Some Open Probability Problems

Hermann Thorisson
Department of Mathematics
University of Iceland

This note presents four independent sets of open problems. The first set suggests an extension of the limit theory for positive recurrent renewal processes to the null recurrent case. The second concerns exact coupling of random walks on the line with step-lengths that are neither discrete nor spread-out. The third concerns the coupling characterization of setwise convergence of distributions of stochastic processes to a stationary limit. The fourth concerns characterizations of mass-stationarity, a concept formalizing the intuitive idea that the origin is a typical location in the mass of a random measure.

1 Null Recurrent Renewal Processes

Consider a renewal process

$$S_n = S_0 + X_1 + \cdots + X_n, \quad 0 \leq n < \infty,$$

where S_0 [the delay] is a nonnegative random variable and X_1, X_2, ... [the recurrence times] are i.i.d. strictly positive and independent of S_0. For $0 \leq t < \infty$, put

$$N_t = \inf\{n \geq 0 : S_n > t\} = \text{the number of renewals in } [0, t]$$

and

$$A_t = t - S_{N_t - 1} = \text{age at time } t,$$

$$B_t = S_{N_t} - t = \text{residual life at time } t,$$

$$D_t = X_{N_t} = A_t + B_t = \text{total life at time } t,$$

$$U_t = A_t/D_t = \text{relative age at time } t.$$

The renewal process is positive recurrent if $\mathbf{E}[X_1] < \infty$ and null recurrent if $\mathbf{E}[X_1] = \infty$. Assume that the distribution of X_1 is non-lattice, that is, $\mathbf{P}(X_1 \in d\mathbb{Z}) < 1$ for all $d > 0$; here $\mathbb{Z}$ denotes the integers. Let U be uniform on $[0, 1]$.

Theorem 1.1. If the renewal process is positive recurrent then U_t tends in distribution to U as $t \to \infty$.

For proof see e.g. [7], Chapter 2, Section 10.

Problem 1.1. Suppose the renewal process is null recurrent. Does U_t still tend in distribution to U as $t \to \infty$?

When the renewal process is positive recurrent, let D have the distribution function $\mathbf{P}(D \le x) = \mathbf{E}[X_1 1_{\{X_1 \le x\}}]/\mathbf{E}[X_1]$, $x > 0$.

Theorem 1.2. If the renewal process is positive recurrent, then D_t tends in total variation [and thus in distribution] to D as $t \to \infty$. On the other hand, if the renewal process is null recurrent, then D_t tends in distribution to infinity as $t \to \infty$.

For proof see e.g. [7], Chapter 2, Section 10.

Problem 1.2. Suppose the renewal process is null recurrent. Is there an non-decreasing function ϕ such that $D_t/\phi(t)$ tends in distribution to a non-degenerate random variable D^ϕ as $t \to \infty$? In particular, does this hold with $\phi(t) = \mathbf{E}[\min\{X_1, t\}]$?

Theorem 1.3. If the renewal process is positive recurrent, then the random pair (D_t, U_t) tends in distribution to (D, U) as $t \to \infty$, where D and U are as above and independent.

For proof see e.g. [7], Chapter 2, Section 10.

Problem 1.3. Suppose the renewal process is null recurrent and that the answers to the above questions are positive. Does $(D_t/\phi(t), U_t)$ then tend in distribution to (D^ϕ, U) as $t \to \infty$, where D^ϕ and U are as above and independent?

If all the answers are positive, then as a corollary we get that in the null-recurrent case $(A_t, B_t)/\phi(t)$ would tend in distribution to $(D^\phi U, D^\phi(1 - U))$ as $t \to \infty$.

2 Coupling of Random Walks on the Line

Let $S = (S_n)_0^\infty$ be a random walk on the line starting at 0, that is,

$$S_n = X_1 + \cdots + X_n, \quad 0 \le n < \infty,$$

where the step-lengths X_1, X_2, ... are i.i.d. Let $S' = (S'_n)_0^\infty$ be a version of S starting at $x \in \mathbb{R}$, that is,

$$S'_n = x + X'_1 + \cdots + X'_n, \quad 0 \le n < \infty,$$

where X'_1, X'_2, ... are i.i.d. with the same distribution as the step-lenghts of S. Say that the random walks admit exact coupling if they can be defined on the same probability space in such a way that there is an a.s. finite random integer T such that

$$S_n = S'_n \text{ for } n \ge T.$$

This holds if and only if $\|\mathbf{P}(S_n \in \cdot) - \mathbf{P}(S'_n \in \cdot)\| \to 0$ as $n \to \infty$ where $\| \cdot \|$ denotes the total variation norm; see e.g. [7], Chapter 4. The existence of exact coupling is known in the following two extreme cases.

The step-lengths are called spread out if there is an n such that the distribution of $X_1 + \cdots + X_n$ is non-singular with respect to Lebesgue measure.

Theorem 2.1. The random walks admit exact coupling for all $x \in \mathbb{R}$ **if and only if** the step-lengths are spread out.

For proof see e.g. [7], Chapter 3, Section 6.

When the step-lengths are discrete (lattice or non-lattice) put

$$A = \{a \in \mathbb{R} : \mathbf{P}(X_1 = a) > 0\}$$

and let G be the smallest additive subgroup of $\mathbb{R}$ containing $A - A$.

Theorem 2.2. Suppose the step-lengths are discrete. Then the random walks admit exact coupling **if and only if** $x \in G$.

For proof see [1].

In the strong lattice case when $A \subseteq d\mathbb{Z}$ for some $d > 0$ and there is an $a \in A$ such that $A - a$ is not contained in any sub-lattice of $d\mathbb{Z}$, then Theorem 2.2 implies that the random walks admit exact coupling if and only if $x \in d\mathbb{Z}$.

In the non-lattice case when for instance the step-lengths take only values in the rationals $\mathbb{Q}$ and each rational value is taken with positive probability, then Theorem 2.2 implies that the random walks admit exact coupling if and only if $x \in \mathbb{Q}$.

Problem 2.1. Suppose the step-lengths are neither discrete nor spread-out. For what initial positions x do the random walks admit exact coupling?

A special case where a solution to this problem is known is the following.

Theorem 2.3. Let $x \in \mathbb{R}$ be given. The random walks admit exact coupling **if** there is an n and a non-trivial measure ν such that $\mathbf{P}(X_1 + \cdots + X_n \in \cdot) \geq \nu + \nu(x + \cdot)$.

For proof, see [1].

3 Setwise Asymptotic Stationarity

Let $X = (X_k)_0^\infty$ and $X' = (X'_k)_0^\infty$ be two discrete-time stochastic processes on the same state space $(E, \mathcal{E})$. For $0 \leq n < \infty$, let θ_n be the shift-maps, that is, $\theta_n X := (X_{n+k})_{k=0}^\infty$. Let $\stackrel{D}{=}$ denote identity in distribution. The process X is stationary if $\theta_n X \stackrel{D}{=} X$ for all n. A set $A \in \mathcal{E}^\infty$ is a tail set if $A \in \theta_n^{-1}(\mathcal{E}^\infty)$ for all n, and invariant if $A = \theta_n^{-1} A$ for all n.

Say that X and X' admit distributional exact coupling if they can be defined on the same probability space in such a way that there are finite random integers T and T' such that $(\theta_T X, T) \stackrel{D}{=} (\theta_{T'} X', T')$. Say that X and X' admit distributional shift-coupling if they can be defined on the same probability space in such a way that there are finite random integers T and T' such that $\theta_T X \stackrel{D}{=} \theta_{T'} X'$.

Theorem 3.1. (a) The processes X and X' admit distributional exact coupling if and only if $\mathbf{P}(X \in A) = \mathbf{P}(X' \in A)$ for all tail sets A and if and only if

$$\mathbf{P}(\theta_n X \in \cdot) - \mathbf{P}(\theta_n X' \in \cdot) \to 0 \text{ in total variation as } n \to \infty.$$

(b) The processes X and X' admit distributional shift-coupling if and only if $\mathbf{P}(X \in A) = \mathbf{P}(X' \in A)$ for all invariant sets A and if and only if

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathbf{P}(\theta_k X \in \cdot) - \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{P}(\theta_k X' \in \cdot) \to 0 \text{ in total variation as } n \to \infty.$$

For proof see [7], Chapters 4 and 5.

Corollary 3.1. If

$$\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{P}(\theta_k X \in A) \to \mathbf{P}(X' \in A) \text{ as } n \to \infty \text{ for all } A \in \mathcal{E}^\infty \tag{3.1}$$

then the limit process X' is stationary, X and X' admit distributional shift-coupling, $\mathbf{P}(X \in A) = \mathbf{P}(X' \in A)$ for all invariant sets A, and

$$\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{P}(\theta_k X \in \cdot) \to \mathbf{P}(X' \in \cdot) \text{ in total variation as } n \to \infty.$$

Proof. The stationarity of X' follows by taking $A \in \mathcal{E}^\infty$ and noting that (3.1) implies the first step in (sending $n \to \infty$)

$$\mathbf{P}(\theta_1 X' \in A) - \mathbf{P}(X' \in A) \leftarrow \frac{1}{n}\sum_{k=0}^{n-1}(\mathbf{P}(\theta_{k+1} X \in A) - \mathbf{P}(\theta_k X \in A))$$

$$= \frac{1}{n}(\mathbf{P}(\theta_n X \in A) - \mathbf{P}(X \in A)) \to 0.$$

Further, (3.1) with A invariant yields $\mathbf{P}(X \in A) = \mathbf{P}(X' \in A)$ and the rest of the corollary follows from Theorem 3.1(b). $\square$

Problem 3.1. If

$$\mathbf{P}(\theta_n X \in A) \to \mathbf{P}(X' \in A) \text{ as } n \to \infty \text{ for all } A \in \mathcal{E}^\infty \tag{3.2}$$

then again X' is stationary. But does it hold that

$$\mathbf{P}(\theta_n X \in \cdot) \to \mathbf{P}(X' \in \cdot) \text{ in total variation as } n \to \infty\,?$$

I guess the answer is negative. However, if it turns out to be positive then the setwise convergence (3.2) would be characterized by X and X' admiting distributional exact coupling.

Problem 3.2. Suppose the answer to the question in Problem 3.1 is negative. What then is the coupling characterization of the convergence (3.2)?

Finally, let E is separable metric with $\mathcal{E}$ its Borel subsets and let d be the product metric on E^∞. Suppose (3.2) only holds for A such that $\mathbf{P}(X \in \text{the boundary of } A) = 0$, that is, suppose $\theta_n X$ converges in distribution to X'. The well-known Skorohod-coupling characterization of this convergence is the following:

> there are processes $X^{(n)}$, $n \geq 0$, such that $X^{(n)} \stackrel{D}{=} \theta_n X$ for each n and
>
> the pointwise limit $\lim_{n\to\infty} X^{(n)}$ exists and has the same distribution as X'.

Problem 3.3. Suppose $\theta_n X$ converges in distribution to X'. Is there a coupling characterization of this convergence which only involves joint construction of the two process X and X' and not a whole family of processes? For instance, can X and X' be constructed on the same probability space in such a way that $d(\theta_n X, \theta_n X') \to 0$ as $n \to \infty$?

4 Mass-stationarity and Allocations

Let ξ be a random measure on a locally compact second countable Abelian group G, for instance $G = \mathbb{R}^d$. Let X be a random element in a space on which G acts, for instance a random field $X = (X_s)_{s \in G}$. The pair (X, ξ) is called stationary if the distribution of (X, ξ) is invariant under deterministic shifts of the origin, that is, if $s(X, \xi) \stackrel{D}{=} (X, \xi)$ for all $s \in G$. Stationarity means intuitively that the origin is a *typical* location in G.

The pair (X, ξ) is called mass-stationary if for all relatively compact Borel subsets C of G,

$$V_C(X, \xi, U_C) \stackrel{d}{=} (X, \xi, U_C)$$

where

$$U_C \text{ is uniform on } C \text{ and independent of } (X, \xi),$$

$$V_C \text{ given } (X, \xi, U_C) \text{ has distribution } \xi(\cdot | C - U_C).$$

Theorem 4.1. The pair (X, ξ) is mass-stationary if and only if it is the Palm version of a stationary pair.

For proof see [5].

Mass-stationarity formalizes the intuitive idea that the origin is a *typical* location in the *mass* of G. A simple example of a mass-stationary random measure is the stationary Poisson process on the line with an extra point added at the origin: shifting the origin to the n^{th} point on the right, – or to the n^{th} point on the left, – does not change the fact that the inter-point distances are i.i.d. exponential. Compared to the simplicity of this example, the above definition of mass-stationarity is rather mysterious. However, for simple point processes it has the following more transparent formulation.

Theorem 4.2. Suppose ξ is a simple point process. Then (X, ξ) is mass-stationary if and only if

$$\Pi(X, \xi) \stackrel{D}{=} (X, \xi) \tag{4.1}$$

for all $\Pi = \pi(X, \xi)$ where π is a measurable G-valued map such that the G-to-G map

$$\tau_{(X,\xi)} : s \mapsto s + \pi(s(X, \xi)) \qquad \text{[with } (X, \xi) \text{ fixed]}$$

preserves the measure ξ, that is, $\xi(\tau_{(X,\xi)} \in \cdot) = \xi$.

For proof see [4].

In the Poisson example we could let $\tau_{(X,\xi)}$ be the map taking each point to the n^{th} point on its right (or on its left) and leaving the other locations where they are. This $\tau_{(X,\xi)}$ maps the point process into itself.

A $\tau_{(X,\xi)}$ as in the above theorem is called a preserving allocation. In general, mass-stationarity cannot be characterized by the property (4.1); see [5]. The counter-example is based on the observation that there could be discrete point-masses of different sizes and an allocation cannot split a discrete point-mass. On the other hand the following is known.

Theorem 4.3. Suppose ξ is diffuse and $G = \mathbb{R}^d$. Then (X, ξ) is mass-stationarity if and only if (4.1) holds with X replaced by (X, Y) for all shift-measurable stationary random fields $Y = (Y_s)_{s \in G}$ that are independent of (X, ξ).

For proof see [6].

Problem 4.1. Suppose ξ is diffuse. Is then mass-stationarity of (X, ξ) equivalent to (4.1) holding without the addition of the stationary independent Y?

Note that if τ is a preserving allocation then $Q_{(X,\xi)}(s, A) := \delta_{\tau_{(X,\xi)}(s)}(A)$ defines a **Markovian** kernel $Q_{(X,\xi)}$ on $G \times G$ which is invariant under joint shifts of s and A. Moreover, $Q_{(X,\xi)}$ is preserving, that is, $\xi Q = \xi$. This particular kernel is trivial in the sense that $Q_{(X,\xi)}(s, \cdot)$ is just the unit-mass at $\tau_{(X,\xi)}(s)$.

Theorem 4.4. The pair (X, ξ) is mass-stationary if and only if for all **bounded** jointly invariant preserving $G \times G$ kernels $Q_{(X,\xi)}$ it holds that

$$\int_G \mathbf{P}(s(X, \xi) \in \cdot) Q_{(X,\xi)}(0, ds) = \mathbf{P}((X, \xi) \in \cdot) \tag{4.2}$$

For proof see [5].

If $Q_{(X,\xi)}$ is not only bounded but has total mass one [is Markovian] then we can introduce a random element Π in G with conditional distribution $Q_{(X,\xi)}(0, \cdot)$ given (X, ξ) and write (4.2) on the probabilistic form (4.1).

Problem 4.2. Does Theorem 4.4. hold if the kernels $Q_{(X,\xi)}$ are restricted to be Markovian?

Problem 4.1 and 4.2 were already posed in [5].

References

[1] Arnaldsson, O. (2010). *On Coupling of Random Walks on the Line.* Master Thesis, Department of Mathemtics, University of Iceland.

[2] Asmussen, S. (2003). *Applied Probability and Queues.* Second edition. Springer.

[3] Heveling, M. and Last, G. (2005). Characterization of Palm measures via bijective point-shifts. *Annals of Probability* **33**, 1698-1715.

[4] Heveling, M. and Last, G. (2007). Point shift characterization of Palm measures on Abelian groups. *Electronic Journal of Probability* **12**, 122-137.

[5] Last, G. and Thorisson, H. (2009). Invariant transports of stationary random measures and mass-stationarity. *Annals of Probability* **37**, 790–813.

[6] Last, G. and Thorisson, H. (2010). Construction of stationary and mass-stationary random measures in $\mathbb{R}^d$. In preparation.

[7] Thorisson, H. (2000). *Coupling, Stationarity, and Regeneration.* Springer, New York.