

universidad nacional de cuyo
departamento de investigaciones científicas

Mi-195

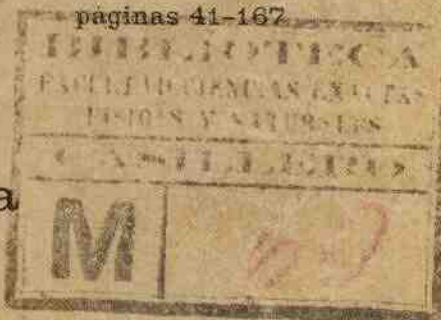
revista matemática cuyana



volumen **1**
1955

fascículo 2

páginas 41-167



instituto de matemática
mendoza
argentina

REVISTA MATEMATICA CUYANA

La REVISTA MATEMÁTICA CUYANA está destinada a la publicación de trabajos originales en los campos de la matemática pura y aplicada, y aparece en forma de fascículos sueltos sin periodicidad fija, anualmente reunidos en un volumen de 250 páginas, aproximadamente.

Castellano, inglés, alemán, francés e italiano, son los idiomas de la Revista.

Los artículos para la Revista deben ser escritos a máquina con doble espacio y enviados a nombre de uno de los miembros del Comité de Redacción, al Instituto de Matemática, Departamento de Investigaciones Científicas, Universidad Nacional de Cuyo, Mendoza, Argentina.

Los colaboradores tienen derecho a 50 tiradas aparte gratis, de sus artículos, y podrán, si lo desean, recibir hasta 150 tiradas aparte a precio de costo.

Comité de Redacción

MISCHA COTLAR.

ANTONIO MONTEIRO.

EDUARDO H. ZARANTONELLO.

En todo lo referente a suscripciones, adquisición de números atrasados, etc., dirigirse al Director del Instituto de Matemáticas, Profesor Mischa Cotlar.

A combinatorial inequality and its applications to L^2 -spaces*

BY M. COTLAR

In this paper we give an estimate of the norm for a certain class of operators T in L^2 -spaces. Since the composition of operators generally improves their norms, our idea is to decompose T into a sum $T = T_1 + \dots + T_N$, in such a way that $T_i T_j$ is very small when $|i - j|$ is great.

This estimate, together with the results of the two following papers, will permit us to unify the theory of Hilbert transforms and ergodic theorems.

1. Generalized integrals. Let $E = \{1, 2, \dots, N\}$ be a finite set of N elements, and μ a set function assigning to each subset $H \subset E$ a non-negative number $\mu(H) \geq 0$. $\mu(H)$ is not assumed to be additive, we only require that $H \subset H'$ implies $\mu(H) \leq \mu(H')$. We shall denote by $\varphi_H(i)$ the characteristic function of the set H : $\varphi_H(i) = 1$ if $i \in H$, and $\varphi_H(i) = 0$ otherwise.

Let $f(i) \geq 0$ be a function defined on E whose values are non-negative integers: $f(1) = \alpha_1, f(2) = \alpha_2, \dots, f(N) = \alpha_N$.

The function $f(i)$ admits a finite number of representations of the form $f(i) = \lambda_1 \varphi_{H_1}(i) + \dots + \lambda_n \varphi_{H_n}(i)$, where the λ_i are non-negative integers, while the sets H_i may overlap. For each such representation we form the sum $s = \lambda_1 \mu(H_1) + \dots + \lambda_n \mu(H_n)$, and define the integral, or sum, of $f(i)$ with respect to $\mu(H)$, by

$$(1) \quad \sum f \Delta \mu = \text{Max } s = \text{Max } \{ \lambda_1 \mu(H_1) + \dots + \lambda_n \mu(H_n) \}.$$

From the definition of $\sum f \Delta \mu$ it is clear that

* Received July 23, 1955.

$$(2) \quad f = f_1 + f_2 \quad \text{implies} \quad \sum f \Delta \mu \geq \sum f_1 \Delta \mu + \sum f_2 \Delta \mu.$$

If μ is additive we obtain the ordinary definition of integral.

LEMMA 1. Let $f(1) = \alpha_1, \dots, f(N) = \alpha_N$, and let $\beta_1 \geq \beta_2 \geq \dots \geq \beta_N$ be the rearrangement in decreasing order of the sequence $\alpha_1, \alpha_2, \dots, \alpha_N$. If H_k denote the set where $f(i) \geq \beta_k$, then

$$\sum f \Delta \mu \geq \sum_{k=2}^N \beta_k (\mu(H_k) - \mu(H_{k-1})) + \beta_1 \mu(H_1).$$

PROOF. Since

$$(2') \quad f(i) = \beta_N \varphi_N + (\beta_{N-1} - \beta_N) \varphi_{N-1} + (\beta_{N-2} - \beta_{N-1}) \varphi_{N-2} + \dots$$

where φ_k stands for φ_{H_k} , it follows that

$$\begin{aligned} \sum f \Delta \mu &\geq \beta_N \mu(H_N) + (\beta_{N-1} - \beta_N) \mu(H_{N-1}) + \dots = \\ &= \beta_N \mu(H_N) + \sum_{k=2}^N \beta_k (\mu(H_k) - \mu(H_{k-1})). \end{aligned}$$

Now we shall consider the particular measure $\mu(H)$ defined as follows: If $H = \{h_1, h_2, \dots, h_m\}$ and $h_1 < h_2 < \dots < h_m$, then

$$(3) \quad \mu(H) = \delta \{ (h_m - h_1) + (h_{m-1} - h_2) + (h_{m-2} - h_3) + \dots \},$$

and if $H_0 = \{h_0\}$, then

$$(3a) \quad \mu(H_0) = 0.$$

From (3) we have that

$$(4) \quad \mu(H) \geq \delta (h_m - h_1), \quad \text{and} \quad \mu(H) \leq m^2,$$

if $m > 1$.

LEMMA 2. Let $\mu(H)$ be the measure defined by (3), and $f(i)$ a function defined on $E = \{1, 2, \dots, N\}$. If $H = \{h_1, \dots, h_r\}$, ($h_1 < h_2 < \dots < h_r$), is the support of $f(i)$, that is the set where $f(i) \neq 0$, and if $f(h_1) = \alpha_1$, then

$$\sum f \Delta \mu \geq \delta (h_r - h_1) + \sum_{i=1}^r i \beta_i - 2(\beta_1 + \beta_2),$$

where $\beta_1 \geq \beta_2 \geq \dots \geq \beta_r$ is the rearrangement in decreasing order of the sequence $\alpha_1, \alpha_2, \dots, \alpha_r$.

PROOF. Let $H' = \{h_1, h_r\}$, $H'' = E - H'$, $f_1 = \chi_{H'}$, $f_2 = f - f_1$, so that $f = f_1 + f_2$. Then

$$(5) \quad \sum f \Delta \mu \geq \sum f_1 \Delta \mu + \sum f_2 \Delta \mu \geq \mu(H') + \sum f_2 \Delta \mu = \\ = 5(h_r - h_1) + \sum f_2 \Delta \mu.$$

Let $\mu'(H)$ be the measure defined as follows: if $H = \{i_1, \dots, i_m\}$, then $\mu'(H) = m^2$ if $m > 1$, and $\mu'(H) = 0$ if $m = 1$. By (4)

$$\mu(H) \geq \mu'(H), \quad \sum f_2 \Delta \mu \geq \sum f_2 \Delta \mu'.$$

Let $f_2(i) = \alpha'_i$, and let $\beta'_1 \geq \beta'_2 \geq \dots$ be the rearrangement in decreasing order of the sequence $\alpha'_1, \alpha'_2, \dots$. If H_k is the set where $f_2(i) \geq \beta'_k$ then H_k contains $\geq k$ elements and

$$\mu'(H_k) \geq k^2, \text{ and } k^2 - (k-1)^2 \geq k+2, \text{ if } k \geq 3.$$

Applying lemma (2') and taking in account that $\beta'_i \geq \beta'_{i+2}$, we obtain

$$\sum f_2 \Delta \mu \geq \sum f_2 \Delta \mu' \geq \sum_k (\beta'_k - \beta'_{k+1}) k^2 \\ \geq \sum_{k=3}^r (k+2) \beta'_{k+2} \geq \sum_{k=1}^r k \beta'_k - 2\beta'_1 - 2\beta'_2.$$

This, together to (5), proves Lemma 2.

2. The main inequality. If $f(i)$ is defined on $E = \{1, 2, \dots, N\}$ and $f(i) = \alpha_i$, we shall write

$$f(i) = \begin{pmatrix} \alpha_1 \alpha_2 \dots \alpha_N \\ 1 \ 2 \ \dots \ N \end{pmatrix}, \quad \sum f \Delta \mu = \sum \begin{pmatrix} \alpha_1 \alpha_2 \dots \alpha_N \\ 1 \ 2 \ \dots \ N \end{pmatrix} \Delta \mu.$$

Let k be a fixed integer and $\lambda > 1$ a real number. Consider all the functions $f(i)$ defined on E such that $f_1(1) + f_1(2) + \dots + f_1(N) = k$ ($f_1(i) =$ non-negative integers), and for each such $f_1(i)$ form the number $\lambda^{-\sum f_1 \Delta \mu}$. We shall give an estimate of the sum $S = \sum \lambda^{-\sum f_1 \Delta \mu}$. More precisely:

LEMMA 3. Let $\mu(H)$ be the measure defined by (3), $E = \{1, 2, \dots, N\}$, N and k fixed integers, and $\lambda > 1$ a real number such that $\lg \lambda > 2^4 k^{-1}$. Then

$$S = \sum_{\alpha_1 + \dots + \alpha_N = k} \frac{k!}{\alpha_1! \dots \alpha_N!} \lambda^{-\sum \binom{\alpha_1 \dots \alpha_N}{1 \ 2 \dots \ N} \Delta \mu} \\ \leq \frac{\lambda^{2k} \cdot k \cdot k^{(k-1)} \cdot N}{(\lambda^2 - 1)^{2k} (\lambda - 1)^k}$$

PROOF. Consider a group of r elements $h_1 < h_2 < \dots < h_r \in H$ and r integers $\alpha_1, \dots, \alpha_r$, such that $\alpha_1 + \dots + \alpha_r = k$ and all the $\alpha_i > 0$.

Since the support of the function $\binom{\alpha_1 \dots \alpha_r}{h_1, \dots, h_r}$ is the set (h_1, \dots, h_r) ,

by Lemma 2

$$\lambda^{-\sum \binom{\alpha_1 \dots \alpha_r}{h_1, \dots, h_r} \Delta \mu} \leq \frac{\lambda^{2k}}{\lambda^{2(h_r - h_1)} \lambda^{2\beta_1 + 2\beta_2 + \dots + r\beta_r}}$$

where $\beta_1 \geq \beta_2 \geq \dots \geq \beta_r$ is the rearrangement in decreasing order of the sequence $\alpha_1, \dots, \alpha_r$.

Let us fix the number r , the elements h_1, \dots, h_r , and the numbers $\beta_1 \geq \dots \geq \beta_r$, and let $\Gamma(\beta_1, \dots, \beta_r)$ denote the set of all groups $\alpha_1, \dots, \alpha_r$, $\alpha_i = \beta_j$, $\alpha_1 + \dots + \alpha_r = k$, $\alpha_i \neq 0$. Then, since $\Gamma(\beta_1, \dots, \beta_r)$ contains at most $r!$ groups,

$$\sum_{\substack{(\alpha_1 \dots \alpha_r) \in \Gamma(\beta_1, \dots, \beta_r) \\ h_1, \dots, h_r \text{ fixed}}} \frac{k!}{\alpha_1! \dots \alpha_r!} \lambda^{-\sum \binom{\alpha_1 \dots \alpha_r}{h_1, \dots, h_r} \Delta \mu} \\ \leq r! \frac{\lambda^{2k}}{\lambda^{2(h_r - h_1)}} \frac{k!}{\alpha_1! \dots \alpha_r!} \left(\frac{1}{\lambda}\right)^{\beta_1} \left(\frac{1}{\lambda^2}\right)^{\beta_2} \dots \left(\frac{1}{\lambda^r}\right)^{\beta_r}.$$

Therefore, if we keep r and h_1, \dots, h_r fixed and let the α_i vary under the condition $\alpha_1 + \dots + \alpha_r = k$, $\alpha_i \neq 0$, we obtain

$$(6) \quad \sum_{\substack{\alpha_1 + \dots + \alpha_r = k \\ \alpha_i \neq 0 \\ r, h_1, \dots, h_r \text{ fixed}}} \frac{k!}{\alpha_1! \dots \alpha_r!} \lambda^{-\sum \binom{\alpha_1 \dots \alpha_r}{h_1, \dots, h_r} \Delta \mu} \leq \\ \frac{r! \lambda^{2k}}{\lambda^{2(h_r - h_1)}} \sum_{\beta_1 + \dots + \beta_r = k} \frac{k!}{\beta_1! \dots \beta_r!} \left(\frac{1}{\lambda}\right)^{\beta_1} \dots \left(\frac{1}{\lambda^r}\right)^{\beta_r} \\ = \frac{r! \lambda^{2k}}{\lambda^{2(h_r - h_1)}} \left(\frac{1}{\lambda} + \dots + \frac{1}{\lambda^r}\right)^k \\ \leq \frac{r! \lambda^{2k}}{\lambda^{2(h_r - h_1)}} \left(\frac{1}{\lambda - 1}\right)^k.$$

We give now another estimate of the left member of (6).
 Since $\lambda^{-\sum_{i=1}^r (a_i + 2s_i + \dots + r s_i)} \leq \lambda^{-(+1+2+\dots+r)} \leq (\sqrt{\lambda})^{r^2}$, we have

$$\begin{aligned}
 (7) \quad & \sum_{\substack{a_1 + \dots + a_r = k \\ a_i \geq 0 \\ r, h_1, \dots, h_r \text{ fixed}}} \frac{k!}{\alpha_1! \dots \alpha_r!} \lambda^{-\sum_{i=1}^r (a_i + 2s_i + \dots + r s_i)} \\
 & \leq \frac{\lambda^{2k}}{\lambda^{5(h_r - h_1)}} \cdot \frac{1}{(\sqrt{\lambda})^{r^2}} \sum_{a_1 + \dots + a_r = k} \frac{k!}{\alpha_1! \dots \alpha_r!} \\
 & = \frac{\lambda^{2k}}{\lambda^{5(h_r - h_1)}} \cdot \frac{r^k}{(\sqrt{\lambda})^{r^2}}
 \end{aligned}$$

If we keep h_1 and h_r fixed and let the h_2, h_3, \dots, h_{r-1} vary under the condition $h_1 < h_2 < \dots < h_r$, we get at most $C_{h_r - h_1}^r$ sums of the form (6) or (7). Therefore, if S_r denotes the sum of all the terms of the form (6) or (7) where only r remained fixed, we obtain from (6) and (7) respectively:

$$\begin{aligned}
 (6a) \quad S_r &= \sum_{\substack{a_1 + \dots + a_r = k \\ a_i \geq 0, r \text{ fixed} \\ 1 \leq h_1 < h_2 < \dots < h_r \leq N}} \frac{k!}{\alpha_1! \dots \alpha_r!} \lambda^{-\sum_{i=1}^r (a_i + 2s_i + \dots + r s_i)} \\
 & \leq \sum_{h_1=1}^N \sum_{h_r=h_1+r}^N C_{h_r - h_1}^r \frac{r! \lambda^{2k}}{\lambda^{5(h_r - h_1)}} \left(\frac{1}{\lambda - 1} \right)^k \\
 & \leq \sum_{h_1=1}^N \left(\sum_{m=r}^N \frac{m(m-1)\dots(m-r+1)}{r!} \frac{1}{\lambda^{5m}} r! \lambda^{2k} \left(\frac{1}{\lambda - 1} \right)^k \right) \\
 & \leq \sum_{h_1=1}^N \frac{\lambda^{2k}}{(\lambda - 1)^k} r! \frac{1}{(\lambda^5 - 1)^k} = \frac{\lambda^{2k}}{(\lambda - 1)^k (\lambda^5 - 1)^k} r! \cdot N.
 \end{aligned}$$

and

$$\begin{aligned}
 (7a) \quad S_r & \leq \sum_{h_1=1}^N \left(\sum_{m=r}^N \frac{m(m-1)\dots(m-r+1)}{r!} \cdot \frac{1}{\lambda^{5m}} \cdot \frac{\lambda^{2k} \cdot r^k}{(\sqrt{\lambda})^{r^2}} \right) \\
 & \leq \frac{\lambda^{2k}}{(\lambda^5 - 1)^k} \frac{r^k}{(\sqrt{\lambda})^{r^2}} \cdot N.
 \end{aligned}$$

If $r < k^{1/2}$, we have from (6a) that

$$(6b) \quad S_r \leq \frac{\tilde{\lambda}^{2k}}{(\tilde{\lambda}^2 - 1)^k (\lambda - 1)^k} k^{(k^{1/2})} \cdot N.$$

If $r \geq k^{1/2}$, then, since $k > (2^k (\lg \tilde{\lambda})^{-k})^2 = 2^{2k} (\lg \tilde{\lambda})^{-2k}$, we have $(\tilde{\lambda})^{r^2} \geq (\tilde{\lambda})^{2k^{1/2}} = (\tilde{\lambda}^{k^{1/2}})^2 \geq r^k$, and from (7a) we get

$$(7b) \quad S_r \leq \frac{\tilde{\lambda}^{2k}}{(\tilde{\lambda}^2 - 1)^k} N.$$

Hence $S = \sum_{r=1}^{k^{1/2}} S_r + \sum_{r=k^{1/2}+1}^N S_r \leq \frac{\tilde{\lambda}^{2k}}{(\tilde{\lambda}^2 - 1)^k (\lambda - 1)^k} k \cdot k^{(k^{1/2})} \cdot N$, and this

proves Lemma 3.

REMARK. Though we shall not use it in this paper, in some cases the following variant of Lemma 3 may be useful.

LEMMA 3a. Let μ be the measure defined by (3), and $\mu_1(H)$ the measure defined as follows: if $H = \{h_1, \dots, h_m\}$, $h_1 < h_2 < \dots < h_m$, then $\mu_1(H) = \mu(H) + h_1$. Then

$$\begin{aligned} \sum_{x_1 + \dots + x_m = k} \frac{k!}{x_1! \dots x_m!} \tilde{\lambda}^{-\sum_{i=1}^m \binom{x_i + \dots + x_m}{1 \ 2 \ \dots \ N} \Delta x_i} \\ \leq \frac{\tilde{\lambda}^{2k} k}{(\tilde{\lambda}^2 - 1)^k} k^{(k^{1/2})}, \end{aligned}$$

so that the number N does not appear in the right hand of the inequality.

The proof is identical to that of Lemma 3. In the present case we will have besides the factor $\tilde{\lambda}^{-h_1}$ in the right side of (6) or (7), so that N will not appear in the last formulas (6a) and (7a).

3. Application to Hilbert and L^2 -spaces. Let $\mathbf{A} = \{T_i\}$ be a commutative normed ring. This means that \mathbf{A} is a set in which a sum $T_1 + T_2 = T_2 + T_1$, a product $T_1 T_2 = T_2 T_1$, and a norm $\|T\| \geq 0$ are defined, in such a way that the following conditions are satisfied:

- a) $(T_1 + T_2) T_3 = T_1 T_3 + T_2 T_3$.
- b) $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$.
- c) $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$.

We shall write $T^2 = TT$, $T^n = T^{n-1} T = T T^{n-1}$.

THEOREM 1. If $T = T_1 + \dots + T_N$ and the T_i satisfy the condition

$$(A) \quad \|T_i T_j\| \leq 2^{-i(i+j)}, \quad \|T_i\| < 1,$$

then

$$(1) \quad \|T^k\| \leq 2^{2k} k! k^{(k-1)k}, \quad N.$$

PROOF. By the property b) of the norm we have

$$\|T^k\| \leq \sum_{\alpha_1 + \dots + \alpha_N = k} \frac{k!}{\alpha_1! \dots \alpha_N!} \|T_1^{\alpha_1} \dots T_N^{\alpha_N}\|.$$

To any term of the form $T_1^{\alpha_1} \dots T_N^{\alpha_N}$ we assign the function

$$f(i) = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_N \\ 1 & 2 & \dots & N \end{pmatrix}.$$

In particular, if $h_1 < \dots < h_m$, to $T_{h_1} T_{h_2} \dots T_{h_m}$ corresponds the characteristic function φ_H of the set $H = \{h_1, \dots, h_m\}$.

From the hypothesis (A) we have that

$$(8) \quad \|T_{h_1} \dots T_{h_m}\| \leq \|T_{h_1} T_{h_m}\| \cdot \dots \cdot \|T_{h_2} T_{h_{m-1}}\| \dots \\ \leq 2^{-(h_m - h_1 - (h_{m-1} - h_2) - \dots)} = \left(\frac{5}{2}\right)^{-\mu(H)},$$

where $\mu(H)$ is the measure defined by (3).

On the other and, to any representation of $f(i)$ of the form

$$f(i) = \gamma_1 \varphi_{H_1}(i) + \gamma_2 \varphi_{H_2}(i) + \dots \quad (\gamma_i = \text{non-negative integers})$$

we make correspond the decomposition of $T_1^{\alpha_1} \dots T_N^{\alpha_N}$ into the factors

$$T_1^{\alpha_1} \dots T_N^{\alpha_N} = (T_{h_1'} T_{h_2'} \dots)^{\gamma_1} (T_{h_1''} T_{h_2''} \dots)^{\gamma_2} \dots,$$

where

$$(h_1', h_2', \dots) = H_1, \quad (h_1'', h_2'', \dots) = H_2, \dots$$

Using (8) we get that

$$\|T_1^{\alpha_1} \dots T_N^{\alpha_N}\| \leq \|T_{h_1'} T_{h_2'} \dots\|^{\gamma_1} \cdot \|T_{h_1''} T_{h_2''} \dots\|^{\gamma_2} \dots \\ \leq \left(\frac{5}{2}\right)^{-\gamma_1 \mu(H_1) - \gamma_2 \mu(H_2) - \dots} \\ \leq \left(\frac{5}{2}\right)^{-\sum \gamma_i \Delta_i} = \left(\frac{5}{2}\right)^{-\sum \binom{\alpha_1 \alpha_2 \dots \alpha_N}{1 \ 2 \ \dots \ N} \Delta^k}.$$

Applying Lemma 3 we obtain

$$\|T^k\| \leq \frac{(\sqrt[5]{2})^{2k} \cdot k \cdot k^{(k-1)/5} \cdot N}{(\sqrt[5]{2^5} - 1)^{2k} (\sqrt[5]{2} - 1)^k} \leq \frac{2^{k/2} \cdot k \cdot k^{(k-1)/5} \cdot N}{(\sqrt[5]{2} - 1)^k},$$

and this proves Theorem 1.

Consider now a (not necessarily complete) real Hilbert space $\mathbf{H} = \{x\}$, and an operator T defined on \mathbf{H} which assigns to any element $x \in \mathbf{H}$ another element $Tx \in \mathbf{H}$. T is not required to be linear, but we assume that T satisfies the Hermitean condition :

$$(9) \quad (Tx, y) = (x, Ty)$$

for any $x, y \in \mathbf{H}$, and that there is a finite number M such that

$$\|Tx\| \leq M \cdot \|x\|, \quad \text{for all } x \in \mathbf{H}.$$

We shall denote by $\|T\|$ the smallest of such numbers M , so that

$$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\| \quad \text{and} \quad \|TT_1\| \leq \|T\| \cdot \|T_1\|$$

PROPOSITION 1. *If T satisfies the condition (9) then*

$$\|T\| = (\|T^{2^m}\|)^{2^{-m}} = (\|T_k\|)^{1/k}, \quad k = 2^m,$$

for any m .

This property is well known for linear operators (Cfr. Gelfand [1]), and subsists without changes for non linear ones. Since the proof is very simple we will reproduce it here. By the Schwarz inequality

$$\begin{aligned} \|Tx\| &= \{(Tx, Tx)\}^{1/2} = \{(x, T^2x)\}^{1/2} \leq \|x\|^{1/2} \|T^2x\|^{1/2} \\ &\leq \|x\|^{1/2} \cdot \|T^2\|^{1/2} \cdot \|x\|^{1/2} = \|T^2\|^{1/2} \cdot \|x\|, \end{aligned}$$

hence $\|T\| \leq \|T^2\|^{1/2}$. On the other hand, $\|T^2\| \leq \|T\| \cdot \|T\| = \|T\|^2$, $\|T^2\|^{1/2} \leq \|T\|$. Thus $\|T\| = \|T^2\|^{1/2}$, and by iteration we obtain $\|T^{2^m}\|^{2^{-m}} = \|T\|$.

THEOREM 2. Let T be an operator satisfying condition (9). If it is possible to decompose T into a sum $T = T_1 + T_2 + \dots + T_N$ such that

$$(A) \quad \|T_i T_j\| \leq 2^{-|i-j|} \mathbb{1}, \quad \|T_i\| \leq 1,$$

$$(B) \quad T_i T_j = T_j T_i,$$

then $\|T\| \leq 8$.

PROOF. Since the operators T_1, \dots, T_N commute they all belong to a commutative normed ring $\mathbf{A} = \{T_\alpha\}$. Applying Theorem 1 and Proposition 1, we obtain, for $k = 2^m$,

$$\|T^k\| = \|T^k\| \leq 2^{mk} \cdot k, \quad k \in \mathbb{N},$$

$$\|T\| \leq 8k^{1/k}, \quad k \in \mathbb{N}.$$

Allowing k to go to infinity, we obtain $\|T\| \leq 8$.

Consider now the particular case $\mathbf{H} = L^2(R^n) =$ the class of functions $f(x)$ which satisfy

$$\|f\|_2 = \left\{ \int_{R^n} |f(t)|^2 dt \right\}^{1/2} < \infty,$$

where R^n is the n -dimensional euclidean space (or more generally a locally compact abelian group).

If $k(x) \in L^1$, that is if $k(x)$ is integrable, it defines on L^2 the linear operator

$$Tf = T_k f = (T_k f)(x) = \int_{R^n} f(x-t) k(t) dt = (f * k)(x),$$

and by a known inequality of Young (Cfr. [2], Chap. IV) we have

$$\|Tf\|_2 \leq \|k\|_1 \cdot \|f\|_2,$$

where

$$\|k\|_1 = \int_{R^n} |k(x)| dx.$$

Thus

$$\|T\| = \|T_k\| \leq \|k\|_1.$$

From Theorem 2 we obtain then at once the following :

THEOREM 2a. Let $k(x) \in L^1$. If it is possible to decompose $k(x)$ into a sum $k(x) = k_1(x) + \dots + k_N(x)$ such that

$$(A) \quad \|k_i \star k_j\|_1 \leq C \cdot 2^{-|i-j|}, \quad \|k_{i+1}\|_1 \leq C,$$

where
$$k_i \star k_j(x) = \int_{\mathbb{R}^n} k_i(x-t)k_j(t) dt,$$

then
$$\|f \star k\|_2 \leq 16C^2 \cdot \|f\|_2$$

holds for every $f \in L^2(\mathbb{R}^n)$.

4. Examples. Let $\mathbb{R}^1 = \{t\}$ be the 1-dimensional euclidean space. For each m we define on $L^2(\mathbb{R}^1)$ the operator H_m by

$$(10) \quad \begin{aligned} H_m f &= H_m f(x) = \int_{-m}^m \frac{f(x-t)}{t} dt + \int_{-m}^{-m} \frac{f(x-t)}{t} dt \\ &= \int_{|m| \leq |t| \leq m} \frac{f(x-t)}{t} dt. \end{aligned}$$

For each i we define the kernels

$$(11) \quad k_i(t) = \begin{cases} 1/t & \text{if } 2^{i-1} \leq |t| \leq 2^i \\ 0 & \text{otherwise} \end{cases}$$

$$(11a) \quad k_{-i}(t) = \begin{cases} 1/t & \text{if } 2^{-i} \leq |t| \leq 2^{-i+1} \\ 0 & \text{otherwise} \end{cases}$$

and the operators

$$(11b) \quad \begin{aligned} T_i f(x) &= f \star k_i(x) = \int_{\mathbb{R}^1} f(x-t)k_i(t) dt \\ &= \int_{2^{i-1} \leq |t| \leq 2^i} \frac{f(x-t)}{t} dt \end{aligned}$$

$$(11c) \quad T_{-i} f(x) = f \star k_{-i}(x) = \int_{2^{-i} < |t| < 2^{-i+1}} \frac{f(x-t)}{t} dt$$

It is clear that if $m = 2^N$, then

$$(12) \quad H_m f = H_{2^N} f = \sum_{i=-N}^N T_i f.$$

It is easy to verify the following properties of the kernels $k_i(t)$:

$$(13) \quad \int_{\mathbb{R}^1} k_i(t) dt = 0, \quad \|k_i\|_1 = 1,$$

$$(14) \quad k_i(t) = 2^{-i} k_1(2^{-i}t),$$

$$(15) \quad \int_{\mathbb{R}^1} |k_1(x-t) - k_1(x)| dx \leq |t|.$$

From (14) and (15) we deduce that:

$$(15a) \quad \int_{\mathbb{R}^1} |k_i(x-t) - k_i(x)| dx = 2^{-i} \int_{\mathbb{R}^1} |k_1(2^{-i}x - 2^{-i}t) - k_1(2^{-i}x)| dx \\ = \int_{\mathbb{R}^1} |k_1(\xi - 2^{-i}t) - k_1(\xi)| d\xi \leq 2^{-i}|t|.$$

Using (13) and (15a) we obtain, for $i > j$,

$$(16) \quad \|k_i * k_j\|_1 = \int_{\mathbb{R}^1} \left| \int_{\mathbb{R}^1} k_i(x-t) k_j(t) dt \right| dx \\ = \int_{\mathbb{R}^1} \left| \int_{\mathbb{R}^1} |k_i(x-t) - k_i(x)| k_j(t) dt \right| dx \\ \leq \int_{2^{j-1} < |t| < 2^j} |k_j(t)| \left| \int_{\mathbb{R}^1} |k_i(x-t) - k_i(x)| dx \right| dt \\ \leq 2^{-i} 2^j \|k_j\|_1 = 2^{-(i-j)}.$$

COROLLARY 1. *The kernels k_i satisfy the condition (Δ) of Theorem 2a. If $H_m f$ is defined by (10), then these operators are uniformly bounded on $L^2(\mathbb{R}^1)$:*

$$(17) \quad \|H_m f\|_2 \leq Cst \cdot \|f\|_2,$$

where the constant is independent of m .

In fact, if $m = 2^N$, it follows from (16), (12) and Theorem 2a that

$$\|H_{2^N} f\|_2 \leq 8 \|f\|_2$$

If $2^N < m < 2^{N+1}$, then $H_m f = H_{2^N} f + H'f$,
where

$$H'f = \int_{m-|t| < 2^{N+1}} + \int_{2^N < |t| < m-1} \frac{f(x-t)}{t} dt,$$

and by Young's inequality

$$\|H'f\|_2 \leq \left\{ \int_{m-|t| < 2^{N+1}} + \int_{2^N < |t| < m-1} \frac{dt}{|t|} \right\} \|f\|_2 \leq 2 \|f\|_2,$$

and this proves (17).

For any step function $f(t)$ (or for any differentiable function with compact support) the operator

$$(18) \quad Hf(x) = \int_{-\infty}^{\infty} \frac{f(x-t)}{t} dt$$

is perfectly defined, and

$$Hf = \lim_{m \rightarrow \infty} H_m f = \sum_{n=-\infty}^{\infty} T_n f.$$

Since the step functions are dense in L^2 and by Corollary 1 the operators H_m are uniformly continuous on $L^2(\mathbb{R}^1)$, we obtain the

COROLLARY 2. *The limit $\lim H_m f = Hf$ exists for any $f \in L^2(\mathbb{R}^1)$ and is a bounded operator. Hence the operator (18) admits a continuous extension to the whole space L^2 .*

The operator Hf is known as the Hilbert transform of f , or the principal value of the integral (18).

Consider now the 2-dimensional euclidean space $\mathbb{R}^2 = \{z\}$; and let us use the complex variable notation :

$$z = x + iy = |z| \cdot e^{i\theta}.$$

