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On a local L^2 -variant of Ikehara's theorem

BY NORBERT WIENER AND AUREL WINTNER *

1. A fundamental theorem of Vivanti and Pringsheim states that if

$$(1) \quad a_n \geq 0$$

holds for all coefficients of a power series

$$(2) \quad p(z) = \sum_{n=0}^{\infty} a_n z^n$$

which converges for $|z| < 1$, then either the point $z = 1$ or no point of the circumference $|z| = 1$ is a singular point of the function $p(z)$. Clearly, the theorem becomes false if (1) is relaxed to

$$(3) \quad s_n \geq 0, \quad \text{where} \quad s_n = \sum_{k=0}^n a_k$$

(for, by changing the value of a_0 alone, $|s_n| < \text{const.}$ can be reduced to (3), whereas trivial examples show $|s_n| < \text{const.}$ is certainly not sufficient for the alternative of Vivanti-Pringsheim). Still less is it possible to relax (1) to the existence of some Cesàro index $m (= 1, 2, \dots)$ satisfying, for the series $\sum a_n$, the condition

$$(4) \quad S_n^m \geq 0 \quad (\text{as } n \rightarrow \infty, \text{ while } m \text{ is fixed}),$$

since (3) is equivalent to the case $m = 0$ of (4).

The Vivanti-Pringsheim theorem has a certain variant which, as an application of his Fourier methods in general Tauberian theory (cf., in particular, the proof of Ikehara's theorem in [2], § 19), one of us proved, but did not publish, some time ago. This theorem (quoted, but not proved, in [1], p. 242 and p. 250, item 12.6f) states that if (1) holds for the coefficients of a power series (2) which con-

* Received Sept. 5-1957.

verges for $|z| < 1$ and is of class L^2 on *some* (no matter how narrow) sector containing the segment $0 \leq z < 1$, then it is of class L^2 on the entire circle. By this is meant that if there exists on *some* arc $-\varepsilon \leq \theta \leq \varepsilon$ of the circumference $|z| = 1$ a measurable function $p_0(\theta)$ satisfying

$$(5) \quad \int_{-\varepsilon}^{\varepsilon} |p(re^{i\theta}) - p_0(e^{i\theta})|^2 d\theta \rightarrow 0 \text{ as } r \rightarrow 1,$$

with $z = re^{i\theta}$ (and $r < 1$) in (2), and if (1) is assumed, then $p_0(\theta)$ can be extended from $[-\varepsilon, \varepsilon]$ to $[-\pi, \pi]$ in such a way that

$$(6) \quad \int_{-\pi}^{\pi} |p(re^{i\theta}) - p_0(e^{i\theta})|^2 d\theta \rightarrow 0 \text{ as } r \rightarrow 1$$

will hold (it being understood that $p_0(e^{i\theta})$ is of class L^2 on $[-\pi, \pi]$).

2. It will be shown in this paper that, in contrast with the Vivanti-Pringsheim theorem itself, *the L^2 -variant can be generalized so as to relax (1) to (3), and even to (4).*

The theorem remains true if the power series (2) is replaced by a Laplace integral and (3) or (4) is adjusted to this general case. We shall give the proof for this more general case. This will have, among other things, the advantage of exhibiting the parallelism with the proof of Ikehara's theorem directly.

3. Let $f(s)$ be a function which is regular in the half-plane $\sigma > 1$, where $s = \sigma + it$, and suppose that there exists on a fixed t -interval $[-a, a]$ a function $f_0(t)$ of class L^2 satisfying

$$(7) \quad \int_{-a}^a |f(\sigma + it) - f_0(t)|^2 dt \text{ as } \sigma \rightarrow 1$$

(by $\sigma \rightarrow 1$ is meant $\sigma \rightarrow 1 + 0$). Then $f(s)$ will be called *of class $L^2(a)$* . In view of the completeness of the space of the L^2 -functions on a t -interval $[-a, a]$, condition (7) is equivalent to

$$(8) \quad \int_{-a}^a |f(\sigma_1 + it) - f(\sigma_2 + it)|^2 dt \rightarrow 0 \text{ as } (\sigma_1, \sigma_2) \rightarrow (1, 1),$$

where $\sigma_1 (> 1)$ and $\sigma_2 (> 1)$ are independent variables.

In this terminology, the main theorem, to be proved, can be formulated as follows:

Let $\mu(u)$, where $0 \leq u < \infty$, be of bounded variation on every finite interval $0 \leq u \leq U (< \infty)$, suppose that the Laplace-Stieltjes transform,

$$(9) \quad f(s) = \int_0^{\infty} e^{-su} d\mu(u),$$

of μ is convergent (but, possibly, not absolutely convergent) on the half-plane $\sigma > 1$, where $s = \sigma + it$, and let

$$(10) \quad \mu(0) = 0 \text{ and } \mu(u) \geq 0, \text{ where } 0 < u < \infty$$

(but

$$(10 \text{ bis}) \quad d\mu(u) \geq 0, \text{ where } 0 \leq u < \infty,$$

is not assumed). Then the function $f(s)$ cannot be of class $L^2(a)$ for any $a = \varepsilon$ unless it is of class $L^2(a)$ for every $a = N$.

In contrast, the proof of Ikehara's theorem ([2], § 19), which assumes (10 bis), cannot be improved so as to relax (10 bis) to (10). In this connection, cf. [3].

4. Conditions (10) and (10 bis) correspond to (3) and (1) respectively, if (9) is identified with (2), by choosing $\mu(u)$ to be a step-function and placing $z = e^{-s}$ (except that the radius of convergence R of (2) then becomes $R = L/e$, instead of $R = 1$).

Correspondingly, the generalization (4) of (3) results if the assumption $\mu(u) \geq 0$ is relaxed to the assumption that there exists some m satisfying $\mu_m(u) \geq 0$ (for large u), where

$$\mu_m(u) = \int_0^u \mu_{m-1}(v) dv, \quad \mu_1(u) = \mu(u).$$

But it will be clear from the proof that the case of an arbitrary m can be treated in the same way as the case $m = 1$, if the partial integration, leading from (9) to (13) and (14) below, is applied m times. For this reason, it will be sufficient to deal with the case of the assumption (10), where $m = 1$ (the assumption (10 bis) would correspond to $m = 0$).

5. In the proof of the theorem, it can be assumed that the given α -value, the value for which $f(s)$ is supposed to be of class $L^2(a)$, is $\alpha = 1$ (this normalization is accomplished by a change of the unit of length on the t -axis). Then the assumption is that

$$(11) \quad \int_{-1}^1 |f(1 + \varepsilon + it) - f(1 + \eta + it)|^2 dt \rightarrow 0 \text{ as } (\varepsilon, \eta) \rightarrow (0, 0)$$

(where $1 + \varepsilon = \sigma_1$ and $1 + \eta = \sigma_2$ in the earlier notations), and the assertion is that

$$\int_{-a}^a |f(1 + \varepsilon + it) - f(1 + \eta + it)|^2 dt \rightarrow 0$$

must hold for every fixed positive $a < \infty$.

To this end, it will be sufficient to show that

$$(12) \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(1 + \varepsilon + it + ic) - f(1 + \eta + it + ic)|^2 dt \rightarrow 0$$

holds for every real c . In fact, (12) can be written in the form

$$\int_{-\frac{1}{2}+c}^{\frac{1}{2}+c} |f(1 + \varepsilon + it) - f(1 + \eta + it)|^2 dt \rightarrow 0.$$

If this is applied to $c = 3/2$ and to $c = -3/2$, and if both of the resulting relations are added to (11), it follows that (11) remains true if its $[-1, 1]$ is increased to $[-2, 2]$.

Clearly, a repetition of this process leads to $[-N, N]$ for any fixed $N > 0$. This is the reason why it will be sufficient to prove that (11) leads to (12) for any fixed real c .

6. If the integral (9) is convergent for $\sigma > 1$, and if, without loss of generality, $\mu(0) = 0$ in (9), then a partial integration shows that

$$(13) \quad f(s) = sF(s),$$

where

$$(14) \quad F(s) = \int_0^{\infty} e^{-su} \mu(u) du,$$

the convergence of the integral (14) for $\sigma > 1$ being part of the statement. [It is worth emphasizing that, whereas the condition (10), which was not used here, and the convergence of (9) for $\sigma > 1$ do not imply the absolute convergence of (9) for $\sigma > 1$, they do imply the absolute convergence of (14) for $\sigma > 1$.]

Since both s and $1/s$ are regular and bounded on every $(\sigma + it)$ -rectangle of the form $1 < \sigma < 2$, $-N < t < N$, it is clear from (13) that (11) and (12) are equivalent to

$$(15) \quad \int_{-1}^1 |F(1 + \varepsilon + it) - F(1 + \eta + it)|^2 dt \rightarrow 0$$

and

$$(16) \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} |F(1 + \varepsilon + it + ic) - F(1 + \eta + it + ic)|^2 dt \rightarrow 0$$

respectively. Hence the assertion of the theorem is equivalent to the statement that, if (10) is assumed, (15) leads to (16) for every real c .

Actually, (16) will prove to be true, not only for every fixed c , but *uniformly* on the *infinite* range $-\infty < c < \infty$. But this additional information is immaterial in the proof of the theorem.

7. The beginning of the proof is about the same as that of Ikebara's theorem; it proceeds as follows:

For any fixed $\sigma > 1$ and for any fixed real c , put

$$F_\sigma(t; c) = (1 - |t|) F(\sigma + it + ic) \text{ if } |t| \leq 1, \quad F_\sigma(t; c) = 0 \text{ if } |t| \geq 1,$$

where $-\infty < t < \infty$, and also put

$$G_\sigma(u; c) = e^{-\sigma u} \mu(u) e^{-icu} \text{ if } u \geq 0, \quad G_\sigma(u; c) = 0 \text{ if } u \leq 0,$$

where $-\infty < u < \infty$ (and $\mu(0) = 0$). Then, since the function

$$(17) \quad S(u) = \left(\sin \frac{1}{2} u \right)^2 / \left(\frac{1}{2} u \right)^2, \quad (S(0) = 1),$$

where $-\infty < u < \infty$, is a positive constant multiple of the Fourier transform of $\max(0, 1 - |t|)$, where $-\infty < t < \infty$, it is readily verified from (14), where $\mu(u) \geq 0$, that the Fourier transform of the function $F_\sigma(t; c)$ of t is the convolution $S(u) \star G_\sigma(u; c)$ (if a

normalizing constant factor, such as $(2\pi)^{-\frac{1}{2}}$, is disregarded). In view of the definitions of $F_\sigma(t; c)$ and $G_\sigma(u; c)$, this means that the identity

$$(18) \quad \int_{-1}^1 e^{ixt} (1 - |t|) F(\sigma + it + ic) dt = H_\sigma(x; c),$$

where

$$(19) \quad H_\sigma(x; c) = \int_0^\infty e^{-\sigma u} \mu(u) e^{-icu} \mathcal{S}(x - u) du,$$

holds for $\sigma > 1$, $-\infty < x < \infty$, $-\infty < c < \infty$ (to a neglected positive constant factor); and that, by Plancherel's relation,

$$\int_{-1}^1 (1 - |t|)^2 |F(\sigma + it + ic)|^2 dt = \int_{-\infty}^\infty |H_\sigma(x; c)|^2 dx$$

(to a constant factor, the square of the preceding constant factor).

If all of this is applied, not to the functions belonging to one $\sigma > 1$, but to the difference of the functions belonging to two σ values, $\sigma = 1 + \varepsilon$ and $\sigma = 1 + \eta$, where $\varepsilon > 0$ and $\eta > 0$, then what results is that the expression

$$(20) \quad \int_{-1}^1 (1 - |t|)^2 |F(1 + \varepsilon + it + ic) - F(1 + \eta + it + ic)|^2 dt$$

is a constant multiple of the expression

$$(21) \quad \int_{-\infty}^\infty |H_{1+\varepsilon}(x; c) - H_{1+\eta}(x; c)|^2 dx.$$

8. Since $(1 - |t|)^2$ has a positive minimum on the interval $-\frac{1}{2} \leq t \leq \frac{1}{2}$, it is clear that the integral on the left of the relation (16) is majorized by a constant multiple of the integral (20). Hence, in order to prove that the relation (16) holds for every fixed c , it is sufficient to show that the function (21) of $(\varepsilon, \eta; c)$ tends to 0 as $(\varepsilon, \eta) \rightarrow (0, 0)$. But this property of (21) can readily be concluded (as a matter of fact, *uniformly* for $-\infty < c < \infty$), as follows:

According to (10) and (17), both functions μ , S are non-negative. Hence it is clear from (19) that

$$\begin{aligned} & |H_{1+\varepsilon}(x; c) - H_{1+\eta}(x; c)| \\ & \leq \left| \int_0^{\infty} (e^{-(1+\varepsilon)u} - e^{-(1+\eta)u}) \mu(u) S(x-u) |e^{-icu}| du \right| \end{aligned}$$

(whether $\varepsilon > \eta$, $\varepsilon < \eta$ or $\varepsilon = \eta$). But since $|e^{-icu}| = 1$, it is also seen from (19) that the last inequality can be written in the form

$$|H_{1+\varepsilon}(x; c) - H_{1+\eta}(x; c)| \leq |H_{1+\varepsilon}(x; 0) - H_{1+\eta}(x; 0)|.$$

Hence, if (ε, η) is fixed, the function (21) of c is majorized by the value attained by (21) at $c = 0$.

Consequently, if (ε, η) is fixed, the function (21) of c is majorized by a constant multiple of the value attained by (20) at $c = 0$, and the constant factor is independent of (ε, η) . But since $(1 - |t|)^2 \leq 1$ if $-1 \leq t \leq 1$, the value of (20) at $c = 0$ is majorized by the integral occurring in (15). Since the limit relation (15), where $(\varepsilon, \eta) \rightarrow (0, 0)$, is assumed this proves that the function (21) of $(\varepsilon, \eta; c)$ tends to 0 as $(\varepsilon, \eta) \rightarrow (0, 0)$, uniformly for $-\infty < c < \infty$.

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Anwendung dualer Quaternionen auf Kinematik

Von WILHELM BLASCHKE (Hamburg) *

Im Folgenden soll gezeigt werden, wie einfach sich die Formeln für die Kinematik des Euklidischen Raumes gestalten lassen, wenn man im Anschluss an EULER die Drehungen mittels der Quaternionen darstellt und dazu noch die sogenannten « dualen Zahlen ».

$$a + \varepsilon b, \quad \varepsilon^2 = 0$$

zur Hilfe nimmt.

Einen Teil des Folgenden habe ich an der Universität in Buenos Aires im Mai 1957 vorgetragen und ich möchte diese Gelegenheit benutzen, um meinen argentinischen Freunden und Kollegen für ihre Gastfreundschaft herzlichst zu danken.

§ 1. QUATERNIONEN

Es seien die q_i zunächst reelle Zahlen, dann schreiben wir eine Quaternion in der Gestalt

$$\mathbf{Q} = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3,$$

und erklären Addition und Multiplikation durch die Formeln

$$(1,2) \quad \begin{aligned} \mathbf{Q} + \mathbf{Q}' &= \sum_0^3 (q_i + q'_i) e_i, \\ \mathbf{Q}\mathbf{Q}' &= \sum_0^3 q_i q'_i e_i e_i \end{aligned}$$

mit

$$(1,3) \quad e_0 e_i = e_i e_0 = e_i,$$

* Eingegangen am 19. Sept. 1957.

sodass wir auch

$$(1,4) \quad e_0 = 1$$

setzen können, und

$$(1,5) \quad \begin{aligned} e_1 e_1 = e_2 e_2 = e_3 e_3 = -1 \\ e_i e_k = -e_k e_i = e_l \end{aligned}$$

für $i, k, l. = 1, 2, 3; 2, 2, 1; 3, 1, 2.$

Dann gilt für die Multiplikation das assoziative Gesetz

$$(1,6) \quad \mathbf{Q}(\mathbf{Q}'\mathbf{Q}'') = (\mathbf{Q}\mathbf{Q}')\mathbf{Q}''.$$

Statt (1,1) schreiben wir auch

$$(1,7) \quad \mathbf{Q} = q_0 + \mathbf{q}, \quad \mathbf{q} = q_1 e_1 + q_2 e_2 + q_3 e_3.$$

Dann soll \mathbf{q} ein *Vektor* heissen und wir führen die *koniugierte Quaternion* ein durch

$$(1,8) \quad \tilde{\mathbf{Q}} = q_0 - \mathbf{q}.$$

Man bestätigt die Rechenregeln

$$(1,9) \quad \mathbf{Q}\mathbf{Q}' = q_0 q'_0 + q_0 \mathbf{q}' + q'_0 \mathbf{q} - \langle \mathbf{q}\mathbf{q}' \rangle + (\mathbf{q} \times \mathbf{q}'),$$

wobei das Skalarprodukt

$$(1,10) \quad \langle \mathbf{q}\mathbf{q}' \rangle = q_1 q'_1 + q_2 q'_2 + q_3 q'_3$$

und das Vektorprodukt

$$(1,11) \quad \mathbf{q} \times \mathbf{q}' = (q_2 q'_3 - q_3 q'_2) e_1 + (q_3 q'_1 - q_1 q'_3) e_2 + (q_1 q'_2 - q_2 q'_1) e_3$$

benutzt ist. Ferner wird

$$(1,12) \quad \mathbf{Q}\tilde{\mathbf{Q}} = q_0^2 + q_1^2 + q_2^2 + q_3^2 = \langle \mathbf{Q}\mathbf{Q} \rangle,$$

$$\frac{1}{2}(\mathbf{Q}\tilde{\mathbf{Q}}' + \mathbf{Q}'\tilde{\mathbf{Q}}) = q_0 q'_0 + q_1 q'_1 + q_2 q'_2 + q_3 q'_3 = \langle \mathbf{Q}\mathbf{Q}' \rangle$$

und

$$(1,13) \quad \tilde{\mathbf{Q}}\mathbf{Q}' = \tilde{\mathbf{Q}}'\tilde{\mathbf{Q}}.$$

Anstelle der reellen q_i werden wir auch «duale» Zahlen verwenden

$$(1,14) \quad q_i = q_i + \varepsilon \bar{q}_i,$$

wobei die q_i, \bar{q}_i reel sind und ε den Regeln genügt

$$(1,15) \quad \varepsilon^2 = 0, \quad e_i \varepsilon = \varepsilon e_i$$

Dabei ist q_i ein Nullteiler für $q_i = 0$. Die division durch Nullteiler ist unzulässig. Wir setzen für duale Quaternionen

$$(1,16) \quad \begin{aligned} \underline{\mathbf{Q}} &= q_0 + \varepsilon \bar{q}_0 + \mathbf{q} + \varepsilon \bar{\mathbf{q}} = \underline{\mathbf{q}}_0 + \underline{\mathbf{q}}; \\ \underline{\mathbf{q}} &= q_1 e_1 + q_2 e_2 + q_3 e_3, \\ \bar{\underline{\mathbf{q}}} &= \bar{q}_1 e_1 + \bar{q}_2 e_2 + \bar{q}_3 e_3 \end{aligned}$$

und führen die Bezeichnungen ein

$$(1,17) \quad \begin{aligned} \tilde{\underline{\mathbf{Q}}} &= q_0 + \varepsilon \bar{q}_0 - \mathbf{q} - \varepsilon \bar{\mathbf{q}} = \underline{\mathbf{q}}_0 - \underline{\mathbf{q}}; \\ \underline{\mathbf{Q}}_x &= q_0 - \varepsilon \bar{q}_0 + \mathbf{q} - \varepsilon \bar{\mathbf{q}}. \end{aligned}$$

Dann wird

$$(1,18) \quad \begin{aligned} 4q_0 &= \underline{\mathbf{Q}} + \tilde{\underline{\mathbf{Q}}} + \underline{\mathbf{Q}}_x + \tilde{\underline{\mathbf{Q}}}_x, \\ 4\bar{q}_0 &= \underline{\mathbf{Q}} + \tilde{\underline{\mathbf{Q}}} - \underline{\mathbf{Q}}_x - \tilde{\underline{\mathbf{Q}}}_x, \\ 4\mathbf{q} &= \underline{\mathbf{Q}} - \tilde{\underline{\mathbf{Q}}} + \underline{\mathbf{Q}}_x - \tilde{\underline{\mathbf{Q}}}_x, \\ 4\bar{\mathbf{q}} &= \underline{\mathbf{Q}} - \tilde{\underline{\mathbf{Q}}} - \underline{\mathbf{Q}}_x + \tilde{\underline{\mathbf{Q}}}_x. \end{aligned}$$

Die duale Quaternion $\underline{\mathbf{Q}}$ heisse gernormt, wenn

$$(1,19) \quad \underline{\mathbf{Q}} \tilde{\underline{\mathbf{Q}}} = 1; \quad \underline{\mathbf{Q}} \tilde{\underline{\mathbf{Q}}} = \langle \underline{\mathbf{Q}} \underline{\mathbf{Q}} \rangle = 1, \quad \langle \underline{\mathbf{Q}} \underline{\mathbf{Q}} \rangle = 0$$

Die Quaternionen hat L. Euler (1707-1783) 1748 eingeführt Spaeter wurden sie von K. F. Gauss (1777-1855) 1819 und insbesondere von W. R. Hamilton (1805-1865) 1840 verwendet. Duale Zahlen dürfte W. K. Clifford (1835-1879) zuerst benutzt haben. Für eine differenzierbare Funktion F der dualen Veränderlichen $q + \varepsilon \bar{q}$ setzt man

$$(1,20) \quad \begin{aligned} F(q + \varepsilon \bar{q}) &= F(q) + \varepsilon \bar{q} F'(q), \\ F'(q) &= \frac{dF(q)}{dq}. \end{aligned}$$

