[
\tilde{A}_{k,p}^{-1} \left[ \sum_{i_1, \ldots, i_k = 1}^n |b_{i_1, \ldots, i_k}|^2 \right]^{\frac{1}{2}} \leq \left[ \mathbb{E} \left( |G_n|^p \right) \right]^{\frac{1}{2}} \leq \tilde{B}_{k,p} \left[ \sum_{i_1, \ldots, i_k = 1}^n |b_{i_1, \ldots, i_k}|^2 \right]^{\frac{1}{2}},
\]

for all \(n \in \mathbb{N}\), where \(G_n = \sum_{i_1, \ldots, i_k = 1}^n b_{i_1, \ldots, i_k} Z_{i_1}^1 \cdots Z_{i_k}^k\).

Decoupling inequalities have evolved as a subject of great interest since their introduction by McConnell and Taqqu [13, 14]. Their motivation was the study of multiple stochastic integrals (see the expository article [3] and the references therein, and also [2, 4, 9] for results on and applications of decoupling inequalities). In these works, the polynomials and multilinear forms involved are generally required to be “tetrahedral”, i.e., that the coefficients \(a_{j_1, \ldots, j_k}\) be zero if \(j_1, \ldots, j_k\) are not all different. For complex gaussian variables, as an immediate consequence of our polynomial Khintchine inequality and its multilinear analogous, we have the following particular case of decoupling inequality, without the tetrahedral assumption:
Theorem 1.8. Let give the following:

variables.

inequality for Steinhaus random variables just as we did in Corollary 1.5 for gaussian

The following are equivalent:

... for all \( n \in \mathbb{N} \).

Now we turn our attention to Steinhaus random variables. Recall that for a uniform random variable \( \phi \) on the interval \([0, 2\pi]\), the (complex) random variable \( e^{i\phi} \) is uniformly distributed on the complex circumference \( S^1 \), and it is called a Steinhaus random variable. For these variables we have:

**Theorem 1.6.** If \( \{\varphi_i\}_{i \in \mathbb{N}} \) is a sequence of independent Steinhaus random variables, then for \( 1 \leq p < \infty \) there are constants \( \tilde{A}_{k,p} \) and \( \tilde{B}_{k,p} \geq 1 \) such that for every symmetric sequence of complex numbers \( \{a_{i_1, \ldots, i_k}\}_{i_1, \ldots, i_k \geq 1} \), we have:

\[
\tilde{A}_{k,p}^{-1} \left[ \sum_{i_1, \ldots, i_k = 1}^{n} |a_{i_1, \ldots, i_k}|^2 \right]^{\frac{1}{2}} \leq \mathbb{E} \left( |F_n|^p \right)^{\frac{1}{p}} \leq \tilde{B}_{k,p} \left[ \sum_{i_1, \ldots, i_k = 1}^{n} |a_{i_1, \ldots, i_k}|^2 \right]^{\frac{1}{2}},
\]

for all \( n \in \mathbb{N} \), where \( F_n = \sum_{i_1, \ldots, i_k = 1}^{n} a_{i_1, \ldots, i_k} \varphi_{i_1} \cdots \varphi_{i_k} \).

Mimicking the proof of Corollary 1.4, we can obtain the following corollary for the multilinear situation:

**Corollary 1.7.** Let \( \{Z_1^i\}_{i \in \mathbb{N}_0}, \ldots, \{Z_k^i\}_{i \in \mathbb{N}} \) a finite set of independent sequences of independent Steinhaus random variables, then for \( 1 \leq p < \infty \) there are constants \( \tilde{A}_{k,p} \) and \( \tilde{B}_{k,p} \geq 1 \) such that for every sequence of complex numbers \( \{b_{i_1, \ldots, i_k}\}_{i_1, \ldots, i_k \geq 0} \), we have:

\[
\tilde{A}_{k,p}^{-1} \left[ \sum_{i_1, \ldots, i_k = 1}^{n} |b_{i_1, \ldots, i_k}|^2 \right]^{\frac{1}{2}} \leq \mathbb{E} \left( |G_n|^p \right)^{\frac{1}{p}} \leq \tilde{B}_{k,p} \left[ \sum_{i_1, \ldots, i_k = 1}^{n} |b_{i_1, \ldots, i_k}|^2 \right]^{\frac{1}{2}},
\]

for all \( n \in \mathbb{N} \), where \( G_n = \sum_{i_1, \ldots, i_k = 1}^{n} b_{i_1, \ldots, i_k} Z_{i_1}^1 \cdots Z_{i_k}^k \).

It is clear that Theorem 1.6 and its Corollary 1.7 together give a decoupling inequality for Steinhaus random variables just as we did in Corollary 1.5 for gaussian variables.

A combination of Theorem 1.3, Corollary 1.4, Theorem 1.6 and Corollary 1.7 give the following:

**Theorem 1.8.** Let \( \{a_{i_1, \ldots, i_k}\}_{i_1, \ldots, i_k \geq 1} \) be a symmetric sequence of complex numbers. The following are equivalent:

(i) \( \sum_{i_1, \ldots, i_k \geq 1} |a_{i_1, \ldots, i_k}|^2 < \infty \) (or any of the equivalent conditions in Theorem 1.2).

(ii) For every sequence (or for some sequence) \( \{X_i\}_{i \in \mathbb{N}} \) of independent standard complex gaussian variables (Steinhaus random variables), and for every \( 1 \leq p < \infty \), the series \( \sum_{i_1, \ldots, i_k = 1}^{n} a_{i_1, \ldots, i_k} X_{i_1} \cdots X_{i_k} \) is convergent in \( L_p \).
(iii) For every independent sequences (or for some independent sequences) of independent standard complex gaussian variables (Steinhaus random variables) \( \{ Y^1_i \}_{i \in \mathbb{N}}, \ldots, \{ Y^k_i \}_{i \in \mathbb{N}} \), and for every \( 1 \leq p < \infty \), the series
\[
\sum_{i_1, \ldots, i_k = 1}^n a_{i_1, \ldots, i_k} Y^1_{i_1} \cdots Y^k_{i_k}
\]
is convergent in \( L_p \).

If the sequence \( \{ a_{i_1, \ldots, i_k} \}_{i_j \geq 1} \) is not symmetric, the equivalence between (i) and (iii) remains true.

Note that as a consequence of Theorems 1.2 and 1.8, almost sure and \( L_p \) convergence for either homogeneous polynomials or the associated multilinear form on gaussian or Steinhaus variables are all equivalent (and equivalent to square summability of the coefficients).

2. Convergence on standard full subspaces

In this section we consider polynomials and multilinear forms whose sets of convergence enjoy some linearity property. In opposition to the previous section, all the results in this one hold for both complex and real variables.

Sjögren [21] studied the convergence of bilinear and quadratic forms of standard gaussian real random variables on what he calls “full subspaces” of \( \mathbb{R}^N \), motivated by the study of convergence of some stochastic integrals. He looks at the sequence \( (X_i)_i \) as an element of \( \mathbb{R}^N \) and, in \( \mathbb{R}^N \), considers a gaussian product measure. A “full subspace” of \( \mathbb{R}^N \) is then a linear subspace with gaussian measure 1. He shows that the convergence on a full subspace (for the gaussian measure) is equivalent to the bilinear form being nuclear on \( \ell^2 \). He also presents a counterexample showing that for trilinear forms this is not true.

In order to study the same problem for \( n \)-linear forms or \( n \)-homogeneous polynomials on \( K^N \), \( K = \mathbb{R} \) or \( \mathbb{C} \), and for more general random variables, we need to restrict somehow the full subspaces considered. We thus define the concept of “standard full subspace” in the construction that follows.

Given a Hilbert-Schmidt injective operator \( T : \ell^2 \to \ell^2 \), we define a norm on \( X_0 \), the set of finite sequences of scalar numbers:
\[
|||x||| = ||Tx||_{\ell^2}.
\]
We denote \( X_T \) the completion of \( X_0 \) with respect to the norm \( ||| \cdot ||| \). We can identify \( X_T \) with a linear subspace of \( \mathbb{K}^N \) whose gaussian measure is 1 (p. 59, [8]). Therefore, \( X_T \) is a full subspace in the sense of Sjögren. We call these spaces “standard full subspaces”.

It is clear that we can continuously extend the operator \( T \) to \( X_T \). We denote \( \hat{T} : X_T \to \ell^2 \) this extension and we have that, for all \( x \in X_T \), \( |||x||| = \|Tx\|_{\ell^2} \). Also, it is straightforward that \( \ell^2 \subset X_T \) and the inclusion \( i : \ell^2 \to X_T \) has the same norm as \( T \).

Note that the standard full subspaces include the following examples: given any sequence \( (\lambda_n)_{n \in \mathbb{N}} \in \ell^2 \) with \( \lambda_n > 0 \) for all \( n \), and \( T(x) = \sum_n \lambda_n x_n e_n \), then \( T \) is an injective Hilbert-Schmidt operator in \( \ell^2 \). The corresponding subspace \( X_T \) is:
\[
X_T = \left\{ (x_n)_{n \in \mathbb{N}} : |||x|||^2 = \sum_n \lambda_n^2 |x_n|^2 < \infty \right\}.
\]

Now we see that standard full subspaces have measure 1 for a great variety of product probabilities. Suppose we are given a probability measure \( \mu_1 \) defined on
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The Borel subsets of $\mathbb{K}$, such that $\int_{\mathbb{K}} |z|^2 d\mu_1(z) = \sigma^2 < \infty$, and let $\mu$ be the induced product measure on $\mathbb{K}^n$. Then we have the following

**Theorem 2.1.** Take an injective Hilbert-Schmidt operator $T : \ell_2 \to \ell_2$ and let $X_T$ be the standard full subspace associated with $T$. If $\mu$ is defined as above, then $\mu(X_T) = 1$.

Our next objective is to relate the convergence of a random polynomial (or multilinear form) on a standard full subspace to properties of the polynomial (or multilinear form) defined by the same coefficients. First we need some definitions.

A mapping $P : E \to \mathbb{K}$ is a $k$-homogeneous polynomial if there exists a $k$-linear form $\Phi : E \times \cdots \times E \to \mathbb{K}$ such that $P(x) = \Phi(x, \ldots, x)$ for all $x \in E$. The space of all continuous $k$-homogeneous polynomials is denoted by $P^k(E)$.

For $x_1, \ldots, x_m \in E$, we define

$$w_r((x_i)_{i=1}^m) = \sup_{x' \in B_{E'}} \left( \sum_i |\langle x', x_i \rangle|^r \right)^{1/r}.$$ 

A polynomial $P \in P^k(E)$ is $r$-dominated if there exists $C > 0$ such that for every finite sequence $(x_i)_{i=1}^m \subset E$ the following holds

$$\left( \sum_{i=1}^m |P(x_i)|^r \right)^{1/r} \leq C w_r((x_i)_{i=1}^m)^k.$$ 

The least of such constants $C$ is called the $r$-dominated quasi-norm of $P$ and will be denoted by $\|P\|_{r-dom}$. The definition for multilinear forms is analogous.

Dominated polynomials verify the following domination property [15]: there exists a probability measure $\nu$ on $B_{E'}$ such that for each $x \in E$ we have:

$$|P(x)| \leq \|P\|_{r-dom} \left( \int_{B_{E'}} |\langle x', x \rangle|^r \, d\nu \right)^{1/r}.$$ 

It is not hard to see that the convergence of a $k$-linear form on the product of $k$ standard full subspaces is equivalent to the convergence on $X \times \cdots \times X$ for some standard full subspace $X$. Therefore, assertion (iii) in next theorem can be also stated as convergence on the product $k$ standard full subspaces. We choose the following formulation for simplicity.

**Theorem 2.2.** Let $\{a_{i_1, \ldots, i_k}\}_{i_1 \geq 1}$ be a symmetric sequence of complex numbers. The following are equivalent:

(i) The series $\sum_{i_1 \leq N_1, \ldots, i_k \leq N_k} a_{i_1, \ldots, i_k} X_{i_1} \cdots X_{i_k}$ converges in a standard full subspace as $N_1, \ldots, N_k \to \infty$.

(ii) $P(x) = \sum_{i_1, \ldots, i_k = 1} a_{i_1, \ldots, i_k} x_{i_1} \cdots x_{i_k}$ defines a 2-dominated $k$-homogeneous polynomial on $\ell_2$.

(iii) The series $\sum_{i_1 \leq N_1, \ldots, i_k \leq N_k} a_{i_1, \ldots, i_k} X_{i_1}^1 \cdots X_{i_k}^k$ converges in a standard full subspace as $N_1, \ldots, N_k \to \infty$. 

(iv) $A(x) = \sum_{i_1, \ldots, i_k = 1}^{\infty} a_{i_1, \ldots, i_k} x_{i_1}^{1} \cdots x_{i_k}^{k}$ defines a 2-dominated $k$-linear form on $\ell_2$.

If the sequence $\{a_{i_1, \ldots, i_k}\}_{i_\geq 1}$ is not symmetric, the equivalence between (iii) and (iv) remains true.

A combination of Theorems 2.1 and 2.2 give the following:

**Corollary 2.3.** Let $\{a_{i_1, \ldots, i_k}\}_{i_\geq 1}$ be a sequence of complex numbers that define a 2-dominated $k$-linear form on $\ell_2$. Then

$$\sum_{i_1 \leq N_1, \ldots, i_k \leq N_k} a_{i_1, \ldots, i_k} X_{i_1} \cdots X_{i_k}$$

converges almost surely as $N_1, \ldots, N_k \to \infty$ for any sequence of independent and identically distributed symmetric random variables $\{X_i\}_{i \in \mathbb{N}}$ with finite variance. If the coefficients are symmetric, the analogous polynomial result holds.

It is a known fact that for degree two, nuclear and dominated polynomials (and multilinear forms) on $\ell_2$ coincide (see for example [5, Section 26.4]). So we can combine Theorem 2.2 with Sjögren’s result [21, Theorem 3] to see that, for gaussian variables and degree two, the convergence of the random 2-homogeneous polynomials or bilinear form on some full subspace implies the convergence on some standard full subspace. However, as we will see in the example below, this is not true for degree greater than 2.

**Theorem 2.4.** The following are equivalent:

(i) For every sequence $\{X_i\}_{i \in \mathbb{N}}$ of independent standard complex gaussian variables, the series $\sum_{i,j} a_{i,j} X_i X_j$ converges in a full subspace.

(ii) The series $\sum_{i,j} a_{i,j} X_i X_j$ converges in a standard full subspace.

(iii) For every sequence $\{X_i\}_{i \in \mathbb{N}}$ and $\{Y_j\}_{j \in \mathbb{N}}$ of independent standard complex gaussian variables, the series $\sum_{i,j} a_{i,j} X_i Y_j$ converges in a full subspace.

(iv) The series $\sum_{i,j} a_{i,j} X_i Y_j$ converges in a standard full subspace.

As a consequence, the convergence of the gaussian 2-homogeneous polynomial on a full subspace implies the almost sure convergence of the random polynomial, for any sequence of independent and identically distributed symmetric random variables $\{X_i\}_{i \in \mathbb{N}}$ with finite variance. This follows from the fact that standard full subspaces measure 1 for the product measure on $\mathbb{K}^N$ induced by these kind of random variables.

Since for degree two, convergence of a Gaussian polynomial in a full subspace implies its convergence on some standard full subspace, one may ask if any full subspace contains a standard full subspace. We answer this for the negative in next example. Sjögren presented an example of a non-nuclear trilinear form that converges on a full subspace. We see that this trilinear form can be chosen so that it is not 2-dominated. Therefore, it does not converge on any standard full subspace. This shows that the previous theorem is not true for higher degrees. Also, it follows that there are full subspaces that contain no standard full subspace.
Example 2.5. A trilinear form that converges on a full subspace but not on any standard full subspace. A full subspace not containing any standard full subspace.

Let \( \rho_1 = 0 \) and \( \rho_{n+1} = \rho_n + n^2 + n + 1 \), for all \( n \in \mathbb{N} \). Sjögren’s example is the following trilinear form:

\[
T(X, Y, Z) = \sum_{n=1}^{\infty} a_n \left( \sum_{1 \leq i, j \leq n} X_{\rho_n+i} Y_{\rho_n+j} Z_{\rho_n+n+i+j} \right),
\]

with \( a_n > 0 \), for all \( n \), verifying

\[
\sum_{n=1}^{\infty} a_n n^7 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} a_n n^2 = \infty.
\]

We consider the sequence \( \{a_n\}_n = \{n^{-2}\}_n \), which satisfies the conditions. We want to prove that in this case the trilinear \( T : \ell_2 \times \ell_2 \times \ell_2 \to \mathbb{C} \) is not 2-dominated.

Let us define the sequences \( \{x^i\}_i \), \( \{y^j\}_j \) and \( \{z^k\}_k \) to be the formed by the standard unit vectors of \( \ell_2 \) in “the same order” as the coordinates of \( X, Y \) and \( Z \) appeared in the definition of \( T \). That is:

\[
\{x^i\}_i = \{e_{\alpha(i)}\}_i, \quad \{y^j\}_j = \{e_{\beta(j)}\}_j, \quad \{z^k\}_k = \{e_{\gamma(k)}\}_k,
\]

where

\[
\alpha = \{\rho_1 + 1, \rho_2 + 1, \rho_2 + 1, \rho_2 + 2, \rho_2 + 2, \rho_3 + 1, \rho_3 + 1, \rho_3 + 1, \rho_3 + 2, \rho_3 + 2, \ldots\}
\]
\[
\beta = \{\rho_1 + 1, \rho_2 + 1, \rho_2 + 2, \rho_2 + 2, \rho_2 + 2, \rho_3 + 1, \rho_3 + 2, \rho_3 + 2, \rho_3 + 3, \rho_3 + 3, \rho_3 + 3 + 1, \rho_3 + 3 + 2, \ldots\}
\]
\[
\gamma = \{\rho_1 + 1 \cdot 1 + 1, \rho_2 + 2 \cdot 1 + 1, \rho_2 + 2 \cdot 1 + 1, \rho_2 + 2 \cdot 1 + 2, \rho_2 + 2 \cdot 2 + 1, \rho_2 + 2 \cdot 2 + 2, \ldots\}
\]

(Observe that there are lots of repetitions in the \( x^i \)'s and in the \( y^j \)'s).

Let \( \theta_n = \frac{n(n+1)(2n+1)}{6} \). Then

\[
\sum_{i=1}^{\theta_n} |T(x^i, y^j, z^k)|^2 = \sum_{i=1}^{\theta_n} \ell^2 a^{rac{2}{3}} = \sum_{i=1}^{\theta_n} \ell^2 \geq \int_0^{n-1} x^{rac{2}{3}} dx = \frac{12}{13} (n - 1)^{\frac{13}{3}}.
\]

It is easy to see that

\[
w_2 \left( \{x^i\}^{\theta_n}_{i=1} \right) = w_2 \left( \{y^j\}^{\theta_n}_{i=1} \right) = \sqrt{n} \quad \text{and} \quad w_2 \left( \{z^k\}^{\theta_n}_{i=1} \right) = 1.
\]

If \( T \) is 2-dominated, we would have

\[
\left( \sum_{i=1}^{\theta_n} |T(x^i, y^j, z^k)|^2 \right)^{\frac{2}{3}} \geq w_2 \left( \{x^i\}^{\theta_n}_{i=1} \right) \cdot w_2 \left( \{y^j\}^{\theta_n}_{i=1} \right) \cdot w_2 \left( \{z^k\}^{\theta_n}_{i=1} \right),
\]

and this would imply

\[
(n - 1)^{\frac{13}{3}} \leq n.
\]

Since this is false, \( T \) cannot be 2-dominated.

Now, the full subspace where the random bilinear form associated to \( T \) converges, cannot be contained in any standard full subspace.

Let us provide a stronger version of Sjögren counterexample, namely, a non-nuclear multilinear form that converges on a standard full subspace. To this end, we denote by \( P_n^{\infty}(E) \) the space of all nuclear \( k \)-homogeneous polynomials on the Banach space \( E \). Both spaces are endowed with their usual norms (see [6] and [16] for details).
Example 2.6. There are non-nuclear multilinear forms that converge on standard full subspaces.

Let us see that, for any natural number $k \geq 3$, there exist $Q \in \mathcal{P}(k \ell_2)$ which is $2$-dominated but it is not nuclear. For $n \geq 1$, in [1] the authors show that there exist polynomials $P_n \in \mathcal{P}(k \ell_2)$ such that
\[
\|P_n\|_{\mathcal{P}(k \ell_2)} \leq C_k n^{1/2}
\]
and
\[
\|P_n\|_{\mathcal{P}(k \ell_2^2)} \geq D_k n^{k-1/2}
\]
for suitable constants $C_k$ and $D_k$, which are independent of $n$.

Consider the following commutative diagram:
\[
\begin{array}{cccc}
\ell_2 & \stackrel{P_n}{\longrightarrow} & \mathbb{C} \\
\downarrow{id} & & & \downarrow{id} \\
\ell_2^n & \stackrel{P_n}{\longrightarrow} & \ell_2^n
\end{array}
\]
Since $\|id : \ell_2^n \to \ell_2^n\| = \sqrt{n}$, applying the little Grothendieck theorem (p. 139, [5]) and the factorization property of dominated polynomials [15, Theorem 10] we obtain an upper bound for $\|P_n\|_{2-dom}$, namely
\[
\|P_n\|_{2-dom} \leq \frac{2}{\sqrt{\pi}} C_k n^{1/2} \sqrt{k} = \tilde{C}_k n^{1/2+k/2},
\]
where $\| : 2-sum$ denotes the 2-summing norm of an operator. For $2^{(k-1)/2} < d < 2^{(2k-3)/2}$ we define $Q_m = (2d)^{-m} P_{2^m} \in \mathcal{P}(k \ell_2^m)$, and obtain lower bounds for their nuclear norms:
\[
\|Q_m\|_{\mathcal{P}(k \ell_2^m)} \geq D_k (2d)^{-m} 2^{m(k-1/2)} > D_k 2^{-m} 2^{-m(k-1)/2} 2^{mk} 2^{-m/2} = D_k 2^{m(k/2-1)} \propto \sum_{m} \|Q_m\|_{2-dom} \leq \tilde{C}_k \sum_{m} (2d)^{-m} 2^{m(1/2+k/2)} < \tilde{C}_k \sum_{m} 2^{-m} 2^{-m(2k-3)/2} 2^{m/2} 2^{mk/2} = \tilde{C}_k \sum_{m} 2^{-m(k/2-1)} < \infty,
\]
and the space of $2$-dominated polynomials is complete in the $2$-dominated quasi-norm.

We end this section with an application of the polynomial Khintchine inequality to extend a result in [15] on dominated polynomials. Note that throughout this section, dominated polynomials were used to characterize some particular kind of
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convergence of random multi-indexed series. Now we take the opposite direction: we will use our results on $L_p$ convergence of random polynomials to obtain properties of dominated polynomials. Theorem 3 in [15] states that for $2 \leq p < \infty$ and $1 \leq r \leq p$, if the polynomial

$$P(x) = \sum_{i_1, \ldots, i_k = 1}^{\infty} a_{i_1, \ldots, i_k} x_{i_1} \cdots x_{i_k}$$

is $r$-dominated on $\ell_p$, then we have $\sum_{i_1, \ldots, i_k = 1}^{\infty} |a_{i_1, \ldots, i_k}|^2 < \infty$. We extend this result to any $r$ and to any Banach sequence space containing $\ell_2$. By a Banach sequence space we understand a Banach lattice over the natural numbers. If a Banach sequence space contains $\ell_2$ (in the sense that each element of $\ell_2$ is a sequence belonging to $E$), then a closed graph argument shows that the formal inclusion $i : \ell_2 \to E$ is continuous.

**Theorem 2.7.** Let $E$ be a Banach sequence space that contains $\ell_2$. If the polynomial

$$P(x) = \sum_{i_1, \ldots, i_k = 1}^{\infty} a_{i_1, \ldots, i_k} x_{i_1} \cdots x_{i_k}$$

on $E$ is $r$-dominated for some $1 \leq r < \infty$, then $\sum_{i_1, \ldots, i_k = 1}^{\infty} |a_{i_1, \ldots, i_k}|^2 < \infty$.

Note that hypotheses of the previous theorem hold for Lorentz sequence spaces $d(w, p)$ with $p \geq 2$ and for any 2-convex Banach sequence space (see [10, 11]).

3. INTEGRAL REPRESENTATION OF HOLOMORPHIC FUNCTIONS

In [20], the authors presented two integral representation formulas: for entire functions and for holomorphic functions on the unit ball of a Banach space. For this, they considered Gaussian measures on Banach spaces and the theory of abstract Wiener spaces. In [17] it is shown that many of results stated for real separable Banach spaces in [8] remains valid in the complex setting [19], which is crucial to our purposes. The papers [18] and [17] were concerned with the study of the classes of holomorphic functions which can be represented using those formulas. We will make use of only a few aspects of the theory. For the sake of completeness we outline the main facts, state the known results we need and prove some new ones. We refer to [6] for the theory of polynomials and analytic functions on infinite dimensional spaces.

Given a separable Hilbert space $H$, if $P$ is a finite-rank orthogonal projector in $H$, a cylinder set in $H$ is a set of the form

$$C = \{x \in H : Px \in \Delta\}$$

where $\Delta$ is a Borel subset of $PH$. We will denote by $\Gamma$ the Gaussian cylinder measure defined on cylinder sets:

$$\Gamma(C) = \frac{1}{\pi^n} \int_{\Delta} e^{-|w|^2} dw,$$
where \( n \) is the complex dimension of \( PH \), and the integral is with respect to the Lebesgue measure. This cylinder measure is not \( \sigma \)-additive, however, integrals of cylinder functions \( F : H \to \mathbb{C} \) of the form \( F = h \circ P \) may be defined by setting

\[
\int_C F \, d\Gamma = \int_\Delta h \, d\Gamma_n,
\]

where \( \Gamma_n \) is standard \( n \)-dimensional Gaussian measure. A norm \( \| . \| \) on \( H \) with the property that for any \( \varepsilon > 0 \) there is a finite-rank orthogonal projector \( P_\varepsilon \) such that for all \( P \perp P_\varepsilon \),

\[
\Gamma \{ x \in H : \| P x \| > \varepsilon \} < \varepsilon,
\]

is called measurable [8]. Examples of measurable norms can be constructed by considering Hilbert-Schmidt operators on \( H \). If \( S : H \to H \) is a injective Hilbert-Schmidt operator, then \( \| x \|_S = \| Sx \|^{1/2} \) is a measurable norm. Upon completing \((H, \| \cdot \|)\) one obtains a Banach space \( X \). The natural inclusion \( \iota : H \hookrightarrow X \) is continuous and dense, and \((\iota, H, X)\) is called an abstract Wiener space. A cylinder set \( C_X \) in \( X \) is one which can be described as

\[
C_X = \{ \gamma \in X : (\varphi_1(\gamma), \ldots, \varphi_n(\gamma)) \in \Delta \}
\]

where \( n \in \mathbb{N} \), \( \{ \varphi_k \}_{k=1}^n \subset X' \) and \( \Delta \) is a Borel set in \( \mathbb{C}^n \). For these sets one consider \( C_H = C_X \cap H \), and defines

\[
\tilde{\Gamma}(C_X) := \Gamma(C_H).
\]

The set function \( \tilde{\Gamma} \) extends to a measure \( W \) (called Wiener measure) on the Borel \( \sigma \)-algebra \( B \) of \( X \).

Since \( \iota' : X' \to H' \) has dense range, we can choose \( \{ z_n \}_{n \in \mathbb{N}} \subset X' \) such that the sequence \( \iota'(z_n) = e'_n \) defines an orthonormal basis \( \{ e'_n \}_{n \in \mathbb{N}} \) of \( H' \) dual to some basis \( \{ e_n \}_{n \in \mathbb{N}} \subset H \). The following proposition is an analogue of Corollary 4.1 (p. 57, [8]), where the real case is studied. Since there are not significative changes on the techniques involved for prove it, we omit the proof.

**Proposition 3.1.** With the previous notations, \( \{ z_n \}_{n=1}^\infty \) is a sequence of independent and identically distributed complex Gaussian random variables with mean 0 and variance 1. Moreover, given \( \varphi \in X' \), then \( \varphi \) is a complex gaussian variable with mean 0 and variance \( \| \iota' \varphi \|^2_{H'} \).

As usual, we can identify \( H' \) with \( H \) via \( I : H' \to H \), where for \( x \in H \) and \( \phi \in H' \), \( \phi(x) = \langle x, I(\phi) \rangle \). Since \( I \) is conjugate linear, in order to preserve analyticity it is necessary to define involutions in \( H \) and \( H' \). If \( x = \sum x_n e_n \) is an element of \( H \), we let \( x^* = \sum \overline{x_n} e_n \). Similarly, if \( \phi \in H' \), define \( \phi^* \) so that \( I(\phi^*) = I(\phi)^* \). Note that \( \langle x^*, y \rangle = \langle x, y^* \rangle \) and \( \phi(x^*) = \overline{\phi(x)} \).

The following diagram will be useful for fixing ideas:

\[
\begin{array}{ccc}
H & \xrightarrow{\iota} & X \\
\overline{\iota} \downarrow & & \downarrow A = \text{Id } \circ \iota' \\
H' & \xleftarrow{\iota'} & X'
\end{array}
\]
This general construction applies to the particular case when $H = \ell_2$ and, fixing a sequence of positive real numbers $(\lambda_n)_{n \in \mathbb{N}} \in \ell_2$, the measurable norm is given by the Hilbert-Schmidt operator

$$S : \ell_2 \to \ell_2$$

$$S((x_n)_n) = (\lambda_n x_n)_n.$$ 

In this way we obtain $\ell_2 \hookrightarrow (\ell_2, \| \cdot \|_s) = B_0 \subset \mathbb{C}^N$ and, since the finite dimensional projectors induce the Gaussian measures $\mu_n$ on the Borel sets of $\mathbb{C}^n$, we conclude that $\bar{\Gamma}$ extends to a measure $W$, which is the same measure that its existence is ensured by the Kolmogorov’s existence theorem. Also, the sequence $\{z_n\}_{n \in \mathbb{N}} \subset X'$ is explicitly determined by the set of linear functionals in $B_0'$, represented via the Riesz theorem by $\left\{ \frac{1}{\sqrt{n}} e_n \right\}_{n \in \mathbb{N}} \subset B_0$. With this choice $\{i'z_n\}_{n \in \mathbb{N}} = \{e_n'\}_{n \in \mathbb{N}}$, the dual basis of the standard orthonormal basis for $\ell_2$.

We do not need the integral formula in its general version, so we just state the following theorem.

**Theorem 3.2.** ([19, Teorema 3.2.7]) If $\{\varphi_j\}_{j=1}^k \subset B_0'$, then

$$\int_{B_0} e^{z(\gamma)} \prod_{j=1}^k \varphi_j(\gamma)dW(\gamma) = \prod_{j=1}^k \varphi_j(1 \circ I \circ i'(z)) \text{ for all } z \in B_0'.$$

Recall that a $k$–homogeneous polynomial $p$ defined on $B_0$ is of finite type if there exist $\{\varphi_j\}_{j=1}^N \subset B_0'$ such that $p(\gamma) = \sum_{j=1}^N \varphi_j(\gamma)$. This space of polynomials is denoted by $\mathcal{P}_f(kB_0)$. From the polarization formula, it can be seen that the product of $k$ different linear functionals is also of finite type.

It is possible to define on $\mathcal{P}_f(kB_0)$ the operator

$$T : \mathcal{P}_f(kB_0) \rightarrow \mathcal{P}_f(kB_0')$$

$$[T(p)](z) = \int_{B_0} e^{z(\gamma)} p(\gamma)dW(\gamma).$$

Since $i' : B_0' \rightarrow \ell_2'$ has dense rank, according to Theorem 3.2, there exists a unique $P \in \mathcal{P}(k\ell_2)$ such that the following diagram commutes:

$$\begin{array}{ccc}
\ell_2 & \xrightarrow{p} & \mathbb{C} \\
\downarrow & & \\
\ell_2' & \xleftarrow{i' \circ i} & B_0' & \xrightarrow{T(p)} \mathbb{C}
\end{array}$$

Namely, since

$$[T(p)](z) = p(i' \circ I \circ i'(z)) = p \circ i([I \circ i'(z)]^2),$$

we have

$$P(x) = p[i(x^*)].$$

From this “extension” property, we can define:

$$T : \mathcal{P}_f(kB_0) \rightarrow \mathcal{P}(k\ell_2)$$

$$[T(p)](x) = p[i(x^*)].$$
Note that we could have defined the operator $T$ independently of the integral representation formula. However, we will see below that this representation approach has some advantages.

We are ready to show the link between random variables and polynomials which will allow us to obtain the polynomial Khintchine inequalities. We will employ Proposition 3.5. The set of polynomials has some advantages.

Given a multi-index $\alpha \in \mathbb{N}_0^{(n)}$, with $|\alpha| = k$, set

$$z^{\alpha}(\gamma) = \prod_{j=1}^{\infty} [z_j(\gamma)]^{\alpha_j} \quad \text{and} \quad (i^* z)^{\alpha}(x) = \prod_{j=1}^{\infty} [e_j'(x)]^{\alpha_j}.$$  

Working on a separable Hilbert space, Dwyer [7] defined Hilbert-Schmidt $k$-functionals and O. Lopushansky and A. Zagorodnyuk study, in [12], the Hilbert space of $k$-homogeneous polynomials $\mathcal{P}_h(k H)$ over $H$, which is intimately related to the operator $T$. We need the following results from [12]:

**Proposition 3.3.** The inclusion $\mathcal{P}_h(k H) \hookrightarrow \mathcal{P}(k H)$ is continuous and $\|P\| \leq \|P\|_h$ for all $P \in \mathcal{P}_h(k H)$.

**Proposition 3.4.** Given $Q_n \in \mathcal{P}_h(n H)$ and $Q_m \in \mathcal{P}_h(m H)$, then we have that $Q_n Q_m \in \mathcal{P}_h(n+m H)$ and $\|Q_n Q_m\|_h \leq \|Q_n\|_h \|Q_m\|_h$.

**Proposition 3.5.** The set of polynomials $\left\{ \sqrt{\alpha!} (i^* z)^{\alpha} \right\}_{|\alpha|=k}$ forms an orthonormal basis of $\mathcal{P}_h(k H)$.

For simplicity of notation the closure of the span of $\{z^{\alpha}\}_{|\alpha|=k}$ in $L^2(W)$ will be denoted by $L^2_k(W)$. The Remark 1 in [18] states that it is possible to define an isomorphism $\tilde{T}: L^2_k(W) \rightarrow \mathcal{P}_h(k \ell^2_2)$, because $\left\{ \frac{z^{\alpha}}{\sqrt{\alpha!}} \right\}_{|\alpha|=k}$ and $\left\{ \frac{\sqrt{|\alpha|!}}{\sqrt{\alpha!}} (i^* z)^{\alpha} \right\}_{|\alpha|=k}$ are orthonormal bases of $L^2_k(W)$ and $\mathcal{P}_h(k \ell^2_2)$ respectively, and we have $T(z^{\alpha}) = (i^* z)^{\alpha}$. Moreover, $\|g_k\|_2 = \sqrt{k!} \|T(g_k)\|_h$ holds for any $g_k \in L^2_k(W)$.

Note that if we are given any linear combination of products of $k$ linear functionals, we can compute its $L^2$--norm in terms of the Hilbertian norm of the polynomial associated via $\tilde{T}$. So, if we think linear functionals as complex gaussian variables, we are able to compute $L^2$--norms of these linear combinations as Hilbertian norms of the associated polynomials.

### 4. The proofs

In this section we present the proofs of the results stated in Sections 1 and 2. In order to prove Proposition 1.1 it is convenient to state the following simple result:

**Lemma 4.1.** Let $X$ be an rotation-invariant complex random variable. Suppose that for some $k \in \mathbb{N}_0$, we have that $\mathbb{E} \left( |X|^{2k} \right) < \infty$, then

$$\mathbb{E} \left( X^m X^n \right) = \delta_{m,n} \mathbb{E} \left( |X|^{2m} \right) \quad \text{for all } m, n \leq k.$$  

In particular, $\mathbb{E}(X^m) = 0$ for $1 \leq m \leq k$. 

Proof: Let \( \theta \in (0, 2\pi) \), since \( e^{i\theta} X \) has the same distribution law than \( X \), it is a matter of fact that

\[
\mathbb{E} \left( X^m \overline{X}^n \right) = \mathbb{E} \left( [e^{i\theta} X]^m \overline{e^{i\theta} X}^n \right) = e^{i(m-n)\theta} \mathbb{E} \left( X^m \overline{X}^n \right).
\]

If \( m, n \leq k \), then \( \left| \mathbb{E} \left( X^m \overline{X}^n \right) \right| \leq \left[ \mathbb{E}(|X|^{2k}) \right]^{(m+n)/2k} < \infty \). So, for \( m \neq n \) we have that \( \mathbb{E} \left( X^m \overline{X}^n \right) \) must be 0, and we obtain the stated result. \( \square \)

**Proof of Proposition 1.1.**

We need to compute

\[
\mathbb{E} \left( |F_n|^2 \right) = \sum_{l_1, \ldots, l_k = 1}^{n \ldots j_k = 1} a_{l_1, \ldots, l_k} \overline{a}_{j_1, \ldots, j_k} \mathbb{E} \left( X_{l_1} \cdots X_{l_k} \overline{X}_{j_1} \cdots \overline{X}_{j_k} \right).
\]

Given \( J = (l_1, l_2, \ldots, l_k) \in \{1, 2, \ldots, n\}^k \), we define \( \mathcal{R}(J) = (\mathcal{R}(J)_m)_{1 \leq m \leq n} \) by \( \mathcal{R}(J)_m = \sum_{r=1}^{k} \delta_{m,l_r} \). This new multi-index counts how many times each number is repeated in \( J \).

Calling \( \alpha = \mathcal{R}(l_1, \ldots, l_k) \), \( \beta = \mathcal{R}(j_1, \ldots, j_k) \), and using the symmetry of the sequence \( \{a_{l_1, \ldots, l_k}\} \), we have

\[
\mathbb{E} \left( |F_n|^2 \right) = \sum_{|\alpha|=k} \sum_{|\beta|=k} \binom{k}{\alpha} \binom{k}{\beta} a_{|\alpha|} \overline{a}_{|\beta|} \mathbb{E} \left( X^\alpha \overline{X}^\beta \right),
\]

where \( X^\alpha \) stands for \( X_1^{\alpha_1} \cdots X_k^{\alpha_k} \). We can use the independence of \( \{X_j\}_{j \in \mathbb{N}} \) and then Lemma 4.1 to obtain:

\[
\mathbb{E} \left( |F_n|^2 \right) = \sum_{|\alpha|=k} \sum_{|\beta|=k} \binom{k}{\alpha} \binom{k}{\beta} a_{|\alpha|} \overline{a}_{|\beta|} \prod_{s=1}^{n} \mathbb{E} \left( X_s^{\alpha_s} \overline{X}_s^{\beta_s} \right) = \sum_{|\alpha|=k} \binom{k}{\alpha}^2 |a_{|\alpha|}|^2 \prod_{s=1}^{n} \mathbb{E} \left( |X_s|^{2\alpha_s} \right).
\]

For the left side inequality, observe that

\[
\prod_{s=1}^{n} \mathbb{E} \left( |X_s|^{2\alpha_s} \right) \geq \prod_{s=1}^{n} \mathbb{E} \left( |X_s| \right)^{2\alpha_s} \geq \left[ \inf_{j \in \mathbb{N}} \mathbb{E} \left( |X_j| \right) \right]^{2k}.
\]

The obvious estimation \( 1 \leq \binom{k}{\alpha} \leq k! \) gives the following:

\[
\mathbb{E} \left( |F_n|^2 \right) \geq \left[ \inf_{j \in \mathbb{N}} \mathbb{E} \left( |X_j| \right) \right]^{2k} \sum_{|\alpha|=k} \binom{k}{\alpha}^2 |a_{|\alpha|}|^2 = \left[ \inf_{j \in \mathbb{N}} \mathbb{E} \left( |X_j| \right) \right]^{2k} \left( \sum_{j_1, \ldots, j_k = 1}^{n} |a_{j_1, \ldots, j_k}|^2 \right).
\]

On the other hand, for any multi-index \( \alpha \), since \( |\alpha| = k \), at most \( k \) numbers of the set \( \{\alpha_1, \ldots, \alpha_n\} \) are different from 0. Moreover, none of them can be greater than
we have:

\[
E \left( |F_n|^2 \right) \leq \sum_{|\alpha| = k} k! \left( \frac{k}{|\alpha|} \right) |a_\alpha|^2 \left[ \sup_{j \in \mathbb{N}} E \left( |X_j|^{2k} \right) \right]^k
\]

\[
= k! \left[ \sup_{j \in \mathbb{N}} E \left( |X_j|^{2k} \right) \right]^k \left( \sum_{i_1, \ldots, i_k = 1}^n |a_{i_1, \ldots, i_k}|^2 \right).
\]

From both inequalities, we can take

\[
A_k^{-1} = \left[ \inf_{j \in \mathbb{N}} E \left( |X_j| \right) \right]^{1/k} \quad \text{and} \quad B_k = \sqrt{k!} \left[ \sup_{j \in \mathbb{N}} E \left( |X_j|^{2k} \right) \right]^{k/2}.
\]

If the variables are gaussian, since

\[
\|F_n\|_2^2 = \sum_{|\alpha| = k} \left( \frac{k}{|\alpha|} \right) |a_\alpha|^2 \prod_{s=1}^n \int_{\mathbb{C}} |X_s(\omega)|^{2\alpha_s} dP(\omega),
\]

we must compute

\[
\prod_{s=1}^n \int_{\Omega} |X_s(\omega)|^{2\alpha_s} dP(\omega) = \prod_{s=1}^n \int_{\mathbb{C}} |w|^{2\alpha_s} e^{-|w|^2} \frac{dw}{\pi}
\]

\[
= \prod_{s=1}^n \int_0^{2\pi} \int_0^{+\infty} \rho^{2\alpha_s+1} e^{-\rho^2} \frac{d\rho d\theta}{\pi} = \alpha!
\]

Then, \(\|F_n\|_2^2 = \sum_{|\alpha| = k} k! \left( \frac{k}{|\alpha|} \right) |a_\alpha|^2 = k! \left( \sum_{i_1, \ldots, i_k = 1}^n |a_{i_1, \ldots, i_k}|^2 \right)\).

In order to prove Theorem 1.2, we need the following lemma, for which we adapt some ideas from [21].

**Lemma 4.2.** Suppose that \(\{X_j\}_{j \in \mathbb{N}}\) is a sequence of independent and rotation-invariant complex random variables which, for some \(k > 1\), satisfies the \((\star)\)-condition. Then, there exists \(\varepsilon > 0\) such that for any sequence of complex numbers \(\{a_j\}_{j \in \mathbb{N}}\), we have

\[
P \left( \sum_{j=1}^n a_j X_j \right)^2 \geq \varepsilon^2 \sum_{j=1}^n |a_j|^2 \geq \varepsilon.
\]

**Proof.** By homogeneity, it is sufficient to prove that the inequality holds assuming that \(\sum_{j=1}^n |a_j|^2 = 1\).

We begin by proving the statement under the additional hypothesis that \(\{a_j\}_{j \in \mathbb{N}}\) is a sequence of real numbers. Since

\[
\left| \sum_{j=1}^n a_j X_j \right|^2 = \left( \Re \sum_{j=1}^n a_j X_j \right)^2 + \left( \Im \sum_{j=1}^n a_j X_j \right)^2
\]

\[
= \left( \sum_{j=1}^n a_j \Re X_j \right)^2 + \left( \sum_{j=1}^n a_j \Im X_j \right)^2
\]
it is enough to prove that

\[ P \left( \left( \sum_{j=1}^{n} a_j \text{Re} X_j \right)^2 \geq \varepsilon^2 \right) \geq \varepsilon. \]

Note that \(\text{Re} X_j = \text{Im}(e^{i \pi} X_j)\) and, using the rotational invariance of \(X_j\), we deduce that \(\text{Re} X_j\) and \(\text{Im} X_j\) are identically distributed. Since \(\mathbb{E}(|X_j|) \leq \mathbb{E}(|\text{Re} X_j|) + \mathbb{E}(|\text{Im} X_j|) = 2\mathbb{E}(|\text{Re} X_j|)\), we have that \(0 < \inf_{j \in \mathbb{N}} \mathbb{E}(|\text{Re} X_j|)\). In addition, since \(|\text{Re} X_j| \leq |X_j|\), it follows that \(\sup_{j \in \mathbb{N}} \mathbb{E}(|\text{Re} X_j|^{2k}) < \infty\).

If we can show that there exists \(\delta > 0\) such that \(P\left(|\text{Re} X_j| \geq \delta\right) \geq \delta\) for all \(j \in \mathbb{N}\), applying [21, Lemma 1], we conclude that there exists \(\varepsilon > 0\) such that

\[ P \left( \left| \sum_{j=1}^{n} a_j \text{Re} X_j \right| \geq \varepsilon \right) = \varepsilon. \]

Suppose that for any \(\delta > 0\), the inequality \(P\left(|\text{Re} X_j| \geq \delta\right) \geq \delta\) does not hold at least for some \(j \in \mathbb{N}\). In particular, choosing a sequence \(\{\delta_s\}_{s \in \mathbb{N}}\) such that \(\delta_s \to 0\), let \(X_j\) verifying \(P\left(|\text{Re} X_j| \geq \delta_s\right) < \delta_s\).

For each \(\delta_s\), and for suitable \(R_s > \delta_s\), we can write

\[
\mathbb{E}(|\text{Re} X_j|) = \int_0^{+\infty} P(|\text{Re} X_j| > t) \, dt = \int_0^{\delta_s} P(|\text{Re} X_j| > t) \, dt + \int_{\delta_s}^{R_s} P(|\text{Re} X_j| > t) \, dt + \int_{R_s}^{+\infty} P(|\text{Re} X_j| > t) \, dt.
\]

Obviously,

\[
\int_0^{\delta_s} P(|\text{Re} X_j| > t) \, dt \leq \delta_s.
\]

Using Chebyshev inequality,

\[
P\left(|\text{Re} X_j| > t\right) \leq \frac{\mathbb{E}\left(|\text{Re} X_j|^2\right)}{t^2} \leq \frac{\sup_{j \in \mathbb{N}} \mathbb{E}\left(|\text{Re} X_j|^2\right)}{t^2} = \frac{M}{t^2}.
\]

So, choosing \(R_s = \frac{M}{\delta_s^{1/2}}\), we obtain

\[
\int_{R_s}^{+\infty} P(|\text{Re} X_j| > t) \, dt \leq \int_{R_s}^{+\infty} \frac{M}{t^2} \, dt = \frac{M}{R_s} = \delta_s^{1/2}.
\]

Finally, since \(t \mapsto P(|\text{Re} X_j| > t)\) is a decreasing function, we have the following bound for the remaining integral:

\[
\int_{\delta_s}^{R_s} P(|\text{Re} X_j| > t) \, dt \leq P\left(|\text{Re} X_j| > \delta_s\right) (R_s - \delta_s) < \delta_s (R_s - \delta_s).
\]

Combining these inequalities,

\[
\mathbb{E}(|\text{Re} X_j|) \leq \delta_s + \delta_s^{1/2} + \delta_s (R_s - \delta_s) = \delta_s - \delta_s^2 + \delta_s^{1/2}(1 + M) \xrightarrow{s \to \infty} 0,
\]

which is a contradiction, because we know that \(\inf_{j \in \mathbb{N}} \mathbb{E}(|\text{Re} X_j|) > 0\).
Now, from [21, Lemma 1], we conclude that

\[
P \left( \sum_{j=1}^{n} a_j X_j \geq \varepsilon^2 \right) \geq P \left( \sum_{j=1}^{n} \Re X_j \geq \varepsilon \right)
\]

\[
= P \left( \sum_{j=1}^{n} \Re X_j \geq \varepsilon \right) \geq \varepsilon.
\]

If we are given any sequence \( \{a_j\}_{j \in \mathbb{N}} \subset \mathbb{C} \), writing \( a_j = |a_j| e^{i \arg(a_j)} \) and defining \( Y_j = e^{i \arg(a_j)} X_j \), we obtain a new sequence \( \{Y_j\}_{j \in \mathbb{N}} \) of independent and rotation-invariant complex random variables, which satisfies the \((*)\)-condition. Therefore,

\[
P \left( \sum_{j=1}^{n} a_j X_j \geq \varepsilon^2 \right) = P \left( \sum_{j=1}^{n} |a_j| e^{i \arg(a_j)} X_j \geq \varepsilon^2 \right)
\]

\[
= P \left( \sum_{j=1}^{n} |a_j| Y_j \geq \varepsilon^2 \right) \geq \varepsilon.
\]

\[\square\]

We also need the following result proved in [21, Lemma 2]:

**Lemma 4.3.** Let \( \{W_j\}_{j \in \mathbb{N}} \) be a sequence of random variables which verifies that there is a constant \( \delta > 0 \) such that

\[
P(|W_j| \geq \delta) \geq \delta,
\]

for all \( j \in \mathbb{N} \). Then, there exists \( \eta > 0 \) such that

\[
P \left( \sum_{j=1}^{n} c_j |W_j|^2 \geq \eta \sum_{j=1}^{n} c_j \right) \geq \eta,
\]

for every \( c_j \geq 0 \) and every \( n \in \mathbb{N} \).

**Proof of Theorem 1.2.**

Let us see that \((i)\) implies \((ii)\). For each \( n \), consider the random variable

\[
Z_n = \sum_{j_1, \ldots, j_k=1}^{n} a_{j_1, \ldots, j_k} X_{j_1} \cdots X_{j_k},
\]

and the \( \sigma \)-algebra \( \mathcal{F}_n = \sigma(X_1, \ldots, X_n) \). We can write

\[
Z_{n+1} = Z_n + X_{n+1} A_1 + X_{n+1}^2 A_2 + \cdots + X_{n+1}^{k-1} A_{k-1} + X_{n+1}^{k} \in \mathcal{F}_n.
\]

where \( Z_n, A_1, \ldots, A_{k-1} \) are \( \mathcal{F}_n \)-measurables. Then,

\[
E(Z_{n+1}|\mathcal{F}_n) = Z_n + A_1 E(X_{n+1}|\mathcal{F}_n) + A_2 E(X_{n+1}^2|\mathcal{F}_n) + \cdots
\]

\[
\cdots + A_{k-1} E(X_{n+1}^{k-1}|\mathcal{F}_n) + a_{n+1} E(X_{n+1}^k|\mathcal{F}_n).
\]

Since \( E(X_{n+1}^j|\mathcal{F}_n) = E(X_{n+1}^j) \), and from Lemma 4.1, \( E(X_{n+1}^j) = 0 \), for all \( j \geq 1 \), it follows that \( \{Z_n\}_n \) is a martingale relative to \( \{\mathcal{F}_n\}_n \).

By \((i)\), the martingale \( \{Z_n\}_n \) is bounded in \( L_2 \), hence it is bounded in \( L^1 \), and so it converges almost surely.
To prove that (iii) implies (iii), we can consider a sequence \( \{X_j\}_{j \in \mathbb{N}} \) of independent rotation-invariant complex random variables, verifying the \((*)\)-condition. We can identify
\[
\{Y_j^r\}_{j \in \mathbb{N}} \sim \{X_{(j-1)k+r}\}_{j \in \mathbb{N}} \quad \text{for } r = 1, 2, \ldots, k.
\]
Given \( k \) natural numbers \( \{l_1, l_2, \ldots, l_k\} \), let
\[
r_1 \equiv l_1 \mod(k), \ r_2 \equiv l_2 \mod(k), \ldots, \ r_k \equiv l_k \mod(k)
\]
for \( r_1, r_2, \ldots, r_k \in \{1, 2, \ldots, k\} \). Setting

- \( b_{l_1,\ldots,l_k} = 0 \) if \( \{r_1, r_2, \ldots, r_k\} \) is not a complete residue system modulo \( k \), or else,

- \( b_{l_1,\ldots,l_k} = \frac{a_{l_1(1)-r_1(1)+k} \cdots a_{l_k(1)-r_k(1)+k}}{k!}, \)

where \( \tau \) is a permutation of \( \{1, \ldots, k\} \) verifying \( 1 = r_{\tau(1)} < r_{\tau(2)} < \cdots < r_{\tau(k)} = k \).

Then, the multilinear random mapping
\[
\sum_{j_1,\ldots,j_k=1}^{n} \ a_{j_1,\ldots,j_k} Y_{j_1}^{1} \cdots Y_{j_k}^{k}
\]
can be viewed as the random polynomial
\[
\sum_{l_1,\ldots,l_k=1}^{nk} b_{l_1,\ldots,l_k} X_{l_1} \cdots X_{l_k}.
\]

Note that \( \{b_{l_1,\ldots,l_k}\}_{l_k \geq 1} \) is a symmetric sequence and, since
\[
\sum_{l_1,\ldots,l_k=1}^{nk} |b_{l_1,\ldots,l_k}|^2 = \frac{1}{k!} \sum_{j_1,\ldots,j_k=1}^{n} |a_{j_1,\ldots,j_k}|^2,
\]
we have \( \sum_{l_1,\ldots,l_k \geq 1} |b_{l_1,\ldots,l_k}|^2 < \infty \). Now, applying \((i) \Rightarrow (ii)\), we deduce that the random polynomial \( \sum_{l_1,\ldots,l_k=1}^{nk} b_{l_1,\ldots,l_k} X_{l_1} \cdots X_{l_k} \) is almost surely convergent and consequently \( \sum_{j_1,\ldots,j_k=1}^{n} a_{j_1,\ldots,j_k} Y_{j_1}^{1} \cdots Y_{j_k}^{k} \) is almost surely convergent too.

It only remains to prove that \((iii)\) implies \((i)\). We will prove by induction that, if \( \{Y_j^1\}_{j \in \mathbb{N}}, \ldots, \{Y_j^k\}_{j \in \mathbb{N}} \) are independent and rotation-invariant complex random variables satisfying the \((*)\)-condition, and \( \{a_{j_1,\ldots,j_k}\}_{j_k \geq 1} \) is a symmetric sequence of complex numbers, then there exists \( \delta > 0 \) such that, for all \( n \in \mathbb{N} \),
\[
P \left( \left\| \sum_{j_1,\ldots,j_k=1}^{n} a_{j_1,\ldots,j_k} Y_{j_1}^{1} \cdots Y_{j_k}^{k} \right\| \geq \delta \sum_{j_1,\ldots,j_k=1}^{n} |a_{j_1,\ldots,j_k}| \right) \geq \delta.
\]

It is clear that from this inequality the result would follow. For \( k = 1 \), if \( \{Y_j\}_{j \in \mathbb{N}} \) is a sequence satisfying the hypothesis, then from Lemma 4.2, there exists \( \varepsilon > 0 \) such that
\[
P \left( \left\| \sum_{i=1}^{n} a_i Y_i \right\| \geq \varepsilon \sum_{i=1}^{n} |a_i| \right) \geq \varepsilon.
\]

Suppose that the result is valid for \( k - 1 \) and let \( \{Y_j^1\}_{j \in \mathbb{N}}, \ldots, \{Y_j^k\}_{j \in \mathbb{N}} \) and \( \{a_{j_1,\ldots,j_k}\}_{j_k \geq 1} \) as in the statement. We have
\[
\sum_{j_1,\ldots,j_k=1}^{n} a_{j_1,\ldots,j_k} Y_{j_1}^{1} \cdots Y_{j_k}^{k} = \sum_{j_1=1}^{n} \left( \sum_{j_2,\ldots,j_k=1}^{n} a_{j_1,\ldots,j_k} Y_{j_2}^{2} \cdots Y_{j_k}^{k} \right) Y_{j_1}^{1}.
\]
If we fix \( Y_{j_2}^{2}, \ldots, Y_{j_k}^{k} \) for \( j_2, \ldots, j_k = 1, \ldots, n \), let \( c_{j_1} = \sum_{j_2,\ldots,j_k=1}^{n} a_{j_1,\ldots,j_k} Y_{j_2}^{2} \cdots Y_{j_k}^{k} \).
Then, as we have already seen,

\[(4.1) \quad P \left( \left| \sum_{j_1=1}^{n} c_{j_1} Y_{j_1} \right|^2 \geq \varepsilon^2 \sum_{j_1=1}^{n} |c_{j_1}|^2 \right) \geq \varepsilon. \]

Also, if we denote \( W_{j_1} = \sum_{j_2,...,j_k=1}^{n} a_{j_1,...,j_k} Y_{j_2}^2 \cdots Y_{j_k}^k \), we can write

\[
\sum_{j_1=1}^{n} \left| \sum_{j_2,...,j_k=1}^{n} a_{j_1,...,j_k} Y_{j_2}^2 \cdots Y_{j_k}^k \right|^2 = \sum_{j_1=1}^{n} \left( \sum_{j_2,...,j_k=1}^{n} |a_{j_1,...,j_k}|^2 \right) |W_{j_1}|^2.
\]

We have, by the inductive hypothesis, that the sequence \( \{W_{j_1}\}_{j_1} \) verifies the hypothesis of Lemma 4.3. Then, there exists \( \eta > 0 \) such that

\[(4.2) \quad P \left( \sum_{j_1=1}^{n} \left( \sum_{j_2,...,j_k=1}^{n} |a_{j_1,...,j_k}|^2 \right) |W_{j_1}|^2 \geq \eta \left( \sum_{j_1=1}^{n} |a_{j_1,...,j_k}|^2 \right) \right) \geq \eta.\]

From (4.1) and (4.2), the result follows with \( \delta = \varepsilon \eta. \)

**Proof of Theorem 1.3.**

Proposition 1.1 shows that \( \|F_n\|_2 = \sqrt{k!} \left( \sum_{i_1,...,i_k=1}^{n} |a_{i_1,...,i_k}|^2 \right)^{1/2}. \) Since for any \( q > 2 \), we have the inclusions

\[ L^q(\Omega_1, \mathfrak{A}_1, P_1) \subset L^2(\Omega_1, \mathfrak{A}_1, P_1) \subset L^1(\Omega_1, \mathfrak{A}_1, P_1), \]

it is sufficient to prove the left inequality for \( p = 1 \) and the right one for \( p > 2 \).

Let us first show that for \( p > 2 \) and \( n \in \mathbb{N} \), we have:

\[
\left\| \sum_{i_1,...,i_k=1}^{n} a_{i_1,...,i_k} X_{i_1} \cdots X_{i_k} \right\|_p \leq B_{k,p} \left( \sum_{i_1,...,i_k=1}^{n} |a_{i_1,...,i_k}|^2 \right)^{1/2}.
\]

If we define the function

\[
\Phi : B_0 \rightarrow \mathbb{C}
\]

\[
\Phi(\gamma) = \sum_{i_1,...,i_k=1}^{n} a_{i_1,...,i_k} z_{i_1}(\gamma) \cdots z_{i_k}(\gamma),
\]

as in the last theorem, \( F_n \) and \( \Phi \) are identically distributed, so it is enough to check

\[
\|\Phi\|_p \leq B_{k,p} \left( \sum_{i_1,...,i_k=1}^{n} |a_{i_1,...,i_k}|^2 \right)^{1/2}.
\]

Choose \( p = 2r \) for a natural number \( r > 1 \).

\[
\|\Phi\|_p = \left[ \int_{B_0} |\Phi(\gamma)|^{2r} dW(\gamma) \right]^{1/2r} = \|\Phi^r\|_2^{2/p}.
\]

As we noted in Section 3, we have a relation between \( \|\psi\|_2 \) and \( \|\tilde{T}\psi\|_h \) for \( \psi \in L^2_h \), so:

\[
\|\Phi\|_p = \|\Phi^r\|_2^{2/p} = \left[ (kr)! \|\tilde{T}(\Phi^r)\|_h^2 \right]^{1/p}.
\]
Using the fact that we know, from Corollary 3.2, how the integral transformation acts over the set of polynomials of finite type, we conclude:

\[
\left[(kr)! \|\bar{T}(\Phi')\|_h^2\right]^{1/p} = \left[(kr)! \|\bar{T}(\Phi)\|_h^2\right]^{1/p}.
\]

From Proposition 3.4,

\[
\left[(kr)! \|\bar{T}(\Phi')\|_h^2\right]^{1/p} \leq \left[(kr)! \|\bar{T}(\Phi)\|_h^2\right]^{1/p} = 2^{\nu} (kr)! \|\bar{T}(\Phi)\|_h^2,
\]

but this is equal to \(2^{\nu} \|\Phi\|_2\).

Since, as in Proposition 1.1, \(\|\Phi\|_2 = \sqrt{k!} \left(\sum_{i_1,\ldots,i_k = 1}^n |a_{i_1,\ldots,i_k}|^2\right)^{1/2}\), we conclude that:

\[
\left\|\sum_{i_1,\ldots,i_k = 1}^n a_{i_1,\ldots,i_k} X_1 \cdots X_k\right\|_p = \|\Phi\|_p \leq 2^{\nu} (kr)! \left(\sum_{i_1,\ldots,i_k = 1}^n |a_{i_1,\ldots,i_k}|^2\right)^{1/2}.
\]

It remains to prove the left inequality for \(p = 1\), but it is a consequence of Hölder inequality:

\[
\|F_n\|_2^2 = \int_{\Omega_1} |F_n(\omega)|^2 dP_1(\omega) = \int_{\Omega_1} |F_n(\omega)|^{2/3} |F_n(\omega)|^{4/3} dP_1(\omega)
\]

\[
\leq \left(\int_{\Omega_1} |F_n(\omega)|^{2/3} dP_1(\omega)\right)^{2/3} \left(\int_{\Omega_1} |F_n(\omega)|^{4/3} dP_1(\omega)\right)^{1/3}.
\]

Hence, we have

\[
\|F_n\|_2^2 = \left(\int_{\Omega_1} |F_n(\omega)|^2 dP_1(\omega)\right)^{3/2} \leq \|F_n\|_1 \|F_n\|_2^2 \leq B_{k,4}^2 \|F_n\|_1 \|F_n\|_p^2,
\]

which is \(\|F_n\|_2 \leq B_{k,4}^2 \|F_n\|_1 \leq B_{k,4}^2 \|F_n\|_p^2\) for \(p \geq 1\).

\(\square\)

**Proof of Corollary 1.4.**

We can construct a sequence \(\{X_i\}_{i \in \mathbb{N}}\) of independent standard complex gaussian variables, just as we did in the proof of Theorem 1.2, such that we can identify \(\{Z_i\}_{i \in \mathbb{N}} \sim \{X_{i(1-r)k+r}\}_{i \in \mathbb{N}}\) for \(r = 1, 2, \ldots, k\). Then, the multilinear random mapping \(G_n\) can be thought, for suitable \(\{a_{i_1,\ldots,i_k}\}_{j \geq 1}\), as the random polynomial

\[
F_n = \sum_{j_1,\ldots,j_k = 1}^{nk} a_{j_1,\ldots,j_k} X_{i_1} \cdots X_{i_k}.
\]

Therefore, \(\|F_n\|_p = \|G_n\|_p\) for any \(1 \leq p < \infty\), and since \(\{a_{i_1,\ldots,i_k}\}_{j \geq 1}\) is a symmetric sequence, we can apply Theorem 1.3, and conclude that

\[
\left(\sum_{j_1,\ldots,j_k = 1}^{nk} |a_{j_1,\ldots,j_k}|^2\right)^{1/2} \leq \frac{1}{\sqrt{k!}} \left(\sum_{i_1,\ldots,i_k = 1}^{n} |h_{i_1,\ldots,i_k}|^2\right)^{1/2},
\]

But,

\[
\left(\sum_{j_1,\ldots,j_k = 1}^{nk} |a_{j_1,\ldots,j_k}|^2\right)^{1/2} = \frac{1}{\sqrt{k!}} \left(\sum_{i_1,\ldots,i_k = 1}^{n} |h_{i_1,\ldots,i_k}|^2\right)^{1/2}.
\]
and we obtain the desired result.

In the proof of Theorem 1.6 we will use the following result, the proof of which is a simple exercise:

**Lemma 4.4.** Given two independent random variables \( \varphi \) and \( X \), \( \varphi \) a Steinhaus variable and \( X \) a standard complex gaussian variable, then \( Y = |\varphi X| \) is a standard complex gaussian variable.

**Proof of Theorem 1.6.**

Following the ideas of Theorem 1.3, it is sufficient to prove that for \( p \geq 2 \), we have:

\[
\left\| \sum_{i_1, \ldots, i_k = 1}^n a_{i_1, \ldots, i_k} \varphi_{i_1} \cdots \varphi_{i_k} \right\|_p \leq B_{k,p} \left( \sum_{i_1, \ldots, i_k = 1}^n |a_{i_1, \ldots, i_k}|^2 \right)^{1/2}.
\]

From Theorem 1.3, we know that for every sequence \( \{Z_i\}_{i \in \mathbb{N}} \) of independent standard complex gaussian variables,

\[
\left( \mathbb{E} \left( \left\| \sum_{i_1, \ldots, i_k = 1}^n b_{i_1, \ldots, i_k} Z_{i_1} \cdots Z_{i_k} \right\|^p \right) \right)^{1/p} \leq B_{k,p} \left( \sum_{i_1, \ldots, i_k = 1}^n |b_{i_1, \ldots, i_k}|^2 \right)^{1/2}.
\]

Take \( \{X_i\}_{i \in \mathbb{N}} \) a sequence of independent standard complex gaussian variables, which is independent from the sequence \( \{\varphi_i\}_{i \in \mathbb{N}} \) of Steinhaus random variables. We will consider a new symmetric sequence defined by

\[
b_{i_1, \ldots, i_k} = \frac{a_{i_1, \ldots, i_k}}{\mathbb{E}(|X_{i_1}| \cdots |X_{i_k}|)}.
\]

From Lemma 4.4, we have that \( \varphi_i |X_i| \sim Z_i \) for all \( i \geq 1 \). So, we can compute

\[
\mathbb{E} \left( \left\| \sum_{i_1, \ldots, i_k = 1}^n b_{i_1, \ldots, i_k} Z_{i_1} \cdots Z_{i_k} \right\|^p \right) = \int_{\Omega_1} \int_{\Omega_2} \left| \sum_{i_1, \ldots, i_k = 1}^n b_{i_1, \ldots, i_k} \prod_{j=1}^k \varphi_{i_j}(w_1) |X_{i_j}(w_2)| \right|^p d\mu_2(w_2) d\mu_1(w_1)
\]

\[
\geq \int_{\Omega_1} \left( \int_{\Omega_2} \left| \sum_{i_1, \ldots, i_k = 1}^n b_{i_1, \ldots, i_k} \prod_{j=1}^k \varphi_{i_j}(w_1) |X_{i_j}(w_2)| \right|^p d\mu_2(w_2) \right) d\mu_1(w_1)
\]

\[
\geq \int_{\Omega_1} \left( \int_{\Omega_2} \left| \sum_{i_1, \ldots, i_k = 1}^n b_{i_1, \ldots, i_k} \prod_{j=1}^k \varphi_{i_j}(w_1) |X_{i_j}(w_2)| \right|^p \right) d\mu_1(w_1)
\]

\[
= \int_{\Omega_2} \left| \sum_{i_1, \ldots, i_k = 1}^n b_{i_1, \ldots, i_k} \mathbb{E}(|X_{i_1} \cdots |X_{i_k}|) \varphi_{i_1}(w_1) \cdots \varphi_{i_k}(w_1) \right|^p d\mu_1(w_1)
\]

\[
= \mathbb{E} \left( \left\| \sum_{i_1, \ldots, i_k = 1}^n a_{i_1, \ldots, i_k} \varphi_{i_1} \cdots \varphi_{i_k} \right\|^p \right).
\]
Therefore,
\[
E \left( \left| \sum_{i_1, \ldots, i_k=1}^{n} a_{i_1, \ldots, i_k} \varphi_{i_1} \cdots \varphi_{i_k} \right|^p \right)^{1/p} \leq B_{k,p} \left( \sum_{i_1, \ldots, i_k=1}^{n} |a_{i_1, \ldots, i_k}|^2 \right)^{1/2} = B_{k,p} \left( \sum_{i_1, \ldots, i_k=1}^{n} \frac{|a_{i_1, \ldots, i_k}|^2}{\mathbb{E}^2(|X_{i_1}| \cdots |X_{i_k}|)} \right)^{1/2} \leq \min_{1 \leq i_1, \ldots, i_k \leq n} \frac{B_{k,p}}{\mathbb{E}(|X_{i_1}| \cdots |X_{i_k}|)} \left( \sum_{i_1, \ldots, i_k=1}^{n} |a_{i_1, \ldots, i_k}|^2 \right)^{1/2} .
\]

\[\square\]

**Proof of Theorem 2.1.**
Let us write \( T \) in its spectral decomposition:
\[
T(x) = \sum_j \lambda_j \langle x, e_j \rangle f_j,
\]
for suitable orthonormal bases \((e_j)_{j \in \mathbb{N}}\) and \((f_j)_{j \in \mathbb{N}}\).

From the following iterated limit:
\[
\lim_{N \to \infty} \left( \lim_{k \to \infty} e^{-\frac{1}{N} \sum_{j=1}^{k} \lambda_j^2 |x_j|^2} \right) = \begin{cases} 
\lim_{N \to \infty} e^{-\frac{1}{N} \sum_{j=1}^{k} \lambda_j^2 |x_j|^2} = 1 & x \in X_T \\
\lim_{N \to \infty} 0 = 0 & x \notin X_T 
\end{cases}
\]
we have:
\[
\lim_{N \to \infty} \left( \lim_{k \to \infty} e^{-\frac{1}{N} \sum_{j=1}^{k} \lambda_j^2 |x_j|^2} \right) = \chi_{X_T}(x).
\]

Fix \( N \in \mathbb{N} \), the sequence of functions
\[
g_{N,k} : \mathbb{R}^n \to \mathbb{R} \\
g_{N,k}(x) = e^{-\frac{1}{N} \sum_{j=1}^{k} \lambda_j^2 |x_j|^2}
\]
converges to the function
\[
g_N(x) = \begin{cases} 
e^{-\frac{1}{N} \sum_{j=1}^{k} \lambda_j^2 |x_j|^2} & x \in X_T \\
0 & x \notin X_T 
\end{cases}
\]
Since we have the bound \( e^{-\frac{1}{N} \sum_{j=1}^{k} \lambda_j^2 |x_j|^2} \leq 1 \) for all \( k \in \mathbb{N} \), applying the Lebesgue dominated convergence theorem, we conclude that:
\[
\int_{\mathbb{R}^n} g_N(x) \, d\mu(x) = \lim_{k \to \infty} \int_{\mathbb{R}^n} g_{N,k}(x) \, d\mu(x).
\]

Also, \( \{g_N(x)\}_{N \in \mathbb{N}} \) is a increasing sequence of non-negative functions converging to \( \chi_{X_T}(x) \). From the monotone convergence theorem we obtain:
\[
\int_{\mathbb{R}^n} \chi_{X_T}(x) \, d\mu(x) = \lim_{N \to \infty} \int_{\mathbb{R}^n} g_N(x) \, d\mu(x) = \lim_{N \to \infty} \left( \lim_{k \to \infty} \int_{\mathbb{R}^n} g_{N,k}(x) \, d\mu(x) \right).
\]
Since $g_{N,k}$ are cylinder functions, these integrals are computed as
\[
\lim_{N \to \infty} \left( \lim_{k \to \infty} \int_{\mathbb{R}^k} g_{N,k}(x) \, d\mu_k(x) \right) = \lim_{N \to \infty} \left( \lim_{k \to \infty} \prod_{j=1}^k e^{-\lambda_j^2|x_j|^2/N} \, d\mu_1(x_j) \right)
\]
Since $t \mapsto e^{-\lambda_j^2 t/N}$ is a convex function, from Jensen inequality we have that
\[
\int_{\mathbb{R}^k} e^{-\lambda_j^2|x_j|^2/N} \, d\mu_1(x_j) \geq e^{-\lambda_j^2 \|x_j\|^2} \, d\mu_1(x_j)/N = e^{-\lambda_j^2 \sigma^2/N}.
\]
From this we conclude that
\[
\lim_{N \to \infty} \left( \lim_{k \to \infty} \int_{\mathbb{R}^k} g_{N,k}(x) \, d\mu_k(x) \right) \geq \lim_{N \to \infty} \left( \lim_{k \to \infty} \prod_{j=1}^k e^{-\lambda_j^2 \sigma^2/N} \right) = 1
\]
Consequently, $\mu(X_T) = 1 = \mu(\mathbb{R}^N)$.

\[\square\]

**Proof of Theorem 2.2.**

First, we prove the equivalence between (i) and (ii). Let $X_T$ a standard full subspace such that $\sum_{i_k \leq N, \ldots, i_k \leq N_k} a_{i_1, \ldots, i_k} X_{i_1} \cdots X_{i_k}$ converges in $X_T$ to a polynomial $\tilde{P}$ as $N_1, \ldots, N_k \to \infty$. By the polynomial Banach-Steinhaus theorem, $\tilde{P}$ is a continuous $k$-homogeneous polynomial on $X_T$.

Let us define on $\tilde{T}(X_T) \subset \ell_2$ the following polynomial $g(\tilde{T}x) = \tilde{P}(x)$. Since
\[
|g(y)| = |g(\tilde{T}x)| = |\tilde{P}(x)| \leq \|\tilde{P}\| \|\|x\|\|^k = \|\tilde{P}\| \|\tilde{T}x\|_{\ell_2}^k = \|\tilde{P}\| \|y\|_{\ell_2}^k,
\]
we can continuously extend $g$ to $\ell_2$, preserving its norm.

Then, $\tilde{P} = g \circ \tilde{T}$ and, we can define $P : \ell_2 \to \mathbb{C}$ by $P = g \circ \tilde{T} \circ i$. Since $\tilde{T} \circ i = T$ and Hilbert-Schmidt operators on $\ell_2$ are absolutely 2-summing, it follows that $P = g \circ T$ is 2-dominated.

Conversely, being $P : \ell_2 \to \mathbb{C}$ 2-dominated, by [15, Theorem 14], there exist a regular Borel probability measure $\mu$ on $(\ell_2, w^*)$ and a $k$-homogeneous polynomial $Q : L_2(\mu) \to \mathbb{C}$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\ell_2 & \xrightarrow{P} & \mathbb{C} \\
 i_X & \downarrow & \downarrow Q \\
 C(\ell_2, w^*) & \xrightarrow{j_2} & L_2(\mu)
\end{array}
\]

The operator $j_2$ is absolutely 2-summing and so is $j_2 \circ i_X$. We can identify the image $j_2 \circ i_X(\ell_2) \subset L_2(\mu)$ with $\ell_2$ and thus we have an injective Hilbert-Schmidt operator $T : \ell_2 \to \ell_2$ that verifies $P = Q|\ell_2 \circ T$.

To see that the series converges in $X_T$ we have to show that there exists a continuous $k$-homogeneous polynomial $\tilde{P} : X_T \to \mathbb{C}$ that coincides with $P$ in $X_0$. Let $\tilde{P} = Q|\ell_2 \circ \tilde{T}$ and from
\[
|\tilde{P}(x)| = |Q(\tilde{T}x)| \leq \|Q\| \|\tilde{T}x\|_{\ell_2}^k \leq \|Q\| \|\|x\|\|^k,
\]
the result follows.

Using polarization formula, as in the proof of Theorem 1.8, it is clear that (i) implies (iii). That (iii) implies (iv) follows just as the implication from (i) to (ii). By [15, Theorem 6], (ii) and (iv) are equivalent, and this ends the proof. \[\square\]
Proof of Theorem 2.7.
Since any \( r \)-dominated polynomials is also \( r' \)-dominated for any \( r' > r \), we can assume \( r \geq n \). Take \( \{X_i\}_{i \in \mathbb{N}} \) a sequence of independent standard complex gaussian variables and put \( Z^n = (X_1, \ldots, X_n, 0, \ldots) \). For each \( n \in \mathbb{N} \) we can use Khintchine’s inequality with \( p = \frac{r}{n} \geq 1 \) to obtain:

\[
\left[ \sum_{i_1, \ldots, i_k = 1}^n |a_{i_1, \ldots, i_k}|^2 \right]^{\frac{1}{2}} \leq A_{k,p} \left[ \mathbb{E} (|P(Z^n)|^p) \right]^{\frac{1}{p}}
\]

\[
\leq A_{k,p} \|P\|_{r-dom} \left[ \mathbb{E} \left( \left[ \int_{B_{E'}} |\langle x', Z^n \rangle |^r d\mu(x') \right]^{\frac{np}{r}} \right]^{\frac{1}{p}}
\]

\[
= A_{k,p} \|P\|_{r-dom} \int_{B_{E'}} \mathbb{E} \left( |\langle x', Z^n \rangle |^r d\mu(x') \right) \frac{1}{p}
\]

\[
= A_{k,p} \|P\|_{r-dom} \int_{B_{E'}} \mathbb{E} \left( |\langle x', Z^n \rangle |^r d\mu(x') \right) \frac{1}{p}
\]

\[
= A_{k,p} \|P\|_{r-dom} \mathbb{E}(|X_1|^r) \frac{1}{p} \int_{B_{E'}} \mathbb{E} \left( |\langle x', Z^n \rangle |^r d\mu(x') \right) \frac{1}{p}
\]

Since the last bound is independent of \( n \), the result follows. \( \square \)

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References


