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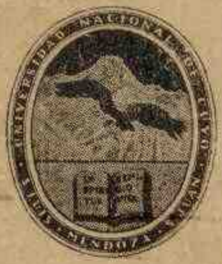

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A combinatorial inequality and its applications to L^2-spaces*

By M. COTLAR

In this paper we give an estimate of the norm for a certain class of operators T in L^2-spaces. Since the composition of operators generally improves their norms, our idea is to decompose T into a sum $T = T_1 + \dots T_N$, in such a way that $T_i T_j$ is very small when $|i-j|$ is great.

This estimate, together with the results of the two following papers, will permit us to unify the theory of Hilbert transforms and ergodic theorems.

1. Generalized integrals. Let $E = \{1, 2, \dots N\}$ be a finite set of N elements, and μ a set function assiggning to each subset $H \subset E$ a non-negative number $\mu(H) \geq 0$. $\mu(H)$ is not assumed to be additive, we only require that $H \subset H'$ implies $\mu(H) \leq \mu(H')$. We shall denote by $\varphi_H(i)$ the characteristic function of the set $H : \varphi_H(i) = 1$ if $i \in H$, and $\varphi_H(i) = 0$ otherwise.

Let $f(i) \geq 0$ be a function defined on E whose values are non-negative integers: $f(1) = \alpha_1, f(2) = \alpha_2, \dots, f(N) = \alpha_N$.

The function $f(i)$ admits a finite number of representations of the form $f(i) = \lambda_1 \varphi_{H_1}(i) + \dots + \lambda_n \varphi_{H_n}(i)$, where the λ_i are non-negative integers, while the sets H_i may overlap. For each such representation we form the sum $s = \lambda_1 \mu(H_1) + \dots + \lambda_n \mu(H_n)$, and define the integral, or sum, of $f(i)$ with respect to $\mu(H)$, by

$$\sum f \Delta\mu = \text{Max } s = \text{Max } \{\lambda_1 \mu(H_1) + \dots + \lambda_n \mu(H_n)\}. \tag{1}$$

From the definition of $\sum f \Delta\mu$ it is clear that

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$$(2) \qquad f = f_1 + f_2 \ \text{ implies } \ \sum f \Delta\mu \geq \sum f_1 \Delta\mu + \sum f_2 \Delta\mu.$$

If μ is additive we obtain the ordinary definition of integral.

LEMMA 1. *Let* $f(1) = \alpha_1, \ldots, f(N) = \alpha_N$, *and let* $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_N$ *be the rearrangement in decreasing order of the sequence* $\alpha_1, \alpha_2 \ldots \alpha_N$. *If* H_k *denote the set where* $f(i) \geq \beta_k$, *then*

$$\sum f \Delta\mu \geq \sum_{k=2}^{N} \beta_k (\mu(H_k) - \mu(H_{k-1})) + \beta_1 \mu(H_1).$$

PROOF. Since

$$(2') \quad f(i) = \beta_N \varphi_N + (\beta_{N-1} - \beta_N)\varphi_{N-1} + (\beta_{N-2} - \beta_{N-1})\varphi_{N-2} + \ldots.$$

where φ_k stands for φ_{H_k}, it follows that

$$\sum f \Delta\mu \geq \beta_N \mu(H_N) + (\beta_{N-1} - \beta_N)\mu(H_{n-1}) + \ldots =$$
$$= \beta_1 \mu(H_1) + \sum_{k=2} \beta_k (\mu(H_k) - \mu(H_{k-1})).$$

Now we shall consider the particular measure $\mu(H)$ defined as follows: If $H = \{h_1, h_2, \ldots, h_m\}$ and $h_1 < h_2 < \ldots < h_m$, then

$$(3) \qquad \mu(H) = 5\{(h_m - h_1) + (h_{m-1} - h_2) + (h_{m-2} - h_3) + \ldots\},$$

and if $H_0 = \{h_0\}$, then

$$(3\,a) \qquad \mu(H_0) = 0.$$

From (3) we have that

$$(4) \qquad \mu(H) \geq 5(h_m - h_1), \quad \text{and} \quad \mu(H) \geq m^2,$$

if $m > 1$.

LEMMA 2. *Let* $\mu(H)$ *be the measure defined by* (3), *and* $f(i)$ *a function defined on* $E = \{1, 2, \ldots, N\}$. *If* $H = \{h_1, \ldots, h_r\}$, $(h_1 < h_2 < \ldots < h_r)$, *is the support of* $f(i)$, *that is the set where* $f(i) \neq 0$, *and if* $f(h_i) = \alpha_i$, *then*

$$\sum f \Delta\mu \geq 5(h_r - h_1) + \sum_{i=1}^{r} i\beta_i - 2(\beta_1 + \beta_2),$$

where $\beta_1 \geq \beta_2 \geq \ldots \beta_r$ *is the rearrangement in decreasing order of the sequence* $\alpha_1, \alpha_2 \ldots \alpha_r$.

PROOF. Let $H' = \{h_1, h_r\}$, $H'' = E - H'$, $f_1 = \varphi_{H'}$, $f_2 = f - f_1$, so that $f = f_1 + f_2$. Then

$$(5) \qquad \sum f \Delta\mu \geq \sum f_1 \Delta\mu + \sum f_2 \Delta\mu \geq \mu(H') + \sum f_2 \Delta\mu = \\ = 5(h_r - h_1) + \sum f_2 \Delta \mu.$$

Let $\mu'(H)$ be the measure defined as follows: if $H = \{i_1, \dots, i^m\}$, then $\mu'(H) = m^2$ if $m > 1$, and $\mu'(H) = 0$ if $m = 1$. By (4)

$$\mu(H) \geq \mu'(H), \qquad \sum f_2 \Delta\mu \geq \sum f_2 \Delta\mu'.$$

Let $f_2(i) = \alpha_i'$, and let $\beta_1' \geq \beta_2' \geq \dots$ be the rearrangement in decreasing order of the sequence $\alpha_1', \alpha_2', \dots$ If H_k is the set where $f_2(i) \geq \beta'_k$ then H_k contains $\geq k$ elements and

$$\mu'(H_k) \geq k^2, \text{ and } k^2 - (k-1)^2 \geq k+2, \text{ if } k \geq 3.$$

Applying lemma (2') and taking in account that $\beta_i' \geq \beta_{i+2}$, we obtain

$$\sum f_2 \Delta\mu \geq \sum f_2 \Delta\mu' \geq \sum_k (\beta_k' - \beta'_{k+1}) k^2$$

$$\geq \sum_{k=3}^{r} (k+2) \beta_{i+2} \geq \sum_{k=1}^{r} k\beta_k - 2\beta_1 - 2\beta_2.$$

This, together to (5), proves Lemma 2.

2. The main inequality. If $f(i)$ is defined on $E = \{1, 2, \dots N\}$ and $f(i) = \alpha_i$, we shall write

$$f(i) = \begin{pmatrix} \alpha_1 \alpha_2 & \dots & \alpha_N \\ 1\ 2 & \dots & N \end{pmatrix}, \qquad \sum f \Delta\mu = \sum \begin{pmatrix} \alpha_1 \alpha_2 & \dots & \alpha_N \\ 1\ 2 & \dots & N \end{pmatrix} \Delta\mu.$$

Let k be a fixed integer and $\lambda > 1$ a real number. Consider all the functions $f_s(i)$ defined on E such that $f_s(1) + f_s(2) + \dots f_s(N) = k$ ($f_s(i)$ = non-negative integers), and for each such $f_s(i)$ form the number $\lambda^{-\sum f_s \Delta\mu}$. We shall give an estimate of the sum $S = \sum \lambda^{-\sum f_s \Delta\mu}$. More precisely:

LEMMA 3. *Let* μ (H) *be the measure defined by* (3), E = {1, 2, . . . N}, N *and* k *fixed integers, and* $\lambda > 1$ *a real number such that* $\lg \lambda > 2^4 k^{-1/6}$. *Then*

$$S = \sum_{\alpha_1 + \ldots \alpha_N = k} \frac{k!}{\alpha_1! \ldots \alpha_N!} \lambda^{-\Sigma \binom{\alpha_1 \alpha_2 \ldots \alpha_N}{1\ 2 \ldots N} \Delta\rho}$$

$$\leq \frac{\lambda^{2k} . k . k^{(k^2/\lambda)} . N}{(\lambda^\rho - 1)^{2k} (\lambda - 1)^k}.$$

PROOF. Consider a group of r elements $h_1 < h_2 < \ldots < h_r \in E$ and r integers $\alpha_1, \ldots, \alpha_r$, such that $\alpha_1 + \ldots + \alpha_r = k$ and all the $\alpha_i > 0$. Since the support of the function $\binom{\alpha_1, \ldots \alpha_r}{h_1, \ldots h_r}$ is the set $(h_1, \ldots h_r)$, by Lemma 2

$$\lambda^{-\Sigma \binom{\alpha_1 \ldots \alpha_r}{h_1 \ldots h_r} \Delta\rho} \leq \frac{\lambda^{2k}}{\lambda^{\rho(h_r - h_1)} \lambda^{\beta_1 + 2\beta_2 + \ldots + r\beta_r}}$$

where $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_r$ is the rearrangement in decreasing order of the sequence $\alpha_1, \ldots \alpha_r$.

Let us fix the number r, the elements $h_1, \ldots h_r$, and the numbers $\beta_1 \geq \ldots \geq \beta_r$, and let $\Gamma(\beta_1, \ldots \beta_r)$ denote the set of all groups $\alpha_1, \ldots \alpha_r$, $\alpha_i = \beta_j$, $\alpha_1 + \ldots + \alpha_r = k$, $\alpha_i \neq 0$. Then, since $\Gamma(\beta_1, \ldots \beta_r)$ contains at most $r!$ groups,

$$\sum_{\substack{(\alpha_1 \ldots \alpha_r) \in \Gamma(\beta_1, \ldots \beta_r) \\ h_1 \ldots h_r \text{ fixed}}} \frac{k!}{\alpha_1! \ldots \alpha_r!} \lambda^{-\Sigma \binom{\alpha_1 \ldots \alpha_r}{h_1 \ldots h_r} \Delta\rho}$$

$$\leq r! \frac{\lambda^{2k}}{\lambda^{\rho(h_r - h_1)}} \frac{k!}{\alpha_1! \ldots \alpha_r!} \left(\frac{1}{\lambda}\right)^{\beta_1} \left(\frac{1}{\lambda^2}\right)^{\beta_2} \cdots \left(\frac{1}{\lambda^r}\right)^{\beta_r}.$$

Therefore, if we keep r and $h_1 \ldots h_r$ fixed and let the α_i vary under the condition $\alpha_1 + \ldots \alpha_r = k$, $\alpha_i \neq 0$, we obtain

$$(6) \qquad \sum_{\substack{\alpha_1 + \cdots \alpha_r = k \\ \alpha_i \neq 0 \\ r, h_1 \ldots h_r \text{ fixed}}} \frac{k!}{\alpha_1! \ldots \alpha_r!} \lambda^{-\Sigma \binom{\alpha_1 \ldots \alpha_r}{h_1 \ldots h_r} \Delta\rho} \leq$$

$$\frac{r! \lambda^{2k}}{\lambda^{\rho(h_r - h_1)}} \sum_{\beta_1 + \cdots \beta_r = k} \frac{k!}{\beta_1! \ldots \beta_r!} \left(\frac{1}{\lambda}\right)^{\beta_1} \cdot \left(\frac{1}{\lambda^r}\right)^{\beta_r}$$

$$= \frac{r! \lambda^{2k}}{\lambda^{\rho(h_r - h_1)}} \left(\frac{1}{\lambda} + \cdots + \frac{1}{\lambda^r}\right)^k$$

$$\leq \frac{r! \lambda^{2k}}{\lambda^{\rho(h_r - h_1)}} \left(\frac{1}{\lambda - 1}\right)^k.$$

We give now another estimate of the left member of (6).

Since $\lambda^{-(\beta_1+2\beta_2+\dots+r\beta_r)} \leq \lambda^{-(+1+2+\dots r)} \leq (\sqrt{\lambda})^{r^2}$, we have

$$(7) \qquad \sum_{\substack{\alpha_1+\dots\alpha_r=k \\ \alpha_i\neq 0 \\ r,\, h_1,\dots h_r \text{ fixed}}} \frac{k!}{\alpha_1!\dots\alpha_r!}\,\lambda^{-\Sigma\binom{\alpha_1\dots\alpha_\mu}{h_1\dots h_r}\Delta_\mu}$$

$$\leq \frac{\lambda^{2k}}{\lambda^{5(h_r-h_1)}}\cdot\frac{1}{\sqrt{\lambda}^{r^2}}\sum_{\alpha_1+\dots\alpha_r=k}\frac{k!}{\alpha_1!\dots\alpha_r!}$$

$$=\frac{\lambda^{2k}}{\lambda^{5(h_r-h_1)}}\cdot\frac{r^k}{(\sqrt{\lambda})^{r^2}}$$

If we keep h_1 and h_r fixed and let the $h_2, h_3, \dots h_{r-1}$ vary under the condition $h_1 < h_2 < \dots < h_r$, we get at most $C^r_{h_r-h_1}$ sums of the form (6) or (7). Therefore, if S_r denotes the sum of all the terms of the form (6) or (7) where only r remained fixed, we obtain from (6) and (7) respectively:

$$(6a) \qquad S_r = \sum_{\substack{\alpha_1+\dots\alpha_r=k \\ \alpha^i\neq 0,\ r \text{ fixed} \\ 1\leq h_1<h_2\dots<h_r\leq N}} \frac{k!}{\alpha_1!\dots\alpha_r!}\,\lambda^{-\Sigma\binom{\alpha_1\dots\alpha_r}{h_1\dots h_r}\Delta_\mu}$$

$$\leq \sum_{h_1=1}^{N}\ \sum_{h_r=h_1+r}^{N} C^r_{h_r-h_1}\frac{r!\,\lambda^{2k}}{\lambda^{5(h_r-h_1)}}\left(\frac{1}{\lambda-1}\right)^k$$

$$\leq \sum_{h_1=1}^{N}\left(\sum_{m=r}^{N}\frac{m(m-1)\dots(m-r+1)}{r!}\,\frac{1}{\lambda^{5m}}\quad r!\ \lambda^{2k}\left(\frac{1}{\lambda-1}\right)^k\right)$$

$$\leq \sum_{h_1=1}^{N}\frac{\lambda^{2k}}{(\lambda-1)^k}\,r!\,\frac{1}{(\lambda^5-1)^k}=\frac{\lambda^{2k}}{(\lambda-1)^k\,(\lambda^5-1)^k}\quad r!\,.\,N.$$

and

$$(7a) \qquad S_r \leq \sum_{h_1=1}^{N}\left(\sum_{m=r}^{N}\frac{m(m-1)\dots(m-r+1)}{r!}\cdot\frac{1}{\lambda^{5m}}\cdot\frac{\lambda^{2k}\,.\,r^k}{(\lambda)^{r^2}}\right.$$

$$\leq \frac{\lambda^{2k}}{(\lambda^5-1)^k}\quad\frac{r^k}{(\sqrt{\lambda})^{r^2}}\cdot N.$$

If $r < k^{1/4}$, we have from (6a) that

$$S_r \leq \frac{\lambda^{2k}}{(\lambda^{\delta}-1)^k(\lambda-1)^k}\, k^{(k^{1/2})} \,.\, N. \tag{6b}$$

If $r \geq k^{1/4}$ then, since $k > (2^4 (\lg \lambda)^{-1})^4 = 2^{16} (\lg \lambda)^{-4}$, we have $(\sqrt[4]{\lambda})^{r^2} \geq (\sqrt[4]{\lambda})^{(k^{1/2})} = (\sqrt[4]{\lambda}^{k^{-1/2}})^k \geq r^k$, and from (7$a$) we get

$$S_r \leq \frac{\lambda^{2k}}{(\lambda^{\delta}-1)^k}\, N. \tag{7b}$$

Hence $S = \sum\limits_{r=1}^{k^{1/4}} S_r + \sum\limits_{r=k^{1/4}+1}^{N} S_r \leq \dfrac{\lambda^{2k}}{(\lambda^{\delta}-1)^k(\lambda-1)^k}\, k \,.\, k^{(k^{1/2})} \,.\, N$, and this proves Lemma 3.

Remark. Though we shall not use it in this paper, in some cases the following variant of Lemma 3 may be useful.

Lemma 3a. *Let* μ *be the measure defined by* (3), *and* $\mu_1(\mathrm{H})$ *the measure defined as follows: if* $\mathrm{H} = \{\mathrm{h}_1, \dots \mathrm{h}_m\}$, $\mathrm{h}_1 < \mathrm{h}_2 < \dots < \mathrm{h}_m$, *then* $\mu_1(\mathrm{H}) = \mu(\mathrm{H}) + \mathrm{h}_1$. *Then*

$$\sum_{\alpha_1+\dots+\alpha_N=k} \frac{k!}{\alpha_1!\dots\alpha_N!}\, \lambda^{-\Sigma\binom{\nu_1\,\nu_2\,\dots\,\nu_N}{1\;\;2\;\dots\;N}\Delta_\nu}$$

$$\leq \frac{\lambda^{2k}k}{(\lambda^{\delta}-1)^k}\, k^{(k^{1/2})},$$

so that the number N *does not appear in the right hand of the inequality.*

The proof is identical to that of Lemma 3. In the present case we will have besides the factor λ^{-h_1} in the right side of (6) or (7), so that N will not appear in the last formulas (6a) and (7a).

3. Application to Hilbert and L²-spaces. Let $\mathbf{A} = \{T\}$ be a commutative normed ring. This means that $\mathbf{A}$ is a set in which a sum $T_1 + T_2 = T_2 + T_1$, a product $T_1 T_2 = T_2 T_1$, and a norm $\| T \| \geq 0$ are defined, in such a way that the following conditions are satisfied:

$$a)\ (T_1 + T_2)\, T_3 = T_1 T_3 + T_2 T_3 \,.\quad b)\ \| T_1 + T_2 \| \leq \| T_1 \| + \| T_2 \| \,.$$
$$c)\ \| T_1 T_2 \| \leq \| T_1 \|\, \| T_2 \|.$$

We shall write $T^2 = TT$, $T^n = T^{n-1}\, T = T\, T^{n-1}$.

THEOREM 1. *If* $T = T_1 + \dots T_N$ *and the* T_i *satisfy the condition*

(A) $$\| T_i T_j \| \leq 2^{-|i-j|}, \quad \| T_i \| < 1,$$

then

(1) $$\| T^k \| \leq 2^{3k} k \,.\, k^{(k^{3/4})} \,.\, N.$$

PROOF. By the property *b*) of the norm we have

$$\| T^k \| \leq \sum_{\alpha_1 + \dots \alpha_N = k} \frac{k\,!}{\alpha_1! \dots \alpha_N!} \| T_1^{\alpha_1} \dots T_N^{\alpha_N} \|.$$

To any term of the form $T_1^{\alpha_1} \dots T_N^{\alpha_N}$ we assign the function

$$f(i) = \begin{pmatrix} \alpha_1 \; \alpha_2 \dots \alpha_N \\ 1 \; 2 \dots N \end{pmatrix}.$$

In particular, if $h_1 < \dots < h_m$, to $T_{h_1} T_{h_2} \dots T_{h_m}$ corresponds the characteristic function φ_H of the set $H = \{ h_1, \dots h_m \}$.

From the hypothesis (A) we have that

(8) $$\| T_{h_1} \dots T_{h_m} \| \leq \| T_{h_1} T_{h_m} \| \cdot \| T_{h_2} T_{h_{m-1}} \| \dots$$
$$\leq 2^{-[(h_m - h_1) + (h_{m-1} - h_2) + \dots]} = (\sqrt[5]{2})^{-\mu(H)},$$

where $\mu(H)$ is the measure defined by (3).

On the other and, to any representation of $f(i)$ of the form

$$f(i) = \gamma_1 \varphi_{H_1}(i) + \gamma_2 \varphi_{H_2}(i) + \dots \qquad (\gamma_i = \text{non-negative integers})$$

we make correspond the descomposition of $T_1^{\alpha_1} \dots T_N^{\alpha_N}$ into the factors

$$T_1^{\alpha_1} \dots T_N^{\alpha_N} = (T_{h_1'} T_{h_2'} \dots)^{\gamma_1} (T_{h_1''} T_{h_2''} \dots)^{\gamma_2} \dots,$$

where

$$(h_1', h_2', \dots) = H_1, \qquad (h_1'', h_2'', \dots) = H_2 \dots$$

Using (8) we get that

$$\| T_1^{\alpha_1} \dots T_N^{\alpha_N} \| \leq \| T_{h_1'} T_{h_2'} \dots \|^{\gamma_1} \,.\, \| T_{h_1''} T_{h_2''} \dots \|^{\gamma_2} \dots$$
$$\leq (\sqrt[5]{2})^{-(\gamma_1 \mu(H_1) + \gamma_2 \mu(H_2) + \dots)}$$
$$\leq (\sqrt[5]{2})^{-\Sigma f \Delta \varphi} = (\sqrt[5]{2})^{-\Sigma \begin{pmatrix} \alpha_1 \; \alpha_2 \dots \alpha_N \\ 1 \; 2 \dots N \end{pmatrix} \Delta \varphi}.$$

Applying Lemma 3 we obtain

$$\| T^k \| \leq \frac{(\sqrt[5]{2})^{2k} \, . \, k \, . \, k^{(k^{1/2})} \, . \, N}{(\sqrt[5]{2^5} - 1)^{2k} \ (\sqrt[5]{2} - 1)^k} \leq \frac{2^{k/2} \, . \, k \, . \, k^{(k^{1/2})} \, . \, N}{(\sqrt[5]{2} - 1)^k},$$

and this proves Theorem 1.

Consider now a (not necessarily complete) real Hilbert space $\mathbf{H} = \{ x \}$, and an operator T defined on $\mathbf{H}$ which assigns to any element $x \in \mathbf{H}$ another element $Tx \in \mathbf{H}$. T is not required to be linear, but we assume that T satisfies the Hermitean condition :

$$(Tx, y) = (x, Ty) \tag{9}$$

for any $x, y \in \mathbf{H}$, and that there is a finite number M such that

$$\| Tx \| \leq M \, . \, \| x \|, \quad \text{for all} \quad x \in \mathbf{H}.$$

We shall denote by $\| T \|$ the smallest of such numbers M, so that

$$\| T_1 + T_2 \| \leq \| T_1 \| + \| T_2 \| \quad \text{and} \quad \| T T_1 \| \leq \| T \| \, . \, \| T_1 \|$$

PROPOSITION 1. *If* T *satisfies the condition* (9) *then*

$$\| T \| = (\| T^{2^m} \|)^{2^{-m}} = (\| T_k \|)^{1/k}, \quad k = 2^m,$$

for any m.

This property is well known for linear operators (Cfr. Gelfand [1]), and subsists without changes for non linear ones. Since the proof is very simple we will reproduce it here. By the Schwarz inequality

$$\| Tx \| = \{ (Tx, Tx) \}^{1/2} = \{ (x, T^2 x) \}^{1/2} \leq \| x \|^{1/2} \{ \| T^2 x \| \}^{1/2}$$
$$\leq \| x \|^{1/2} . \| T^2 \|^{1/2} . \| x \|^{1/2} = \| T^2 \|^{1/2} . \| x \|,$$

hence $\| T \| \leq \| T^2 \|^{1/2}$. On the other hand, $\| T^2 \| \leq \| T \| . \| T \| = \| T \|^2$, $\| T^2 \|^{1/2} \leq \| T \|$. Thus $\| T \| = \| T^2 \|^{1/2}$, and by iteration we obtain $\| T^{2m} \|^{2^{-m}} = \| T \|$.

THEOREM 2. *Let* T *be an operator satisfying condition* (9). *If it is possible to decompose* T *into a sum* $T = T_1 + T_2 + \dots T_N$ *such that*

$$\text{(A)} \qquad \| T_i T_j \| \leq 2^{-|i-j|}, \ \|T_i\| \leq 1,$$

$$\text{(B)} \qquad T_i T_j = T_j T_i,$$

then $\| T \| \leq 8$.

PROOF. Since the operators $T_1 \dots T_N$ commute they all belong to a commutative normed ring $\mathbf{A} = \{ T_\alpha \}$. Applying Theorem 1 and Proposition 1, we obtain, for $k = 2^m$,

$$\| T \|^k = \| T^k \| \leq 2^{3k} \,.\, k \,.\, k^{(k^1/2)} \,.\, N,$$

$$\| T \| \leq 8 k^{1/k} \,.\, k^{(1:k^1/2)} \,.\, N^{1/k}.$$

Allowing k to go to infinity, we obtain $\| T \| \leq 8$.

Consider now the particular case $\mathbf{H} = L^2(R^n) =$ the class of functions $f(x)$ which satisfy

$$\| f \|_2 = \left\{ \int_{R^n} |f(t)|^2 \, dt \right\}^{1/2} < \infty,$$

where R^n is the n-dimensional euclidean space (or more generally a locally compact abelian group).

If $k(x) \in L^1$, that is if $k(x)$ is integrable, it defines on L^2 the linear operator

$$Tf = T_k f = (T_k f)(x) = \int_{R^n} f(x - t) \, k(t) \, dt = (f * k)(x),$$

and by a known inequality of Young (Cfr. [2], Chap IV) we have

$$\| Tf \|_2 \leq \| k \|_1 \cdot \| f \|_2,$$

where

$$\| k \|_1 = \int_{R^n} | k(x) | \, dx.$$

Thus

$$\| T \| = \| T_k \| \leq \| k \|_1.$$

From Theorem 2 we obtain then at once the following:

THEOREM 2*a*. *Let* $\mathrm{k}(\mathrm{x}) \in \mathrm{L}^1$. *If it is possible to decompose* $\mathrm{k}(\mathrm{x})$ *into a sum* $\mathrm{k}(\mathrm{x}) = \mathrm{k}_1(\mathrm{x}) + \ldots + \mathrm{k}_N(\mathrm{x})$ *such that*

$$\|k_i * k_j\|_1 \leq C \, . \, 2^{-|i-j|}, \qquad \|k_i\|_1 \leq C. \tag{A}$$

where

$$k_i * k_j(x) = \int_{R^n} k_i(x-t)\, k_j(t)\, dt.$$

then

$$\|f * k\|_2 \leq 16 C^2 \, . \, \|f\|_2$$

holds for every $\mathrm{f} \in \mathrm{L}^2(\mathrm{R}^n)$.

4. **Examples.** Let $R^1 = \{t\}$ be the 1-dimensional euclidean space. For each m we define on $L^2(R^1)$ the operator H_m by

$$H_m f = H_m f(x) = \int_{1/m}^{m} \frac{f(x-t)}{t}\, dt + \int_{-m}^{-1/m} \frac{f(x-t)}{t}\, dt \tag{10}$$

$$= \int_{1/m \leq |t| \leq m} \frac{f(x-t)}{t}\, dt.$$

For each i we define the kernels

$$k_i(t) = \begin{cases} 1/t & \text{if } \; 2^{i-1} \leq |t| \leq 2^i \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

$$k_{-i}(i) = \begin{cases} 1/t & \text{if } \; 2^{-i} \leq |t| \leq 2^{-i+1} \\ 0 & \text{otherwise.} \end{cases} \tag{11a}$$

and the operators

$$T_i f(x) = f * k_i(x) = \int_{R^1} f(x-t)\, k_i(t)\, dt \tag{11b}$$

$$= \int_{2^{i-1} \leq |t| \leq 2^i} \frac{f(x-t)}{t}\, dt$$

$$T_{-i} f(x) = f * k_{-i}(x) = \int_{2^{-i} < |t| < 2^{-i+1}} \frac{f(x-t)}{t}\, dt \tag{11c}$$

It is clear that if $m = 2^N$, then

$$H_m f = H_{2^N} f = \sum_{i=-N}^{N} T^i f. \tag{12}$$

It is easy to verify the following properties of the kernels $k_i(t)$:

$$\int_{R^1} k_i(t)\,dt = 0, \qquad \|k_i\|_1 = 1, \tag{13}$$

$$k_i(t) = 2^{-i} k_1(2^{-i}t), \tag{14}$$

$$\int_{R^1} |k_1(x-t) - k_1(x)|\,dx \leq |t|. \tag{15}$$

Form (14) and (15) we deduce that:

$$\int_{R^1} |k_i(x-t) - k_i(x)|\,dx = 2^{-i} \int_{R^1} |k_1(2^{-i}x - 2^{-i}t) - k_1(2^{-i}x)|\,dx \tag{15a}$$

$$= \int_{R_1} |k_1(\xi - 2^{-i}t) - k_1(\xi)|\,d\xi \leq 2^{-i}|t|.$$

Using (13) and (15a) we obtain, for $i > j$,

$$\|k_i * k_j\|_1 = \int_{R^1} \left| \int_{R^1} k_i(x-t)\,k_j(t)\,dt \right| dx \tag{16}$$

$$= \int_{R_1} \left| \int_{R_1} [k_i(x-t) - k_i(x)]\,k_j(t)\,dt \right| dx$$

$$\leq \int_{2^{j-1} < |t| < 2^j} |k_j(t)| \left| \int_{R^1} |k_i(x-t) - k_i(x)|\,dx \right| dt$$

$$\leq 2^{-i} 2^j \|k_j\|_1 = 2^{-(i-j)}.$$

COROLARY 1. *The kernels* k_i *satisfy the condition* (Λ) *of Theorem 2a. If* $H_m f$ *is defined by* (10), *then these operators are uniformly bounded on* $L^2(R^1)$:

$$\|H_m f\|_2 \leq Cst.\,\|f\|_2, \tag{17}$$

where the constant is independent of m.

In fact, if $m = 2^N$, it follows from (16), (12) and Theorem 2*a* that

$$\| H_{2^N} f \|_2 \leq 8 \| f \|_2$$

If $2^N < m < 2^{N+1}$, then $H_m f = H_{2^N} f + H' f$,
where

$$H'f = \int\limits_{m<|t|<2^{N+1}} + \int\limits_{2^{-N}<|t|<m^{-1}} \frac{f(x-t)}{t}\,dt,$$

and by Young's inequality

$$\| H'f \|_2 \leq \left\{ \int\limits_{m<|t|<2^{N+1}} + \int\limits_{2^{-N}<|t|<m^{-1}} \frac{dt}{|t|} \right\} \| f \|_2 \leq 2 \| f \|_2,$$

and this proves (17).

For any step function $f(t)$ (or for any differentiable function with compact support) the operator

$$Hf(x) = \int_{-\infty}^{\infty} \frac{f(x-t)}{t}\,dt \tag{18}$$

is perfectly defined, and

$$Hf = \lim_{m \to \infty} H_m f = \sum_{n=-\infty}^{\infty} T_n f.$$

Since the step functions are dense in L^2 and by Corolary I the operators H_m are uniformly continuous on $L^2(R^1)$, we obtain the

Corolary 2. *The limit lim* $H_m f = Hf$ *exists for any* $f \in L^2(R^1)$ *and is a bounded operator. Hence the operator* (18) *admits a continuous extension to the whole space* L^2.

The operator Hf is known as the Hilbert transform of f, or the principal value of the integral (18).

Consider now the 2-dimensional euclidean space $R^2 = \{z\}$, and let us use the complex variable notation:

$$z = x + iy = |z| \cdot e^{i\theta}.$$

Let $w(\theta)$ be a function defined on $(0, 2\pi)$ which satisfies the two following conditions:

$$\int_0^{2\pi} w(\theta)\, d\theta = 0, \tag{19}$$

$$\int_0^{2\pi} | w(\theta - d(\theta)) - w(\theta) |\, d\theta \leq C \,.\, d, \tag{20}$$

for any differentiable function $d(\theta)$ such that $| d(\theta) | \leq d$.

For each m we define the operator

$$H_m f(u) = \iint\limits_{1/m < |z| < m} \frac{f(u - z)\, w(\theta)}{|z|^2}\, dx dy\,, \tag{21}$$

where

$$z = |z|\, e^{i\theta} = x + iy,$$

and for each i we define the kernels

$$k_i(z) = \begin{cases} \dfrac{w(\theta)}{|z|^2} \text{ if } 2^{i-1} \leq |z| < 2^i, \quad z = |z|\, e^{i\theta} \\ 0 \qquad \text{otherwise} \end{cases} \tag{22}$$

$$k_{-i}(z) = \begin{cases} w(\theta)/|z|^2 \text{ if } \quad 2^{-i} < |z| < 2^{-i+1} \\ 0 \qquad \text{otherwise} \end{cases} \tag{22a}$$

and the operators:

$$T_i f(u) = f * k_i(u) = \iint\limits_{2i-1 < |z| < 2^i} \frac{f(u - z)\, w(\theta)}{|z|^2}\, dx dy \tag{22b}$$

$$T_{-i} f(u) = f * k_{-i}(u), \tag{22b}$$

so that for $m = 2^N$ we have

$$H_m f = H_{2^N} f = \sum_{i=-N}^{N} T_i f. \tag{12a}$$

Using (19) and (20) it is easy to verify the following properties of the kernels k_i:

$$(13a)\qquad \iint_{R^2} k_i(z)\,dxdy = 0, \qquad (z = x + iy),$$

$$(14a)\qquad k_i(z) = 2^{-2i} k_1(2^{-i} z)$$

$$(15a)\qquad \iint_{R^2} |k_1(z-u) - k_1(z)|\,dzdy \leq Cst.\,|u|$$

Only (15*a*) needs a proof. Let $z_1 = z - u = \rho_1 e^{i\theta_1}$, $z = \rho e^{i\theta}$, and suppose that $1 \leq \rho_1 \leq 2$ and $1 \leq \rho \leq 2$. Then

$$|k_1(z_1) - k_1(z)| = \left|\frac{w(\theta_1)}{\rho_1^2} - \frac{w(\theta)}{\rho^2}\right| \leq$$
$$\frac{|w(\theta_1) - w(\theta)|}{\rho_1^2} + \frac{2\,|w(\theta)|\,|\rho_1 - \rho|}{\rho_1^2\rho^2}.$$

Therefore if A denotes the set of the z such that $1 \leq |z| \leq 2$, $1 \leq |z-u| \leq 2$, we shall have

$$\iint_A |k_1(z-u) - k_1(z)|\,dzdy \leq$$
$$\int_1^2 \rho\,d\rho \left\{ \int_0^{2\pi} \frac{|w(\theta_1) - w(\theta)|}{\rho_1^2}\,d\theta + \int_0^{2\pi} \frac{2\,|w(t)|\,(\rho_1 - \rho)}{\rho_1^2\rho^2}\,d\theta \right\}$$
$$\leq Cst. \int_1^2 \left\{ \int_0^{2\pi} |w(\theta) - w(\theta - d(\theta))|\,d\theta \right\} d\rho + Cst.\,(\rho_1 - \rho),$$

where $|d(\theta)| \leq cst.\,|u|$, and $\rho_1 - \rho \leq |u|$. Hence by (20),

$$(15b)\qquad \iint_A |k_1(z-u) - k_1(z)|\,dzdy \leq Cst.\,|u|.$$

Since the set where $|k_1(z-u) - k_1(z)| \neq 0$ is equal to $A + A'$ where A' is a set of measure $\leq 2|u|$, the desired inequality (15*a*) follows from (15*b*) and (14).

From (13*a*), (14) and (15*a*) we deduce, as above, the following results:

COROLARY 1*a*. *The kernels* k_i *defined by* (22) *satisfy the condition* (A) *of Theorem 2a, and the operators* H_m *defined by* (22*b*) *are uniformly bounded on* $L^2(R^2)$.

COROLARY 2*a*. *The lim* $H_m f = Hf$ *exists for any* $f \in L^2(R^2)$ *and is a bounded operator which gives the principal value of the integral*

$$Hf(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(u-z)\,w(\theta)}{|z|^2}\,dxdy$$

The Corolary 2*a* was first proved by Zygmund and Calderon [3] using the theory of Fourier transforms.

The operators H_m in the case R^n, $n \geq 3$, are defined in a similar way and the above corolaries hold for any n.

In the following paper we shall give applications to the ergodic theorems.

REMARK. The first direct proof (without using Fourier Analysis) of Corolary 1, for the case R^1, was given by Besicovitch and improved by Lusin [4]. Lusin's method is a very special case of our general Theorem 2 and does not apply to the n-dimensional operators H_m, if $n \geq 2$. However, Professor Zygmund and I showed that Lusin's proof still applies for the particular 2-dimensional kernel $k(z) = z^{-2} = e^{-2i\theta} / |z|^2$, that is if $w(\theta) = e^{-2i\theta}$.

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BIBLIOGRAPHY

1. I. GELFAND, *Normierte Ringe*. Recueil Mathématique, 9 (1941).
2. A. ZYGMUND, *Trigonometrical Series*. Monogr. Math. Warsaw, 1935.
3. A. CALDERÓN and A. ZYGMUND, *On the existence of certain singular integrals*. Acta Mathematica, 88 (1952).
4. N. LUSIN, *On a singular integral*. In the book « Integral and trigonometrical series », Moscow-Leningrad (1951).

A general interpolation theorem for linear operations*

By M. COTLAR [1]

Let $\mathbf{D} = \{f(x)\}$ be the set of all step functions defined on the n-dimensional eucliden space R^n, and $T = Tf$ a linear (or semilinear) operator defined on $\mathbf{D}$, which assigns to each $f \in \mathbf{D}$ an arbitrary function $Tf(x) = (Tf)(x)$. If (1) $\| Tf \|_p \leq C_p \cdot \| f \|_p$, $p \ngtr 1$, and (2) $\| Tf \| \leq C_1 \cdot \| f \|_1$, hold for all $f \in \mathbf{D}$, then from a theorem of M. Riesz ([2], p. 198) it follows that (3) $\| Tf \|_r \leq C_r \cdot \| f \|_r$, for every $1 < r < p$. The aim of this paper is to show that the norms $\| Tf \|_1$ in (2) can be replaced by considerably smaller values. We modify the norm $\| Tf \|_1$ in the following ways: *a*) by replacing the operator T by a smaller one, *b*) by replacing the range of integration by smaller sets, and *c*) by introducing small variations of the function $f(x)$. Finally we confine the functions to cubes with centers at a fixed point x.

Since we give direct proofs, the knowledge of Riesz's theorem is not assumed here, and the paper is self contained.

1. A covering lemma. We begin by proving a lemma of Vitali's type, which will be used in this and the following papers ([6]).

LEMMA 1. *Let* $Q_1, Q_2, \ldots, Q_k \ldots$ *be a sequence of n-dimensional cubes (with sides paralel to the coordinate axes), and let* l_k *denote the length of the side, and* x_k *the center of* Q_k. *If*

$$a) \qquad l_k \geq l_{k+1}, \ \text{and } b) \ x_k \overline{\in} \mathbf{U}_{i=1}^{k-1} Q_i,$$

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* Received August 15, 1955.

[1] The essential parts of this and the following papers are taken from the author's Doctoral Disertation (University of Chicago 1953).

I wish to express my gratitude to Professor A. Zygmund for his valuable and generous advice and suggestions.

then every point x *of the space belongs to at most* 2^n *cubes* Q_k (n = *dimension of the space* R^n).

PROOF. Consider first the case $n=1$. Let $x_0 \in R^1$ and let Q_i, Q_j $(i<j)$ be the first two cubes of the sequence which contain the point x_0. Since the center x_j of Q_j does not belong to Q_i and $l_j \leq l_i$, it is clear that the distance of x_0 to the set $R^1 - (Q_i \cup Q_j)$ is $> l_j/2$. Any other cube Q_k, $k>j$, has its center x_k in the set $R^1 - (Q_i \cup Q_j)$ and the length of its side is $l_k \leq l_j$, hence it cannot contain the point x_0. This proves the lemma for $n=1$.

We shall now prove the lemma by induction on n. Assume that the lemma is true for R^{n-1}. Consider a point $x_0 \in R^n$, a hyperplane π passing through x_0 and perpendicular to one of the axes, and let E_1, E_2 be the two half-spaces determined by π. Let $Q' = Q_{i_1}$, $Q'' = Q_{i_2}, \dots$, $(i_1 < i_2 < \dots)$ be the cubes with centers $x', x'', \dots$ in the half-space E_1, which contain x_0, and let $P', P'', \dots$ be the intersections of $Q', Q'', \dots$ with π. The intersections $P', P'', \dots$ are $(n-1)$-dimensional cubes with sides paralel and equal to those of $Q', Q'', \dots$, and the center $z^{(k)}$ of $P^{(k)}$ is the projection of the center $x^{(k)}$ of $Q^{(k)}$ on π. Thus, the sequence $P', P'', \dots$ satisfies condition a); it also satisfies condition b), because if it were $z_k \in P^{(i)}$, $i<k$, since $z^{(k)}$ is the projection of $x^{(k)}$ and the distance $(z^{(k)}, x^{(k)}) \leq l_k \leq l_i$, it would be $x^{(k)} \in Q^{(i)}$, in contradiction with the hypothesis. Thus, $P', P'', \dots$ is a sequence of $(n-1)$ dimensional cubes satisfying a) and b), and by the inductive assumption there are at most 2^{n-1} cubes $P^{(k)}$, and hence at most 2^{n-1} cubes $Q^{(k)}$ containing x_0. Similarly there are at most 2^{n-1} cubes containing x_0 and with centers belonging to E_2, and this proves the lemma.

The following lemma sharpens a lemma used by N. Wiener [1] and other authors.

LEMMA 2. *If for each point* x *of a compact set* $S \subset R^n$ *there is given a cube* $Q(x)$ *with center at* x, *then, given* $\varepsilon > 0$, *it is possible to select a sequence of these cubes* $Q_1 = Q(x_1)$, $Q_2 = Q(x_2), \dots$ *such that:*

1) *The measure of the set* $S - (\bigcup_i Q_i) \cap S = S - \bigcup_i Q_i$ is $< \varepsilon$. *That is, the points of* S *no covered by the* Q_i *belong to a set of measure* $< \varepsilon$.

2) *Every point* x *of the space belongs to at mos* 2^n *cubes* Q_i.

3) *The cubes* $\frac{1}{2}Q_i$ *do not overlap, where* $\frac{1}{2}Q$ *is the cube with the same center as* Q *but of half the side.*

4) *There is a system of disjoint sets* E_i *such that* $\frac{1}{2}Q_i \subset E_i \subset Q_i$, *and* $\bigcup_i E_i = \bigcup_i Q_i$.

PROOF. Let $l(x)$ = length of the side of the cube $Q(x)$, and $s_1 = \sup\limits_{x \in S} l(x)$.

Since S is compact the theorem is evident if $s_1 = \infty$, thus let us assume $s_1 < \infty$. Take a point $x_1 \in S$ such that $s_1^n < (1 + \varepsilon) l_1^n$, where, $l_1 = l(x_1)$, and let Q^1 be the cube with center at x_1 and side $= s_1$. Then $Q^1 \supset Q(x_1) = Q_1$ and $|Q^1 - Q_1| < \varepsilon s_1^n$, where $|Q|$ denotes the measure of Q. Consider all the points $x \in S - Q^1$, and let $s_2 = \sup\limits_{x \in S - Q^1} l(x)$, $s_2 \leq s_1$.

Take a point $x_2 \in S - Q^1$ such that $s_2^n < (1 + 2^{-1}\varepsilon)\ l_2^n, l_2 = l(x_2)$, and let Q^2 be the cube with center at x_2 and side $= s_2 \leq s_1$. Then, $Q^2 \supset Q(x_2) = Q_2$, and $|Q^2 - Q_2| < 2^{-1}\varepsilon s_2^n, \ldots$. Continuing in this way, for all $i = 1, 2, \ldots$ we obtain two sequences of cubes $Q^1, Q^2, \ldots$, and $Q_1, Q_2, \ldots$, such that :

a) The sequence Q^i satisfies the two conditions of Lemma 1.

b) Q^i and Q_i have the same center x_i, $Q^i \supset Q_i$, and $|Q^i - Q_i| < 2^{-i}\varepsilon$.

(*c*) $x_j \bar{\in} Q^i$, if $j > i$.

(*d*) If $x \in S - \bigcup_{i=1}^{k} Q^i$, then $l(x) \leq s_k$.

We shall show that the sequence Q_i satisfies conditions 1), 2) and 3).

By Lemma 1 (*a*) and (*b*), it is clear that Q_i satisfy condition 2). Since $s_k \leq s_i$ and $x_k \bar{\in} Q^i$ if $k > i$, it is clear that the cubes $\frac{1}{2} Q^i$, and hence the cubes $\frac{1}{2} Q_i$, do not overlap. We shal prove now that $S \subset \bigcup_i Q^i$. In fact, let $x' \in S - \bigcup_i Q^i$. Then by (*d*) we must have $0 < l(x') \leq s_i$, and $|\frac{1}{2} Q^i| \geq 2^{-n} l(x')^n = a > 0$, for any $i = 1, 2, \ldots$ Since the cubes $\frac{1}{2} Q^i$ do not overlap and S is compact, this implies that the sequence $Q^1, Q^2, \ldots$ contains a finite number m of cubes. Therefore $s_{m+1} = 0$, $l(x') \leq s_{m+1} = 0$, and we arrive to a contradiction.

Thus, $S \subset \bigcup_i Q^i$, and by (*b*) it follows that $|S - \bigcup_i Q_i| < \sum 2^{-i}\varepsilon = \varepsilon$, so that condition 1) is satisfied.

Finally, if the sequence E_i is defined by induction as follows :

$$E_1 = Q_1 - \bigcup_{i \neq 1} (\tfrac{1}{2} Q_i), \quad E_k = (Q_k - \bigcup_{i < k} E_i) - \bigcup_{i \neq k} (\tfrac{1}{2} Q_i),$$

it is easy to see that this sequence E_i satisfies condition 4).

This proves Lemma 2.

REMARK 1. From the above proof it is clear that if $l(x) =$ length of side of $Q(x)$, is a lower semicontinuous function of x, then condition 1) of lemma 2 can be replaced by the following one:

1*a*) S is covered by the cubes $Q_i : S \subset \mathbf{U}_i Q_i$.

REMARK 2. By a similar argument the following generalizations can be proved:

LEMMA 1*a*. Let Q_1, Q_2, be a sequence of n-dimensional cubes and let $l_k =$ length of side of Q_k, $x_k =$ center of Q_k. If there is a number $c \geq 1$ such that:

$$a) \quad l_{k+1} \leq c l^k; \qquad\qquad b) \quad x_{k+1} \bar{\in} \mathbf{U}_{i=1}^{k} Q_i;$$

then every point x of the space R^n belongs to at most $(2c)^n$ cubes Q_i.

LEMMA 2*a*. If for each point x of a compact set S there is given a cube $Q(x)$ with center at x, then it possible to select a sequence of these cubes $Q_1 = Q(x_1)$, $Q_2 = Q(x_2)$, ..., such that

1*a*) S is covered by the cubes $Q_i : S \subset \mathbf{U}_i Q_i$.

2*a*) Every point x of the space belongs to at most 2^{2n} cubes Q_i.

3*a*) The cubes $\frac{1}{4} Q_i$ do not overlap, where $\frac{1}{4} Q$ is the cube with the same center as Q and $\frac{1}{4}$ of side.

4*a*) There is a system of disjoint sets E_i such that $\frac{1}{4} Q_i \subset E_i \subset Q_i$ and $\mathbf{U}_i E_i = \mathbf{U}_i Q_i$.

2. The associate operators $\mathbf{P}_m$ and $\mathbf{P}_M$.

Let $R^n = \{x\}$ be the n-dimensional space, $L^r = L^r(R^n)$, $0 < r < \infty$, the set of all measurable (real) functions $f(x)$ defined on R^n and such that

$$\|f\|_r = \Big(\int_{R^n} |f(x)|^r \, dx \Big)^{1/r} < \infty,$$

and $\mathbf{D} = \{f(x)\}$ the set of all step functions defined on R^n, that is the set of all functions of the form $f(x) = \sum_{i=1}^{m} \lambda_i \varphi_{Q_i}(x)$, where Q_i, are n-dimensional cubes with sides paralel to the axes, and φ_Q the characteristic function of the set Q. Each function $f \in \mathbf{D}$ belongs to the spaces L^r for every r, $0 < r < \infty$, and $\mathbf{D}$, as a subset of L^r, is dense in L^r.

If $f(x)$ is a step function it takes a finite number of values $\neq 0$ and its support, that is the set $\{x : |f(x)| > 0\}$, is composed of a finite number of cubes. We shall use the following notations:

$$(1) \qquad S(f) = \{x : |f(x)| > 0\} = \text{support of } f,$$

$$(1a) \qquad m(f) = \operatorname*{Min}_{x \in S(f)} |f(x)|,$$

$$(1b) \qquad M(f) = \operatorname*{Max}_{x \in S(f)} |f(x)|,$$

so that, if $f \neq 0$, $0 < m(f) \leq M(f)$.
For non negative functions $f(x) \geq 0$ we shall use the following notations:

$$(2) \qquad f^{[\lambda]}(x) = \begin{cases} f(x) \text{ if } f(x) \geq \lambda,\ \lambda \geq 0, \\ 0 \text{ otherwise} \end{cases}$$

$$(2a) \qquad f_{[\lambda]}(x) = f(x) - f^{[\lambda]}(x) \quad \therefore \quad f(x) = f^{[\lambda]}(x) + f_{[\lambda]}(x),$$

$$(2b) \qquad E[f > \lambda] = \text{the set of the points } x \text{ for which } f(x) \geq \lambda,$$

$$(2c) \qquad |\mathrm{E}[f \geq \lambda]| = \mu(\lambda; f) = \text{measure of } E[f \geq \lambda].$$

Thus, $\mu(\lambda) = \mu(\lambda; f)$ is the distribution function of $f(x)$.
A general tool frequently used in Analysis for the evaluation of integrals, is the following formula

$$(3) \qquad \int_{R^n} |g(x)|^r \, dx = -\int_0^\infty \lambda^r d\mu(\lambda) = -\int_0^\infty \lambda^r d_\lambda |E[|g| \geq \lambda]|.$$

which is an easy consequence of the definition of the Lebesgue integral *.

Let T be an operator defined on **D** which assigns to any function $f \in \mathbf{D}$ an arbitrary function $Tf(x) = (Tf)(x)$, $x \in R^n$. T has not to be linear; we only require the following conditions:

$$|T(f_1 + f_2)(x)| \leq |Tf_1(x)| + |Tf_2(x)|.$$

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* The proof can be found in Zygmund's book [**2**], Chap. IX.

T is said to be of the type p if

$$\| Tf \|_p \leq M \,.\, \| f \|_p \,, \quad \text{for all} \quad f \in \mathbf{D}.$$

where M does not depend on f. If T is linear it is of the type p if and only if it can be extended to a bounded operator on the whole space L^p.

A theorem of M. Riesz [2] (2) asserts that if T is simultaneously of the type p_1 and of the type p_2, $1 \leq p_1 < p_2$, that is if

$$(4) \qquad \int_{R^n} | Tf(x) |^{p_1} \, dx \leq O_{p_1} \,.\, \int_{R^n} | f(x) |^{p_1} \, dx$$

and

$$(4\,a) \qquad \int_{R^n} | Tf(x) |^{p_2} \, dx \leq O_{p_2} \,.\, \int_{R^n} | f(x) |^{p_2} \, dx,$$

then T is also of the type p :

$$(4\,b) \qquad \int_{R^n} | Tf |^{p} \, dx \leq O_{p} \,.\, \int_{R^n} | f |^{p} \, dx \,,$$

for every p such that $p_1 < p < p_2$.

Here and in the following O_p will denote a constant which depends only on T and on p, but not on f.

The purpose of this paper is to show that (4 b) still holds if we replace in (4) and (4 a) the norms $\| Tf \|_{p_1}$ and $\| Tf \|_{p_2}$ by smaller values. In order to obtain these smaller values we proceed to modify $\| Tf \|_p$ as indicated in the introduction. In this section we will apply the first procedure as follows:

DEFINITION 1. *To each operator* T *we assign two operators* $\mathrm{P_m}$ *and* $\mathrm{P_M}$ *defined as follows :* $\mathrm{P_m f(x)} = \mathrm{m(f)}$ *if* $| \mathrm{Tf(x)} | \geq \mathrm{m(f)}$, *and zero otherwise,* $\mathrm{P_M f(x)} = \mathrm{M(f)}$ *if* $| \mathrm{Tf(x)} | \geq \mathrm{M(f)}$ *and zero otherwise.*

---

(2) Riesz's theorem asserts moreover that

$$(O_p)^{(p_2 - p_1)/p} \leq (O_{p_1})^{(p - p_1)/p_1} (O_{p_2})^{(p_2 - p)/p_2}$$

Riesz considers also more general types (p, r) (see [2] page 198). However, these other aspects of the theorem will not be considered here and we hope to do it in a future paper.

We shall say that T *is of the* m-*type* p_1 *if*

$$\int_{R^n} |P_m f(x)|^{p_1}\, dx \leq 0_{p_1} \cdot \int_{R^n} |f(x)|^{p_1}\, dx, \tag{5}$$

and of the M-*type* p_2, if

$$\int_{R^n} |P_M f|^{p_2}\, dx \leq 0_{p_2} \cdot \int_{R^n} |f|^{p_2}\, dx. \tag{5a}$$

Since $P_m f(x) \leq |Tf(x)|$, $P_M f(x) \leq |Tf(x)|$, the conditions (5) and (5*a*) are considerably weaker than conditions (4) and (4*a*).

PROPOSITION 1. *a*) T *is of the* m-*type* p, *if and only if*

$$|E[\,|Tf| \geq \lambda\,]| \leq O_p \cdot \frac{1}{\lambda^p} \int_{R^n} |f|^p\, dx \tag{6}$$

holds for every $\lambda \leq$ m (f). *b*) T *is of* M-*type* p, *if and only if* (6) *holds for every* $\lambda \geq$ M (f). *c*) T *is simultaneously of the* m-*type* p_1 *and* M-*type* p_2, $1 \leq p_1 < p_2$, *if and only if* (6) *holds for every* $\lambda > 0$ *and for every* p *such that* $p_1 \leq p \leq p_2$.

PROOF. *a*) Since $\int_{R^n} |P_m f(x)|^p\, dx = (m(f))^p\, |E[\,|Tf| \geq m(f)\,]|$, it is obvious that (6) (for every $\lambda \leq m(f)$) implies (5). Conversely, assume that (5) is true, and let $\lambda \leq m(f)$. Take a cube Q, out of the support of f and of measure $|Q| = \varepsilon$, and define the function $g(x)$ as follows : $g(x) = \lambda$ if $x \in Q$, and zero otherwise. Then $\lambda = m(f+g) = m(g)$, and by hypothesis :

$$|E[\,|T(f+g)| \geq \lambda/2\,]| \leq \frac{2^{p_1} O_{p_1}}{\lambda^{p_1}} \int_{R^n} |f+g|^{p_1}\, dx$$

$$\leq \frac{2^{p_1+1} O_{p_1}}{\lambda^{p_1}} \int_{R^n} |f|^{p_1}\, dx + \frac{2^{p_1+1} O_{p_1}}{\lambda^{p_1}} \cdot \varepsilon \lambda^{p_1},$$

$$|E[\,|Tg| \leq \lambda/2\,]| \leq \frac{2^{p_1} O_{p_1}}{\lambda^{p_1}} \int_{R^n} |g|^{p_1}\, dx \leq \frac{2^{p_1} O_{p_1}}{\lambda^{p_1}} \cdot \varepsilon \lambda^{p_1}.$$

Therefore, since

$$|Tf| \leq |T(f+g)| + |Tg|,$$

$$E[|Tf| \geq \lambda] \subset E[|T(f+g)| \geq \lambda/2] \cup E[|Tg| \geq \lambda/2],$$

$$|E[|Tf| \geq \lambda]| \leq |E[|T(f+g)| \geq \lambda/2]| + |E[|Tg| \geq \lambda/2]|$$

$$\leq \frac{2^{p_1+1} O_{p_1}}{\lambda^{p_1}} \int_{R^n} |f|^{p_1} dx + \varepsilon_1,$$

where ε_1 is arbitrarly small. The part *b*) is proved in a similar way. *c*) Assume that T is simultaneously of the *m*-type p_1 and of the *M*-type p_2, $1 \leq p_1 < p_2$, and let $p_1 \leq p < p_2$. We may assume that $f \geq 0$.

Taking in account that for any λ it is true that

$$m(f^{[\lambda]}) \geq \lambda \geq M(f_{[\lambda]}),$$

ann applying *a*), *b*), we shall have:

$$|E[|Tf| \geq \lambda]| \leq |E[|T(f^{[\lambda/2]})| \geq \lambda/2]| + |E[|T(f_{[\lambda/2]})| \geq \lambda/2]|$$

$$\leq \frac{2^{p_1} O_{p_1}}{\lambda^{p_1}} \int_{R^n} |f^{[\lambda/2]}|^{p_1} dx + \frac{2^{p_2} O_{p^2}}{\lambda^{p_2}} \int_{R^n} |f_{[\lambda/2]}|^{p_2} dx$$

$$\leq \frac{2^{p_1} O_{p_1}}{\lambda^{p_1}} \int |f^{[\lambda/2]}|^{p_1} \left(\frac{|f^{[\lambda/2]}|}{\lambda/2}\right)^{p-p_1} dx$$

$$+ \frac{2^{p_2} O_{p_2}}{\lambda^{p_2}} \int_{R^n} |f_{[\lambda/2]}|^{p_2} \left(\frac{|f_{[\lambda/2]}|}{\lambda/2}\right)^{p-p_2} dx$$

$$\leq O_p \frac{1}{\lambda^p} \int_{R^n} |f(x)|^p dx,$$

and this proves Proposition 1.

DEFINITION 1*a*. *We shall say that* T *satisfies the condition* (p), *if* T *satisfies condition* (6) *for all* $\lambda > 0$.

The following proposition is a special case of an interpolation theorem due to Marcinkiewicz [3] (3).

Proposition A. (Of Marcinkiewicz). *If* T *satisfies simultanously the condition* (p_1) *and the condition* (p_2), *then* T *is of the type* p, *for every* p *such that* $\mathrm{p}_1 < p < \mathrm{p}_2$.

Proof. It is sufficient to consider the case $f \geq 0$. Let us denote $|E[|Tf| \geq \lambda]|$ by $\mu(\lambda)$.

Integrating (3) by parts we obtain

$$(3a) \quad \int_{R^n} |Tf|^p \, dx = -\int_0^\infty \lambda^p d\mu(\lambda) = -\lim_{t \to \infty} t^p \mu(\lambda) + \lim_{t \to \infty} p \int_0^t \lambda^{p-1} \mu(\lambda) \, d\lambda$$

$$= p \int_0^\infty \lambda^{p-1} \mu(\lambda) \, \alpha\lambda.$$

Indeed, if one of the two members of $(3a)$ is finite then both members are finite, $t^p \mu(t) \to 0$, and $(3a)$ is true. In both members are infinite there is nothing to prove. Hence $(3a)$ holds in any case.

From

$$|Tf(x)| \leq |T(f^{[\lambda]})(x)| + |T(f_{[\lambda]})(x)|$$

we deduce

$$\mu(2\lambda) = |E[|Tf| \geq 2\lambda]| \leq |E[|T(f^{[\lambda]})| \geq \lambda]| + |E[|T(f_{[\lambda]})| \geq \lambda]|$$
$$\leq \mu_1(\lambda) + \mu_2(\lambda).$$

Since T satisfies condition (p_1) and condition (p_2), we obtain from $(3\,a)$

$$\int_{R^n} |Tf|^p \, dx = p \int_0^\infty \lambda^{p-1} \mu(\lambda) \, d\lambda = 2^p p \int_0^\infty \lambda^{p-1} \mu(2\lambda) \, d\lambda.$$

---

(3) Marcinkiewicz's interpolation theorem embraces also the case of the more general types (p, r) and gives a relation between the constants O_p. Marcinkiewicz stated his results without proofs. Professor Zygmund supplied the proof and indicated some interesting applications to Fourier Analysis. I used Proposition A in my Doctoral Thesis without knowing Marcinkiewicz's theorem, and I am obliged to Professor Zygmund for letting me know the results of Marcinkiewicz.

$$\leq 2^p p \int_0^\infty \lambda^{p-1} \mu_1(\lambda)\, d\lambda + 2^p p \int_0^\infty \lambda^{p-1} \mu_2(\lambda)\, d\lambda$$

$$\leq 2^p p \,.\, O_{p_1} \int_0^\infty \left\{ \lambda^{p-1} \frac{1}{\lambda^{p_1}} \int_{R^n} |f^{[\lambda]}(x)|^{p_1}\, dx \right\} d\lambda$$

$$+\, 2^p p. \int_0^\infty \left\{ \lambda^{p-1} \frac{O_{p_2}}{\lambda^{p_2}} \int_{R^n} |f_{[\lambda]}(x)|^{p_2}\, dx \right\} d\lambda$$

$$= 2^p{}_p\, O_{p_1} \int_{R^n} dx \left\{ \int_0^\infty \lambda^{p-1-p_1} |f^{[\lambda]}(x)|^{p_1}\, d\lambda \right\}$$

$$+\, 2^p p\, O_{p_2} \int_{R^n} dx \left\{ \int_0^\infty \lambda^{p-1-p_2} |f_{[\lambda]}(x)|^{p_2}\, d\lambda \right\}$$

$$= 2^p p\, O_{p_1} \int_{R^n} dx \left\{ \int_0^{|f(x)|} \lambda^{p-1-p_1} |f(x)|^{p_1}\, d\lambda \right\}$$

$$+\, 2^p p\, O_{p_2} \int_{R^n} dx \left\{ \int_{|f(x)|}^\infty \lambda^{p-1-p_2} |f(x)|^{p_2}\, d\lambda \right\}$$

$$= \frac{2^p p\, O_{p_1}}{p - p_1} \int_{R^n} |f(x)|^{p-p_1} \,.\, |f(x)|^{p_1}\, dx$$

$$+\, 2^p p\, O_{p_2} \,.\, \int_{R^n} |f(x)|^{p-p_2} \,.\, |f(x)|^{p_2}\, dx \leq O_p \int_{R^n} |f(x)|^p\, dx.$$

And this proves the proposition.

From proposition A and Proposition 1 we obtain at once the following

THEOREM 1. *If* T *is simultaneously of the* m-*type* p_1 *and of the* M-*type* p_2, *then* T *is of the type* p *for every* p *such that* $p_1 < p < p_2$. *In other words, the conditions* (5) *and* (5 *a*) *imply the condition* (4 *b*) *for every* p *such that* $p_1 < p < p_2$.

The theorem 1 is not true for $p \leq p_1$. For this case we have the following weaker propositions.

THEOREM 2. *The following two conditions on* T *are equivalent :*

a) T *is simultaneously of the* m-*type* p_1 *and of the* M-*type* p_2; $1 \leq p_1 < p_2$.

b) T *satisfies the condition*

$$\int_S |Tf|^\alpha \, dx \leq \frac{O_p}{p-\alpha} \cdot |S|^{1-\frac{\alpha}{p}} \left\{ \int_{R^n} |f(x)|^p \, dx \right\}^{\alpha/p} \tag{4 c}$$

for every set S *of finite mesure, and* α, p, *such that* $0 < \alpha < p_1$, $p_1 \leq p \leq p_2$.

PROOF. *a*) implies *b*): Let $y > 0$ be a fixed positive number, $\varphi_S(x)$ the characteristique function of the set S, and let us denote

$$| E | \varphi_S(x) . | Tf(x) | \geq \lambda | |$$

by $\mu_S(\lambda)$. Then it is clear that $\mu_S(\lambda) \leq |S|$ = measure of S, and $\mu_S(\lambda) \leq | E | Tf(x) | \geq \lambda | |$. Hence, using Proposition 1 and $(3a)$, we obtain

$$\int_S |Tf(x)|^\alpha \, dx = \int_{R^n} \varphi_S(x) \, Tf(x) |^\alpha \, dx = \alpha \int_0^\infty \lambda^{\alpha-1} \mu_S(\lambda) \, d\lambda = \alpha \int_0^y + \alpha \int_y^\infty$$

$$\leq \alpha \int_0^y \lambda^{\alpha-1} |S| \, d\lambda + \alpha \int_y^\infty \left\{ \lambda^{\alpha-1} . \frac{O^p}{\lambda^p} \int_{R^n} |f(x)|^p \, dx \right\} d\lambda.$$

$$\leq y^\alpha . |S| + \frac{\alpha}{p-\alpha} O_p \, y^{\alpha-p} \int_{R^n} |f|^p \, dx.$$

Taking

$$y = \left\{ \frac{1}{|S|} \int_{R^n} |f(x)|^p \, dx \right\}^{1/p},$$

we obtain (4*c*)

b) implies *a*): Let S be any set of finite measure contained in $E[| Tf | \geq \lambda]$, and let p be any number such that $p_1 \leq p \leq p_2$. Take $\alpha < p_1$ and apply (4*c*):

$$\lambda^{\alpha} | S | \leq \int_S | Tf |^{\alpha} \, dx \leq \frac{O_p}{p-\alpha} | S |^{1-\frac{\alpha}{p}} \Big\{ \int_{R^n} | f(x) |^p \, dx \Big\}^{\alpha/p}$$

hence

$$| S | \leq O_p' \cdot \frac{1}{\lambda^p} \int_{R^n} | f(x) |^p \, dx.$$

Therefore T satisfies condition (p) for every $p, p_1 \leq p \leq p_2$, and this proves the theorem.

The above proof gives also the following

THEOREM 2*a*. *The following two conditions are equivalent :* a_1) T *satisfies the condition* (p_0) *for a fixed* $p_0 \geq 1$. b_1) T *satisfies* (4c) *for every* $\alpha < p_0$. *In particular the two following conditions are equivalent:* a_1) T *satisfies the « Kolmogorof inequality ».*

$$\text{(6a)} \qquad | E[| Tf | \geq \lambda] | \leq \frac{O_1}{\lambda} \int_{R^n} | f(x) | \, dx, \quad \text{for every } \lambda > 0.$$

b_2) T *satisfies the inequality*

$$\text{(4d)} \qquad \int_S | Tf(x) |^{\alpha} \, dx \leq \frac{O_1}{1-\alpha} | S |^{1-\alpha} \Big\{ \int_{R^n} | f(x) | \, dx \Big\}^{\alpha},$$

for every set S *and every* $\alpha < 1$ *(or only for one value* $\alpha < 1$*).*

From the equivalence of the conditions a_2) and b_2) it follows that if (4*d*) is true for one value of $\alpha < 1$ then it is true for every $\alpha < 1$.

THEOREM 2*b*. *If* T *is simultaneously of the* m*-type* p_1 *and of the* M*-type* p_2, $1 < p_1 < p_2$, *then :*

$$\text{(4e)} \quad \int_S | Tf(x) |^{p_1} \, dx \leq O_{p_1, p_2} \Big\{ | S | + \int_{R^n} | f(x) |^{p_1} (1 + \log^{+} | f(x) |) \, dx \Big\}.$$

In particular if T *is simultanously of the* m*type 1 and of the* M*-type* p, $p > 1$, *then*

$$\text{(4f)} \qquad \int_S | Tf(x) | \, dx \leq O_p \Big\{ | S | + \int_{R^n} | f(x) | (1 + \log^{+} | f(x) |) \, dx \Big\}$$

Proof. With the same notations as in theorem 2 and Proposition A, we have that $\mu_S(2\lambda) \leq |S|$, $\mu_S(2\lambda) \leq \mu_1(\lambda) + \mu_2(\lambda)$, hence:

$$\int_S |Tf|^{p_1}\,dx \leq 2^{p_1}p_1 \int_0^\infty \lambda^{p_1-1}\,\mu_S(2\lambda)\,d\lambda \leq 2^{p_1}p_1 \int_0^1 \lambda^{p_1-1}\,|S|\,d\lambda \;+$$

$$2^{p_1}p_1 \int_1^\infty \left\{ \lambda^{p_1-1} \cdot \frac{O_{p_1}}{\lambda^{p_1}} \int_{R^n} |f^{[\lambda]}(x)|^{p_1}\,dx \right\} d\lambda \;+$$

$$+\; 2^{p_1}p_1 \int_1^\infty \left\{ \lambda^{p_1-1}\,\frac{O_{p_2}}{\lambda^{p_2}} \int_{R^n} |f_{[\lambda]}(x)|^{p_2}\,dx \right\} d\lambda$$

$$= 2^{p_1}|S| + \int_{R^n} dx \int_1^{|f(x)|} \frac{O_{p_1}}{\lambda}\,|f(x)\,^{p_1}\,d\lambda + \int_{R^n} dx \int_{|f(x)|}^\infty O_{p_2}\,\lambda^{p_1-p_2-1}\,|f(x)|^{p_2}\,d\lambda$$

$$\leq 2^{p_1}|S| + O_{p_1} \int_{R^n} |f(x)|^{p_1} \log^+ |f(x)|\,dx + \frac{O_{p_2}}{p_2-p_1} \int_{R^n} |f(x)|^{p_1}\,dx,$$

and this proves the theorem.

We still observe the following property: Let T be of the M-type p_0, $p_0 > 1$. Then T satisfies the « Kolmogoroff inequality » $(5d)$, if and only if it satisfies the following inequality

$$(5e) \qquad |E[\,|Tf^{(\lambda)}| > \lambda\,]| \leq \frac{O_1}{\lambda} \int_{R^n} |f(x)|\,dx, \quad \text{for all } \lambda > 0.$$

3. The main theorem. In this section we shall confine our attention to the case $p_1 = 1$ only. Without loss of generality we may assume that $p_2 = 2$.

In this section and henceforth we shall assume that the operator T satisfies the condition (2):

$$(7) \qquad |E[\,|Tf| < \lambda\,]| \leq O_2 \cdot \frac{1}{\lambda^2} \int_{R^n} |f(x)|^2\,dx$$

for every $\lambda > 0$, and every $f \in \mathbf{D}$.

We know already that condition (7) together with condition

$$(7a) \qquad \int_{R^n} |Tf(x)|\,dx = \|Tf\|_1 \leq O_1 \int_{R^n} |f(x)|\,dx = O_1 \|f\|_1 ,$$

implies that

$$(4b) \qquad \int_{R^n} |Tf|^p\,dx \leq O_p \int_{R^n} |f|^p\,dx ,$$

for every p such that $1 < p < 2$.

Now, while keeping condition (7) fixed, we shall weaken the condition $(7a)$ by modifying the value of the norm $\|Tf\|_1$.

For this purpose we shall replace, in the first place, the range of integration R^n by the smaller set $R^n - S(f)$, where $S(f)$ is the support of f, and more generaly by sets of the form $R^n - S_L(f)$, where $S_L(f)$ is a « generalized support » defined as follows. The ordinary support $S(f)$ may be defined as the set of the points x for which $|If(x)| \geq m(f)$, where $If = f$ is the identity operator. The identity operator I satisfies, obviously, the following conditions:

a) If is of the m-type 1, and of the M-type 2.

b) $|I(f+g)| < |If| + |Ig|$.

DEFINITION 2. *Let* $L = Lf$ *be a fixed operator satisfying the conditions* a) *and* b). *We define the generalized support* $S_L(f)$ *of* f *as the set of the points* x *for which* $|Lf(x)| \geq m(f)$.

Now we will introduce small variations of the function f, in the following sense: We will modify the function f only within the support S and by values not exceeding the minimum $m(f)$. More precisely:

DEFINITION 3. For each number $\Delta > 0$ and for each operator L satisfying a) and b), and each $f \in \mathbf{D}$, we define $V(f, L, \Delta) = \{\delta(x)\}$, to be the set of all the function $\delta(x)$ such thet

$$(\delta_1) \qquad \delta(x) = 0, \quad \text{if} \quad x \in R^n - S_L(f),$$

$$(\delta_2) \qquad |\delta(x)| \leq \Delta m(f), \text{ for all } x \in R^n.$$

From the definition it is clear that if $\delta \in V(f, L, \Delta)$ and $\delta' \leq \delta$, then also $\delta'(x) \in V(f, L, \Delta)$. We will replace now f by the function $f + \delta$ which gives the minimum value to the integral. We arrive thus to the following definition.

DEFINITION 3*a*. *For any operator* L *satisfying* a) *and* b), *and for any number* Δ *we define the « modified norm ».*

$$\| Tf \|_{L,\Delta} = \inf_{\delta \in V(f,\, L,\, \Delta)} \Big\{ \int_{R^n - S_L(f)} | T(f - \delta)(x) | \, dx \Big\}.$$

The notation $\| \ \|_{L,\Delta}$ does not mean that $\| \ \|_{L,\Delta}$ is a norm, but only that this number was obtained by a certain modification of the norm $\| \ \|_1$.

Since the null function $\delta \equiv 0$ belongs to $V(f, L, \Delta)$, we have

$$\| Tf \|_{L,\Delta} \leq \int_{R^n - S_L(f)} | Tf(x) | \, dx \leq \| Tf \|_1.$$

In general $\| Tf \|_{L,\Delta}$ is considerably less then $\| Tf \|_1$.

PROPOSITION 2. *Let* T *be an operator satisfying the condition* (7) *and the condition :*

$$\| Tf \|_{L,\Delta} \leq O_1 \cdot \| f \|_1, \tag{7b}$$

for some operator L, *some constant* Δ, *and for all* f∈**D**. *Then* T *is of the* m-*type 1, and hence of the type* p, *for every* p *such that* $1 < p < 2$.

PROOF. Let $\varepsilon < 0$. By hypothesis, there exists a function $\delta(x) \in V(f, L, \Delta)$ such that

$$\int_{R^n - S_L(f)} | Tg(x) | \, dx \leq O_1 \int_{R^n} | f(x) \, dx + \varepsilon, \tag{7c}$$

where $\qquad g(x) = f(x) - \delta(x), \quad f(x) = g(x) + \delta(x).$

Let us denote the set $E[| Tg | > {}^1\!/_2 \, m(f)]$ by E_g. Then, by (7 *c*)

$$| E_g \cap (R^n - S_L(f)) | \leq \frac{2 \, O_1}{m(f)} \int_{R^n} | f(x) | \, dx + \varepsilon \cdot \frac{2}{m(f)}. \tag{7 d}$$

Since the operator L is simultaneously of the m-type 1 and of the M-type 2, and therefore satisfies condition $-(p)$ for every $1 < p < 2$, we obtain

$$(7\,e) \qquad |\,S_L(f)\,| = |\,E[\,|\,Lf\,| \geq m(f)\,]\,| \leq \frac{O_1}{m\,(f)} \int\limits_{R^n} |f(x)\,|\,dx.$$

From $(7\,d)$ and $(7\,e)$ it follows

$$(7\,f) \qquad |\,E_g\,| = |\,E_g \cap S_L(f)\,| + |\,E_g \cap (R^n - S_L(f)\,|$$

$$< \frac{4\,O_1}{m\,(f)} \int\limits_{R^n} |f(x)\,|\,dx + \frac{2\,\varepsilon}{m\,(f)}.$$

Using (7), $(7\,e)$, (δ_1) and (δ_2) we have

$$(7\,g) \qquad |\,E_\delta\,| = \left|\,E\left[\,|\,T\delta\,| \geq \frac{m\,(f)}{2}\right]\right| \leq \frac{2\,O_2}{m(f)^2} \int\limits_{R^n} |\,\delta\,(x)\,|^2\,dx$$

$$\leq \frac{2\,O_2\,\Delta}{m\,(f)^2}\,|\,S_L(f)\,]\,.\,m\,(f)^2 = 2\,O^2\,\Delta\,[\,S_L(f)\,]$$

$$\leq 2\,O_2\,.\,\Delta\,.\,\frac{O_1}{m\,(f)} \int\limits_{R^n} |f(x)\,|\,dx = \frac{O_1'}{m\,(f)} \cdot \int\limits_{R^n} |f(x)\,|\,dx.$$

Since $f = g + \delta$,

$$|\,E[\,|\,Tf \geq m\,(f)\,] \leq \left|\,E\left[\,|\,Tg\,| \geq \frac{m\,(f)}{2}\right]\right| + |\,E\left[\,|\,T\delta\,| \geq \frac{m\,(f)}{2}\right]$$

$$= |\,E_g\,| + |\,E_\delta\,].$$

and since $\varepsilon < 0$ is arbitrary, we obtain from $(7\,g)$ and $(7\,f)$,

$$|\,E\,[\,|\,Tf\,| \geq m\,(f)\,]\,| \leq \frac{O_1''}{m\,(f)} \int\limits_{R^n} |\,f(x)\,|\,dx.$$

Hence T is of the m-type 1, and this proves the proposition. If we define $|\,T\,|_1$ and $\|\,T\,\|_{L.\,\Delta}$ by

$$\|\,T\,\|_1 = \sup_{f \in \mathbf{D}} \frac{\|\,Tf\,\|_1}{\|f\,\|_1}, \qquad \|\,T\,\|_{L.\,\Delta} = \sup_{f \in \mathbf{D}} \frac{\|\,Tf\,\|_{L.\,\Delta}}{\|f\,\|_1},$$

the conditions (7 *a*) and (7 *b*) may be written in the following form:

(7 *a*) $$\| T \|_1 \leq O_1$$

(7 *b*) $$\| T \|_{L,\Delta} \leq O_1.$$

The Proposition 2 replaces the norm $\| T \|_1$ by the smaller number $\| T \|_{L,\Delta}$. We shall now replace $\| T \|_{L,\Delta}$ by a still smaller number. For this purpose, we will give first another expression for $\| T \|_{L,\Delta}$.

Let Q be a cube with center at a fixed point x_0 and with sides paralel to the axes, $\frac{1}{2} Q$ the cube wih the same center but of half side, and let E be a set composed of a finite number of cubes and such that $\frac{1}{2} Q \subset E \subset Q$. Let $\delta_0 \in V(f, L, \Delta)$ be such that

$$\| Tf \|_{L,\Delta} = \int_{R^n - S_L(f)} | T(f - \delta_0) | \, dx - \varepsilon.$$

Then, f and δ_0 being fixed, if Q is sufficientely large, we will have $\varphi_E(x) f(x) = \varphi_Q(x) f(x) = f(x)$, $\varphi_Q(x) \delta_0(x) = \delta_0(x)$, where φ_E is the characteristic function of the set E. Hence, for large Q, and any E such that $\frac{1}{2} Q \subset E \subset Q$, we have:

$$\inf_{\delta \in V(f, L, \Delta)} \int_{R^n - S_L(f)} | T(f\varphi_E - \delta\varphi_Q) | \, dx \leq$$

$$\int_{R^n - S_L(f)} | T(f - \delta_0) | \, dx \leq \| Tf \|_{L,\Delta} + \varepsilon,$$

$$\sup_{E, \frac{1}{2} Q \subset E \subset Q} \; \inf_{\delta \in V(f, L, \Delta)} \int_{R^n - S_L(f)} | T(f\varphi_E - \delta\varphi_Q) | \, dx \leq \| Tf \|_{L,\Delta} + \varepsilon.$$

On the other hand, for any $\delta \in V(f, L, \Delta)$ is also $\delta\varphi_Q \in V(f, L, \Delta)$, hence the left side of the last inequality is $\geq$ than

$$\inf_{\delta \in V(f, L, \Delta)} \int_{R^n - S_L(f)} | T(f - \delta) | \, dx = \| Tf \|_{L,\Delta}.$$

Thus we obtain that

$$(8a)\qquad \| Tf |_{L,\Delta} = \lim_{|Q| \to \infty} \left\{ \sup_{E,\ \frac{1}{2} Q \subset E \subset Q}\ \inf_{\delta \in V(f,L,\Delta)} \int_{R^n - S_L(f)} | T(f\varphi_E - \delta\varphi_Q)(x) |\, dx \right\}$$

This suggest the following definition.

DEFINITION 4. For each cube $Q = Q(x_0)$ with center at x_0 we consider all the sets E such that $\frac{1}{2} Q \subset E \subset Q$, and define :

$$(9)\qquad | Tf \|_{L,\Delta,Q} = \sup_{\frac{1}{2} Q \subset E \subset Q}\ \inf_{\delta \in V(f,L,\Delta)} \int_{R^n - S_L(f)} | T(f\varphi_E - \delta\varphi_Q)(x) |\, dx$$

Then, by (8a) $\| Tf \|_{L,\Delta}$ and $\| T \|_{L,\Delta}$ may be written in the following form : .

$$(8b)\qquad \| Tf \|_{L,\Delta} = \lim_{|Q(x_0)| \to \infty} \| Tf \|_{L,\Delta,Q(x_0)},$$

$$(8c)\qquad \| T \|_{L,\Delta} = \sup_{f\in \mathbf{D}}\ \lim_{|Q(x_0)| \to \infty} \frac{\| Tf \|_{L,\Delta,Q(x_0)}}{\| f\varphi_Q \|_1}$$

for any point $x_0 \in R$. We shall now replace « lim » by « inf » :

DEFINITION 4a. *For each point* $x_0 \in R^n$, *we consider all the cubes* $Q = Q(x_0)$ *with center at* x_0, *and define*

$$(9a)\qquad \| T \|_{L,\Delta,x_0} = \sup_{f\in \mathbf{D}}\ \inf_{Q = Q(x_0)} \frac{\| Tf \|_{L,\Delta,Q}}{\| f\varphi_Q \|_1}.$$

From (8c) it is clear that $\| T \|_{L,\Delta,x_0} \leq | T \|_{L,\Delta}$.

THEOREM 3. *Let* Δ *be a positive number and* L *an operator satisfying the above conditions* a) b). *If* T *satisfies the two conditions:*

$$(7)\qquad | E[\, | Tf | > \lambda] | \leq \frac{O_2}{\lambda^2} \int_{R^n} | f |^2\, dx, \quad \text{for all } \lambda > 0,$$

and

$$(10)\qquad \| T \|_{L,\Delta,x_0} \leq O_1,$$

for all $x_0 \in R^n$, *then* T *satisfies the inequality*

$$\| T \|_{L,\, 2^n \Delta} \leq O'_1 = 2^n O_1. \tag{7b}$$

Furthemore, by Proposition 2, T *is of the* m-*type 1 and of the type* p, *for every* p *such that* $1 < p < 2$.

PROOF. Let $f \in \mathbf{D}$. By assumption, for every $x_0 \in S(f)$ there is a cube $Q(x_0) = Q$, such that

$$\| Tf \|_{L,\, \Delta,\, Q} \leq O_1 \| f \varphi_Q \|_1,$$

hence, by definition of $\| Tf \|_{L,\, \Delta,\, Q}$, for any set E such that $\frac{1}{2} Q \subset E \subset Q$ there is a function $\delta_E \in V(f, L, \Delta)$, such that

$$\int\limits_{R^n - S_L(f)} | T(f \varphi_E - \delta_E \varphi_Q)(x) | \, dx \leq O_1 \| f \varphi_Q \|_1, \quad Q = Q(x_0). \tag{10a}$$

By Lemma 2, it is possible to select a sequence of these cubes $Q_1 = Q(x_1)$, $Q_2 = Q(x_2)$, ... and a sequence of sets E_1, E_2, ... with the following properties: There is a set $S_1 = S_1(f) \subset S(f)$ such that $| S(f) - S_1 | < \varepsilon$ and: 1) $S_1(f) \subset \bigcup_i Q_i = \bigcup_i E_i$; 2) any point $x \in R_n$ belongs to at most 2^n cubes Q_i; 3) the sets E_i are disjoint and $\frac{1}{2} Q_i \subset E_i \subset Q_i$.

By (10 *a*) for each i there exists a function $\delta_i = \delta_{Ei} \in V(f, T, \Delta)$ such that

$$\int\limits_{R^n - S_L(f)} | T(f \varphi_{Ei} - \delta_i \varphi_{Qi}) | \, dx \leq O_1 \| f \varphi_{Qi} \|_1. \tag{10 b}$$

Let us define

$$\delta(x) = \sum_i \delta_i(x) \, \varphi_{Qi}(x) = \sum_i \delta_{Ei}(x) \, \varphi_{Qi}(x).$$

Since all the δ_i vanish outside of $S_L(f)$, we have

$$\delta(x) = 0, \quad \text{if} \quad x \in R^n - S_L(f).$$

Since any point x belongs to at most 2^n cubes Q_i, and since each

$$| \delta_i(x) | \leq \Delta m(f),$$

we obtain that

$$| \delta(x) | \leq 2^n \Delta \, . \, m(f), \quad \text{for all} \quad x \in R^n.$$

Thus, $\delta(x)$ satisfies the conditions (δ_1) and (δ_2) with the constant $2^n \Delta$ instead of Δ, that is $\delta(x) \in V(f, L, 2^n\Delta)$.

Since the sets E_i are disjoint, denoting $f(x)\varphi_{S_i}(x)$ by $f'(x)$,

$$f'(x) = \sum_i f(x)\varphi_{E_i}(x),$$

$$f'(x) - \delta(x) = \sum_i (f\varphi_i - \delta_i\varphi_{Q_i})(x),$$

and we obtain

$$(10\,c) \qquad |T(f' - \delta)(x)| \leq \sum_i |T(f\varphi_i - \delta_i\varphi_{Q_i})(x)|.$$

From (10 c) and (10 b) we deduce that

$$\int_{R^n - S_L(f)} |T(f' - \delta)(x)|\,dx \leq \sum_i \int_{R^n - S_L(f)} |T(f\varphi_i - \delta_i\varphi_{Q_i})(x)|\,dx$$

$$\leq O_1 \sum_i \|f\varphi_{Q_i}\|_1 = O_1 \sum_i \int_{Q_i} |f(x)|\,dx.$$

Since any point $x \in R^n$ belongs to at most 2^n cubes Q_1,

$$\sum_i \int_{Q_i} |f(x)|\,dx \leq 2^n \int_{R^n} |f(x)|\,dx,$$

and we obtain

$$\int_{R^n - S_L(f)} |T(f' - \delta)(x)|\,dx \leq 2^n O_1 \int_{R^n} |f(x)|\,dx.$$

Since $\delta \in V(f, L, 2^n\Delta)$ and since $\varepsilon > 0$ is arbitrary, using (7) it is easy to see that the last inequality holds with f instead of f', hence $\| Tf \|_{L,\,2^n\Delta} \leq 2^n O_1 \|f\|_1$.

Hence, by proposition 2, T is of the m-type 1, and of type p, for $1 < p < 2$. This proves the theorem.

4. The modified norms $\| Tf \|_r$. The generality of the Theorem 3 makes it difficult to apply it directly. For this reason, we give in this section an intermediate theorem which is more easy to handle.

We shall call « a square-support » of $f(x)$, any cube $Q(f)$ such that $Q(f) \supset S(f)$, where $S(f)$ is the ordinary support of f.

For each square-support $Q(f)$ we define the mean value

$$(11) \qquad \mu(f, Q) = \frac{1}{|Q|} \int_{Q} |f(x)|\, dx, \qquad Q = Q(f).$$

DEFINITION 5. *We define* $W(f, Q(f), \Gamma) = \{\gamma(x)\}$, *to be the class of all the functions* $\gamma \in \mathbf{D}$ *such that*

$$(\gamma_1) \qquad \gamma(x) = 0, \qquad \text{if} \qquad x \in R^n - Q(f),$$

$$(\gamma_2) \qquad |\gamma(x)| \leq \Gamma \cdot \mu(f, Q(f)), \text{ for all } x \in R^n,$$

Γ *being a fixed positive number, and* $Q(f)$ *a square-support of* f. *Next, for each square-support* $Q = Q(f)$, *we define*

$$(11a) \qquad \| Tf \|_{Q,\Gamma} = \inf_{\gamma \in W(f, Q, \Gamma)} \int_{R^n - 2Q} | T(f - \gamma)(x) |\, dx,$$

where 2Q *is the cube with the same center as* Q *and of twice the side. Finally we define*

$$(11b) \qquad \| Tf \|_{\Gamma} = \sup_{Q = Q(f)} \| Tf \|_{Q,\Gamma}.$$

Let Λ be the Hardy-Littlewood maximal operator, defined by

$$(\Lambda) \qquad \Lambda f(x) = \sup_{Q(x)} \left\{ \frac{1}{|Q(x)|} \int_{Q(x)} |f(t)|\, dt \right\},$$

where the sup is taken for all the cubes $Q(x)$ containing the point x. By a known theorem of Hardy and Littlewood (the proof of this, and more general theorems, is given in the following paper [6]) the operator $L = \Lambda f$ satisfies the conditions a), b) required in Definition 2.

THEOREM 4. *Let* T *be an operator satisfying* (7). *If* T *satisfies the inequality*

$$(12) \qquad \| Tf \|_{\Gamma} \leq O_1 \| f \|_1, \quad \text{for all } f \in \mathbf{D},$$

then T *satisfies also the inequality*

$$(12a) \qquad \| Tf \|_{L,\Lambda} \leq O_1' \cdot \| f \|_1,$$

with $\mathrm{L} = 2^{2n+1}\Lambda$ *and* $\Delta = {}^1/_2\Gamma$, *where* Λ *is the Hardy-Littlewood maximal operator. Hence* T *is of the* m*-type 1 and of the type* p, *for every* $1 < \mathrm{p} < 2$.

PROOF. By theorem 3 it is enough to prove that

$$\| T \|_{x_0, L, \Delta} \leq O_1,$$

for every point x_0. Or, what is the same, it is enough to prove that given $f \in \mathbf{D}$ and $x_0 \in \mathrm{R}^n$, there is a cube $Q = Q(x_0)$, such that for every set E, $\frac{1}{2}Q \subset E \subset Q$, there exists a function $\delta_E \in V(f, L, \Delta)$ such that

$$(12b) \qquad \int_{R^n - S_L(f)} | T(f\varphi_E - \delta_E \varphi_Q)(x) | \, dx \leq O_1 \int_Q | f(x) | \, dx,$$

If x_0 is outside of $\mathbf{S}(f)$, (12*b*) will be trivially satisfied if we take Q sufficientely small and $\delta_E \equiv 0$. Therefore, let us assume that $x_0 \in \mathbf{S}(f)$.

Then we have

$$\lim_{|Q(x_0)| \to 0} \frac{1}{|Q(x_0)|} \int_{Q(x_0)} | f(x) | \, dx = | f(x_0) | \geq m(f).$$

For very small cubes $Q(x_0)$ we have

$$\frac{1}{|Q(x_0)|} \int_{Q(x_0)} | f(x) | \, dx \geq \frac{m(f)}{2}$$

while for very large cubes $Q(x_0)$ this quotient is arbitrary small. Hence there is a cube $Q = Q(x_0)$ with center at x_0 satisfying

$$(13) \qquad \frac{1}{|Q|} \int_Q | f(x) | \, dx = \frac{m(f)}{2} \quad \therefore \quad \mu(f, Q) = \frac{m(f)}{2}.$$

We shall prove that this cube $Q = Q(x)$ is the desired one. Let E be a set such that $\frac{1}{2}Q \subset E \subset Q$ and let g be defined by $g(x) = f(x)\varphi_E(x)$. Then $Q \supset S(g) =$ support of g, so that Q is a square-support for g, and

$$(14) \qquad \mu(g, Q) \leq \mu(f, Q) = \frac{m(f)}{2}.$$

Hence by hypothesis $\| Tg \|_\Gamma \leq 0_1 \| g \|_1$, and since $\| Tg \|_\Gamma = \sup_Q \| Tg \|_{Q.\Gamma}$, we obtain $\| Tg \|_{Q.\Gamma} \leq O_1 \| g \|_1$.

By definition of $\| Tg \|_{Q.\Gamma}$, it follows that there exists a function $\gamma(x)$ such that

$$\gamma(x) = 0 \quad \text{if} \quad x \in R^n - Q, \tag{15}$$

$$|\gamma(x)| \leq \Gamma \mu(g, Q) \leq \frac{\Gamma}{2} m(f), \tag{15 a}$$

and

$$\int_{R^n - 2Q} |T(g - \gamma)(x)| \, dx \leq 2\, O_1 \| g \|_1. \tag{16}$$

By (15), $\gamma(x) = \gamma(x) \varphi_Q$, and by definition $g(x) = f(x) \cdot \varphi_E(x)$, so that (16) may be written in the following form

$$\int_{R^n - 2Q} |T(f\varphi_E - \gamma\varphi_Q)(x)| \, dx \leq 2\, O_1 \cdot \| f\varphi_Q \|_1 . \tag{17}$$

If $x \in 2Q$, there is a cube Q' of measure $|Q'| = 2^{2n} |Q|$, such that $x \in Q'$ and $Q \subset Q'$. Hence, by (13)

$$\frac{1}{|Q'|} \int_{Q'} |f(x)| \, dx \geq \frac{1}{2^{2n} |Q|} \int_Q |f| \, dx = \frac{m(f)}{2^{2n+1}}.$$

This shows that $2^{2n+1} \Lambda f(x) \geq m(f)$ for every $x \in 2Q$, so that by definition of $S_L(f) = S_{2^{2n+1}\Lambda}(f)$,

$$R^n - S_L(f) \subset R^n - 2Q \subset R^n - Q. \tag{18}$$

Therefore, by (17)

$$\begin{aligned} \int_{R^n - S_L(f)} & |T(f\varphi_E - \gamma\varphi_Q)(x)| \, dx \\ & \leq \int_{R^n - 2Q} |T(f\varphi_E - \gamma\varphi_Q)| \, dx \leq 2\, O_1 \| f\varphi_Q \|_1. \end{aligned} \tag{17 a}$$

From (15), (15*a*) and (18) we deduce that $\gamma \in V(f, L, \tfrac{1}{2}\Gamma)$. Hence from (17*a*) we obtain (12*b*), and this proves the theorem.

COROLARY 1. *Let* T *be an operator satisfying condition* (7) *and the condition :*

$$(\mu) \qquad \int\limits_{R^n - 2Q} \left| T(f - \mu'(f, Q)\,\varphi_Q)(x) \right| dx \le O_1 \int\limits_{R^n} |f(x)|\, dx, \qquad Q = Q(f),$$

for every square-support $Q = Q(f)$ *and for every* $f \in \mathbf{D}$, *and where* $\mu'(f, Q)$ *is defined by*

$$\mu'(f, Q) = \frac{1}{|Q|} \int\limits_Q f(x)\, dx.$$

Then T *is of the m-type 1 and hence satisfies the inequality*

$$(4b) \qquad \int\limits_{R^n} |Tf(x)|^p\, dx \le O_p \int\limits_{R^n} |f(x)|^p\, dx,$$

for every p *such that* $1 < p < 2$.

In the following section we give an application of this Corolary.

5. Examples. The Hilbert operator

$$(19) \qquad Hf(x) = \int\limits_{-\infty}^{\infty} \frac{f(x-t)}{t}\, dt = \lim_{\varepsilon \to 0} H_\varepsilon f(x)$$

$$= \lim_{\varepsilon \to 0} \int\limits_{\varepsilon^{-1} > |t-x| > \varepsilon} \frac{f(x-t)}{t}\, dt,$$

is perfectly well defined, for every step function $f \in \mathbf{D}$.

We have seen in the precedent paper [5] that H is of the type 2 :

$$(20) \qquad \int\limits_{-\infty}^{\infty} |Hf(x)|^2\, dx \le O_2 \int\limits_{-\infty}^{\infty} |f(x)|^2\, dx,$$

so that H satisfies obviously condition (7).

PROPOSITION 3. *The operator* H *satisfies the condition* (μ) *of the precedent Corolary. The operators* H_ε *satisfy the same condition uniformly, that is with the same constant* O_1 (≤ 4).

PROOF. Let $Q = Q(f) = (-a, a)$ be a square support of $f(x)$, and let $\mu'(x) = \mu'(f, Q)\,\psi_Q(x)$. Then

$$\int_{-\infty}^{\infty} \mu'(t)\,dt = \mu'(f, Q) \cdot |Q| = \int_Q f(t)\,dt, \tag{21}$$

$$\int_{-\infty}^{\infty} (f(t) - \mu'(t))\,dt = \int_{-a}^{a} (f(t) - \mu'(t))\,dt = 0,$$

and

$$\int_{-\infty}^{\infty} |\mu'(t)|\,dt = |\mu'(f, Q)| \cdot |Q| \leq \|f\|_1. \tag{21a}$$

Using (21) we obtain

$$|H(f - \mu')(x)| = \left| \int_{-\infty}^{\infty} \frac{f(x-t) - \mu'(x-t)}{t}\,dt \right| = \left| \int_{-\infty}^{\infty} \frac{f(t) - \mu'(t)}{x-t}\,dt \right| \tag{22}$$

$$= \left| \int_Q \frac{f(t) - \mu'(t)}{x-t}\,dt \right| = \left| \int_{-a}^{a} \frac{f(t) - \mu'(t)}{x-t}\,dt - \frac{1}{x} \cdot 0 \right| =$$

$$\left| \int_{-a}^{a} \frac{f(t) - \mu'(t)}{x-t}\,dt - \int_{-a}^{a} \frac{f(t) - \mu'(t)}{x}\,dt \right| = \left| \int_{-a}^{a} \frac{[f(t) - \mu'(t)]\,t}{x\,(x-t)}\,dt \right|$$

$$\leq a \int_{-a}^{a} \frac{|f(t)|\,dt}{|x\,(x-t)|} + a \int_{-a}^{a} \frac{|\mu'(t)|\,dt}{|x|\,|x-t|}.$$

If $x \in R^1 - 2Q$, that is if $|x| \geq 2a$, and if $|t| < a$, then $|x|\,|x-t| \geq \frac{1}{2}|x|^2$, and by (22) and (21a) we obtain

$$|H(f - \mu')(x)| \leq 4a\,\|f\|_1 : |x|^2 \tag{22a}$$

for every $x \in R^1 - 2Q$. Hence

$$\int_{R^1 - 2Q} |H(f - \mu')(x)|\,dx \leq 2 \cdot \int_{2a}^{\infty} \frac{4a \cdot \|f\|_1}{x^2}\,dx = 4a \cdot \|f\|_1,$$

and this proves the proposition for the operator H. The same argument applies to the H_ε.

Proposition 4. *The operators H and H_ε are of the type* p *and*

$$\int_{R^1} | Hf(x) |^p \, dx \leq O_p \int_{R^1} | f(x) |^p \, dx, \tag{23}$$

$$\int_{-\infty}^{\infty} | H_\varepsilon f(x) |^p \, dx \leq O_p \, . \int_{-\infty}^{\infty} | f |^p \, dx, \quad (O_p \text{ independent of } \varepsilon) \tag{23}$$

for every $1 < \mathrm{p} < \infty$. *For* $\mathrm{p} \leq 1$ *these relations should be replaced by the following weaker inequalities :*

$$| E [| Hf | \geq \lambda] | \leq \frac{O_1}{\lambda} \int_{-\infty}^{\infty} | f(x) | \, dx, \quad (\lambda > 0), \tag{23a}$$

$$\int_S | Hf(x) |^\alpha \, dx \leq \frac{O_1}{1-\alpha} | S |^{1-\alpha} \left\{ \int_{-\infty}^{\infty} | f | \, dx \right\}^\alpha, \quad (\alpha < 1). \tag{23b}$$

$$\int_S | Hf(x) | \, dx \leq O_1 \left\{ | S | + \int_{-\infty}^{\infty} | f(x) | \, (1 + \log^+ | f(x) |) \, dx \right\}, \tag{23c}$$

and similarly for the operators H_ε.

Proof. From Corolary 1, Theorem 2, 2*a*) and 2*b*), and from (20) and Proposition 3, it follows that (23*a*), (23*b*), (23*c*) are true, and that (23) is true for $1 < p \leq 2$. If $p > 2$, and q is the conjugate number, $1/p + 1/q = 1$, then, since L^p and L^q are conjugate spaces, by a well known theorem,

$$\| Tf \|_p = \sup_g \left\{ \left| \int_{-\infty}^{\infty} Tf(x) \, . \, g(x) \, dx \right| : \| g \|_q \right\},$$

for all step functions $g \varepsilon \mathbf{D}$. By interchanging the order of integrations it is easy to check that

$$\left| \int_{-\infty}^{\infty} Tf(x) \, . \, g(x) \, dx \right| = \left| \int_{-\infty}^{\infty} g(x) \, dx \int_{-\infty}^{\infty} \frac{f(x-t)}{t} \, dt \right|$$

$$= \left| \int_{-\infty}^{\infty} Tg(x) \, . \, f(x) \, dx \right| \leq \| Tg \|_q \, . \, \| f \|_p .$$

Since $q < 2$, T is of the type q and we obtain

$$\| Tf \|_p \leq \frac{\| Tg \|_q \cdot \| f \|_p}{\| g \|_q} \leq \frac{O_q \cdot \| g \|_q \cdot \| f \|_p}{\| g \|_q} = O_p \cdot \| f \|_p,$$

and this proves Proposition 3.

The inequality (23) is due to M. Riesz who proved it by using the theory of analytic functions. The inequalities (23 *a*), (23 *b*) are due to Kolmogoroff.

Consider now, as in the precedent paper [**5**], the 2-dimensional euclidean space, $R^2 = \{ z = x + iy = | z | e^{i\theta} \}$, a function $w(\theta)$ defined on $(O, 2\pi)$ which satisfies the following conditions:

$$\int_0^{2\pi} w(\theta)\, d\theta = 0, \qquad \int_0^{2\pi} | w(\theta - d(\theta)) - w(\theta) |\, d\theta \leq \mathbf{C} \cdot d,$$

if $| d(\theta) | \leq d$, and the operator Hf defined by

$$(24) \qquad Hf(u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(u - z)\, w(\theta)}{| z |^2}\, dx dy = \lim_{\varepsilon \to 0} H_\varepsilon f(u)$$

$$= \lim_{\varepsilon \to 0} \iint_{\varepsilon < | z - u | < \varepsilon^{-1}} \frac{f(u - z)\, w(\theta)}{| z |^2}\, dx dy.$$

By exactly the same argument as in Proposition 3 and 4 (using the inequality (15 *b*) of the precedent paper) we obtain:

PROPOSITION 3 *a*. *The operators* H *and* H_ε, *defined by* (24), *satisfy the condition* (μ) *uniformly, that is with a common constant* O_1.

PROPOSITION 4 *a*. *The operator* H *defined by* (24) *is of the type* $\mathfrak{p}$, *that is satisfies* (23), *for every* $\mathfrak{p}$ *such that* $1 < \mathfrak{p} < \infty$; H *satisfies also the conditions* (23 *a*), (23 *b*) *and* (23 *c*). *The corresponding operators* H_ε *fullfil the same conditions uniformly, with a common constant* $\mathrm{O}_\mathfrak{p}$.

The Proposition 4 *a* is due to Zygmund and Calderón [4] who proved it by a different method.

The same argument applies to the case R^n, $n \geq 3$, and the Propositions 3 *a* and 4 *a* are true for any *n*-dimensional kernel $K(t) = \dfrac{w(\theta)}{| z |^n}$, where $w(\theta)$ does not depend on $|z|$ and satisfies the above conditions. More general results will be given in the following paper on Hilbert transforms and ergodic theorems.

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BIBLIOGRAPHY

1. N. Wiener, *The ergodic theorem.* Duke Math. J., 5 (1939).
2. A. Zygmund, *Trigonometrical Series.* Monog. Math. Warsaw., 1935.
3. A. Marcinkiewicz, *Comptes Rendus Ac. Sci.* Paris, July (1939).
4. A. Zygmund and A. Calderón, *On the existence of certain singular integrals.* Acta Mathematica, 88 (1952).
5. M. Cotlar, *A combinatorial inequality and its application to L^2-spaces.* Revista Matemática Cuyana, I (1955).
6. — *Some generalizations of Hardy-Littlewood's maximal theorem.* Ibid., I (1955).

Some generalizations of the Hardy-Littlewood maximal theorem *

BY M. COTLAR

If $T = Tf$ is an operator of the type p, or of the m-type p (see the definition in the precedent paper), and if $M = Mf$ is another operator such that $|Mf| \leq |Tf|$ for each f, then it is obvious that Mf is also of the type p, or the m-type p. However, the situation becomes less trivial if we replace the ordinary relation $\leq$ by more general ones.

In this paper we consider two relations $|M| \prec |T|$ and $|M| << |T|$, which express that Mf «is locally in mean» $\leq |Tf|$. The maximal theoren of Hardy and Littlewood ([1], p. 244) corresponds to the case $|M| \prec |I|$ where I is the identity operator.

We also consider the case of product operators $Mh(x, y) = M_1^x (M_2^y h(x, y)$. In this case the situation remains the same if $p > 1$, but some modifications should be introduced if $p \leq 1$. Finally we give a general maximal theorem for double transformations. We give direct proofs and the Hardy-Littlewood theorem is not assumed to be known.

1. Local subordinations of operators. — Let $R^n = \{x\}$ be the n-dimensional space and $\mathbf{D} = \{f(x)\}$ a set of functions dense in all the $L^p(R^n)$-spaces; for instance, $\mathbf{D}$ may be the set of all step functions. Let $T = Tf$ be an operator defined on $\mathbf{D}$ and such that

$$|T(f+g)(x)| \leq |Tf(x)| + |Tg(x)| \tag{1}$$

for each $x \in R^n$, and any $f, g \in \mathbf{D}$.

As in the precedent paper [2], we shall say that T is of the type p, if

$$\|Tf\|_p^p = \int_{R^n} |Tf|^p \, dx \leq O_p \int_{R^n} |f|^p \, dx = O_p \cdot \|f\|_p^p,$$

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* Received August 22, 1955.

for all $f \in \mathbf{D}$. T will be said to satisfy condition (p) if

$$|E[|Tf| \geq \lambda]| \leq \frac{O_p}{\lambda^p} \int_{R^n} |f|^p \, dx, \tag{2}$$

for any $\lambda > O$ and any $f \in \mathbf{D}$. Here $|E[\ ]|$ denotes the measure of the set $E[\ ]$. We have seen in the precedent paper that (2) is equivalent to

$$\int_S |Tf|^\alpha \, dx \leq 0_\alpha \, |S|^{1-\alpha/p} \Big\{ \int_{R^n} |f|^p \, dx \Big\}^{\alpha/p}, \text{ for every } \alpha < p. \tag{3}$$

If $M = Mf$ is another operator defined on R^n and satisfying (1), we shall write $|M| \leq 0_1 |T|$, if $|Mf(x)| \leq O_1 |Tf(x)|$ for every $f \in D$ and almost all $x \in R^n$, O_1 being independent of f and x. The following proposition is trivial and we state it for the sake of completness.

PROPOSITION 1. Let $|M| \leq O_1 |T|$. If T is of the type p, then M is also of the type p. If T satisfies condition (p), then M also satisfies this condition.

DEFINITION 1. *We shall write* $|\mathrm{M}| \prec O_1 |\mathrm{T}|$, *if for each* $\mathrm{f} \in \mathbf{D}$ *and for each* $\mathrm{x} \in \mathrm{R}^n$, *there is a (n-dimensional) cube* $\mathrm{Q}(\mathrm{x})$ *with center at* x *(and with sides paralel to the axes), such that*

$$|Mf(x)| \leq \frac{O_1}{|Q(x)|} \int_{Q(x)} |Tf(t)| \, dt = \frac{O_1}{|Q(x)|} \int \varphi_{Q(x)}(t) \, |Tf(t)| \, dt, \tag{A}$$

where $\varphi_{\mathrm{Q}}(\mathrm{t})$ *is the characteristic function of the set* Q, *and* $|\mathrm{Q}|$ *the measure of* Q.

The inequality (A) expresses that $Mf(x)$ is «locally subordinated» to $|Tf(x)|$, and the term «locally» refers to the point x. We consider now another interpretation of the term «locally subordinated», refering both to the point x and the function f.

DEFINITION 2. *We shall write* $|\mathrm{M}| << \mathrm{O}_1 |\mathrm{T}|$, *if for each* $\mathrm{f} \in \mathbf{D}$ *and* $x \in \mathrm{R}^n$, *there is a cube* $\mathrm{Q}(\mathrm{x})$, *with center at* x *and such that*

$$|Mf(x)| \leq \frac{O_1}{|\mathrm{Q}(x)|} \int_{\mathrm{Q}(x)} |T(\varphi_{\mathrm{Q}(x)} \cdot f)(t)| \, dt. \tag{B}$$

We shall write $|\mathrm{M}|^\alpha << |\mathrm{T}|^\alpha$, *if* $|\mathrm{M}_\alpha| << |\mathrm{T}_\alpha|$, *where* $|\mathrm{M}_\alpha f| = |\mathrm{Mf}|^\alpha$ *and* $|\mathrm{T}_\alpha f| = |\mathrm{Tf}|^\alpha$.

If $I = If$ is the identity operator, $|M| \prec O_1 |I|$ is equivalent to

$$|Mf(x)| \leq O_1 \cdot \underset{Q(x)}{Sup} \left\{ \frac{1}{|Q(x)|} \int_{Q(x)} |f(t)|\, dt \right\},$$

where the sup is taken for all the cubes with center at x.

Let Λ be the Hardy-Littlewood maximal operator (see [1], p. 244), defined by

$$(\Lambda) \qquad \Lambda f(x) = \underset{Q(x)}{Sup} \left\{ \frac{1}{|Q(x)|} \int_{Q(x)} |f(t)|\, dt \right\}.$$

Then $|M| \prec O_1 |T|$ is equivalent to

$$(\mathrm{A}_1) \qquad |Mf(x)| \leq O_1 \cdot \Lambda f(x).$$

PROPOSITION 2. (Of Hardy and Littlewood). *The operator Λ possesses the following properties:*

a) *For any* $\lambda > 0$, $E_\lambda = E[\Lambda f \geq \lambda] \subset H_\lambda$, *where* $H_\lambda \subset R^n$ *is a set satisfying*

$$(4) \qquad |E_\lambda| \leq |H_\lambda| \leq \frac{2^n}{\lambda} \int_{H_\lambda} |f(x)|\, dx \leq \frac{2^n}{\lambda} \int_{R^n} |f|\, dx.$$

b) *If* $f \geq 0$ *then*

$$(5) \qquad |E[\Lambda f \geq 2\lambda]| \leq |H_\lambda| \leq \frac{2^n}{\lambda} \int_{H_\lambda} f^{(\lambda)}(x)\, dx,$$

where $f^{(\lambda)}(x) = f(x)$ *if* $f(x) \geq \lambda$, *and zero otherwise.*

c) *If* $p > 1$, *then* $|\Lambda f|^p \leq \Lambda(|f|^p)$.

d) Λ *satisfies condition* 1, *and is of the type* p *for every* $p > 1$.

PROOF: For each point $x \in E_\lambda = E[\Lambda f \geq \lambda]$ there is a cube $Q(x)$ such that

$$\lambda \leq \Lambda f(x) \leq \frac{(1+\varepsilon)}{|Q(x)|} \int_{Q(x)} |f|\, dt,$$

$$\lambda |Q(x)| \leq (1+\varepsilon) \int_{Q(x)} |f(t)|\, dt.$$

By Lemma 2 of the precedent paper [2] we can select a sequence of these cubes $Q_i = Q(x_i)$ such that $E_\lambda \subset \cup_i Q_i = H_\lambda$, and each point of R_n belongs to at most 2^n cubes Q_i. Therefore

$$\lambda\,|E_\lambda| \leq \lambda\,|H_\lambda| \leq \lambda \sum_i |Q_i| \leq (1+\varepsilon) \sum_i \int_{Q_i} |f(t)|\,dt$$

$$\leq (1+\varepsilon)\, 2^n \int_{\Sigma Q_i} |f|\, td = (1+\varepsilon)\, 2^n \int_{H_\lambda} |f(t)|\,dt,$$

and this proves part a).

In order to prove b), it is sufficient to observe that if $x \in E_{2\lambda}$, there is a cube $Q(x)$ such that (assuming $f \geq 0$, $f_{[\lambda]} = f - f^{[\lambda]}$),

$$2\lambda \leq \frac{1+\varepsilon}{|Q(x)|} \int_{Q(x)} |f(t)|\,dt \leq \frac{1+\varepsilon}{|Q(x)|} \int_{Q(x)} f^{[\lambda]} dt + \frac{1+\varepsilon}{|Q(x)|} \int_{Q(x)} f_{[\lambda]}\, dt.$$

Since

$$\frac{1+\varepsilon}{|Q(x)|} \int_{Q(x)} f_{[\lambda]}\, dt \leq (1+\varepsilon)\,\lambda,$$

$$(1-\varepsilon)\,\lambda \leq \frac{1+\varepsilon}{|Q(x)|} \int_{Q(x)} f^{[\lambda]}(t)\,dt,$$

and we repeat now the argument used in a).

In order to prove c), it is sufficient to observe that, by Hölder's inequality,

$$\frac{1}{|Q|} \int_Q |f|\,dt \leq \frac{1}{|Q|} \cdot |Q|^{1-1/p} \left\{ \int_Q |f|^p\,dt \right\}^{1/p} = \left\{ \frac{1}{|Q|} \int_Q |f|^p\,dt \right\}^{1/p} =$$

$$\leq \{ \Lambda(|f|^p) \}^{1/p}.$$

From c) and a) it follows that:

$$|E[\Lambda f \geq \lambda]| = |E[\{\Lambda f\}^p \geq \lambda^p]| \leq |E[\Lambda(|f|^p) \geq \lambda^p]|$$

$$\leq \frac{O_p}{\lambda^p} \int_{R^n} |f|^p\,dt, \quad (p > 1).$$

This shows that Λ satisfies condition (p) for every $p > 1$, and by a) Λ satisfies condition 1. Hence, by Theorem 1 of the precedent paper [2], Λ is of the type p, for every $p > 1$.

This proves Proposition 1.

THEOREM 1. a) *If* $|\mathrm{M}|^\alpha \prec O_\alpha |\mathrm{T}|^\alpha$, *and* T *satisfies condition* (p) *(respectively, if* T *is of the type* p*), and* $\mathrm{p} < \alpha$, *then* M *also satisfies condition* (p) *(or is of the type* p*)*.

b) *If* $|\mathrm{M}|^\alpha << |\mathrm{T}|^\alpha$, *and* T *satisfies condition* (p), $\mathrm{p} < \alpha$, *then* M *also satisfies condition* (p), *and is of the type* q, *for any* $\mathrm{q} > \mathrm{p}$.

PROOF. *a*) Let $|M|^\alpha \prec O_\alpha |T|^\alpha$, so that $|Mf|^\alpha \le O_\alpha \Lambda(|Tf|^\alpha)$. By Proposition 2, *a*),

$$E[|Mf| \ge \lambda] = E[|Mf|^\alpha \ge \lambda^\alpha] \subset E[O_\alpha \,.\, \Lambda(|Tf|^\alpha) > \lambda^\alpha] \subset H_\lambda,$$

and

$$|H_\lambda| \le \frac{O_\alpha}{\lambda^\alpha} \int_{H_\lambda} |fT|^\alpha \, dt.$$

If T satisfies condition (p), then by (3),

$$|H_\lambda| \le \frac{O_\alpha}{\lambda^\alpha} \int_{H_\lambda} |Tf|^\alpha \, dt \le \frac{O_{\alpha'}}{\lambda^\alpha} \cdot |\mathrm{H}_\lambda|^{1-\alpha/p} . \left\{ \int_{R^n} |f|^p \, dt \right\}^{\alpha/p},$$

and we obtain $|E[|Mf| \ge \lambda]| \le |H_\lambda| \le \dfrac{O_{\alpha,p}}{\lambda^p} \displaystyle\int_{R^n} |f|^p \, dt.$

Hence M satisfies condition (p), and this proves the first assertion of *a*). On the other hand, since $|Mf|^\alpha \le O_\alpha \Lambda(|Tf|^\alpha)$, by Proposition 2,*c*)

$$\int_{R^n} |Mf|^p \, dt \le O_\alpha^{p/\alpha} \int_{R^n} \{\Lambda(|Tf|^\alpha)\}^{p/\alpha} \, dt \le O_{\alpha,p} \int_{R^n} |Tf|^p \, dt.$$

Hence, if T is of the type p, we obtain

$$\int_{R^n} |Mf|^p \, dt \le O_p \int_{R^n} |f^p \, dt,$$

and M is also of the type p.

b) Let $|M|^\alpha << O_\alpha |T|^\alpha$, so that for each f and each $x \in R^n$,

$$|Mf(x)|^\alpha \le \frac{O_\alpha}{|Q(x)|} \int_{Q(x)} |T(\varphi_{Q(x)} \cdot f)(t)|^\alpha \, dt.$$

If T satisfies condition (p), $p > \alpha$, then by (3)

$$|Mf(x)|^\alpha \leq \frac{O_\alpha}{|Q(x)|} \cdot |Q(x)|^{1-\alpha/p} \left\{ \int_{R^n} |\varphi_{Q(x)}(t) f(t)|^p \, dt \right\}^{\alpha/p}$$

$$= O_\alpha \cdot \left(\frac{1}{Q(x)}\right)^{\alpha/p} \left\{ \int_{Q(x)} |f(t)|^p \, dt \right\}^{\alpha/p},$$

hence

$$|Mf(x)|^p \leq \frac{O_p}{|Q(x)|} \int_{Q(x)} |f|^p \, dt \leq O_p \, . \, \Lambda(|f|^p). \tag{4}$$

Therefore, $|M|^p \prec O_p |I|^p$, where I denotes the identity operator, and by a), M is of the type q, for every $q > p$.

On the other hand, since $E[|Mf| \geq \lambda] = \mathrm{E}[|Mf|^p \geq \lambda^p]$, by (4) and by Proposition 2, a),

$$|E[|Mf| \geq \lambda]| < |E[O_p \Lambda(|f|^p) \geq \lambda^p]| \leq \frac{O_p}{\lambda^p} \int_{R^n} |f|^p \, dt.$$

Hence M satisfies condition (p), and this proves Theorem 1.

REMARK. $|M|^\alpha \prec |T|^\alpha$ implies also $|M|^p \prec |T|^p$, for any $p > \alpha$. Similarly, $|M|^\alpha << |T|^\alpha$ implies $|M|^p << |T|^p$. In fact, if $|M|^\alpha << |T|^\alpha$, then $|Mf(x)|^\alpha \leq \Lambda(|T(\varphi_{Q(x)} f)|^\alpha)(x)$, and by Proposition 2, b), $|Mf(x)|^\alpha \leq [\Lambda(|T(\varphi_{Q(x)} f)|^p)(x)]^{\alpha/p}$. Hence $|Mf(x)|^p \leq \Lambda(|T(\varphi_{Q(x)} f)|^p)(x)$, that is, $|M|^p << |T|^p$.

COROLARY 1. *Let* M, T_1, T_2, T_3, *be operators defined on* **D**, *such that for each* $\mathrm{f} \in \mathbf{D}$ *and each* $\mathrm{x} \in \mathrm{R}^n$, *there exist two cubes* Q(x) *and* Q′(x), *with center at* x *and satisfying*

$$|Mf(x)|^\alpha \leq O_1 |T_1 f(x)|^\alpha + \frac{O_2}{|Q(x)|} \int_{Q(x)} |T_2 f(x)|^\alpha \, dt \tag{C}$$

$$+ \frac{O_3}{|Q'(x)|} \int_{Q'(x)} |T_3(\varphi_{Q'(x)} \cdot f)(t)|^\alpha \, dt, \quad (\alpha < 0).$$

Then : a) *If* T_1, T_2, T_3, *satisfy condition* (p), $\mathrm{p} < \alpha$, *then* M *also satisfies condition* (p).

b) *If* T_2 *satisfies condition* (p_1), *and if* T_1, T_3 *are of the type* p_2, $\alpha < \mathrm{p}_1 < \mathrm{p}_2$, *then* M *also satisfies condition* (p_1) *and is of the type* p_2.

PROOF. Condition (C) implies that, for each $x \in \mathrm{R}^n$, one at least of the three following inequalities must be true

$$| M f(x) |^\alpha \leq 3\, O_1 \,| T_1 (x) |^\alpha, \tag{c_1}$$

$$| M f(x) |^\alpha \leq 3\, O_2 \,| Q (x) |^{-1} \int\limits_{Q(x)} | T_2 f(t) |^\alpha \, dt, \tag{c_2}$$

$$| M f(x) |^\alpha \leq 3\, O_3 \,| Q' (x) |^{-1} \int\limits_{Q'(x)} | T_3 (\mp_{Q'(x)} \cdot f) (t) |^\alpha \, dt. \tag{c_3}$$

Therefore, $R^n \subset E_1 \cup E_2 \cup E_3$, where E_i is the set of the points x satisfying (C_i^i), $i = 1, 2, 3$. Define $M_i f(x) = | M f(x) |$ if $x \in E_i$, and zero otherwise, $i = 1, 2, 3$. Then $| M f(x) \leq M_1 f(x) + M_2 f(x) + M_3 f(x)$, and $M_1^\alpha \leq 3\, O_1 \,| T_1 |^\alpha$, $| M_2 |^\alpha \leq 3\, O_2 \,| T_2 |^\alpha$, $| M_3 |^\alpha << 3\, O_3 | T_3 |^\alpha$. If T_1, T_2, T_3, satisfy condition (p), $p > \alpha$, then by Proposition 1 and Theorem 1, M_1, M_2 and M_3 satisfy condition (p), hence M satisfies the same condition. The part *b*) is proved in a similar way.

COROLARY 2. *Let* M, T_1, T_2, T_3, *be operators defined on* **D**, *such that for each* $\mathrm{f} \in \mathbf{D}$ *and for each* $\mathrm{x} \in \mathrm{R}^n$, *there is a cube* $\mathrm{Q}(\mathrm{x})$ *with center at* x *and satisfying*

$$| M f(x) | \leq O_1 \,| T_1 f(x) | + O_2 \,| T_2 f(x_1) | + O_3 \,| T_3 (\mp_{Q(x)} f) (x_1) |, \tag{D}$$

for every point $\mathrm{x}_1 \in {}^1/_2\, \mathrm{Q}(\mathrm{x})$ (= *the cube with the same center as* Q, *and half of side*).

Then :

a) *If* T_1, T_2, T_3, *satisfy condition* (p), *then* M *also satisfies condition* (p).

b) *If* T_2 *satisfies condition* (p), *and if* T_1, T_3 *are of the type* p_1, $\mathrm{p} < \mathrm{p}_1$, *then* M *also satisfies condition* (p) *and is of the type* p_1.

PROOF. Let $\alpha < p$. Since (D) holds for each point $x_1 \in {}^1/_2\, \mathrm{Q}(x)$, then, raising to the α-power and integrating over $\frac{1}{2}\, Q(x)$ we obtain :

$$| M f(x) |^\alpha = \frac{2^n}{| Q(x) |} \int\limits_{Q(x)/_2} | M f(x) |^\alpha \, dx_1 \leq 2^n\, O_0 \Big\{ | \mathrm{T}_1 f(x) |^\alpha +$$

$$+ \frac{1}{| Q(x) |} \int\limits_{Q(x)} | T_2 f(x_1) |^\alpha \, dx_1 + \frac{1}{| Q(x) |} \int\limits_{Q(x)} | T_3 (\mp_{Q(x)} f)(x_1) |^\alpha \, dx_1 \Big\}.$$

Hence by Corolary 1 we obtain Corolary 2.

$R^{n+m} = R^n \times R^m = \{(x, y)\}$, $L^p(R^{n+m})$ will denote the set of the functions $h(x, y)$ defined on R^{n+m} and such that

$$\| h \|_p^p = \int_{R^{n+m}} | h(x, y) |^p \, dxdy = \int_{R^m} dy \int_{R^n} | h(x, y) |^p \, dx < \infty,$$

and $L^p(R^n)$, $L^p(R^m)$), the corresponding spaces for R^n and R^m, respectively.

Let $T = Tf = Tf(x)$ (respectively $S = Sg(y)$) be an operator defined on the set $\mathbf{D}(R^n)$ (respectively, $\mathbf{D}(R^m)$) of all step functions $f(x)$ (respectively $g(y)$) of R^n (of R^m). We assume that T and S are linear operators.

Let $\mathbf{D}(R^n \times R^m) = \mathbf{D}(R^n) \times \mathbf{D}(R^m) = \{ h(x, y) \}$ be the set of all step functions of R^{n+m} of the form

$$h(x, y) = \lambda_1 f_1(x) g_1(y) + \dots + \lambda_k f_k(x) g_k(y),$$

where $f_i \in \mathbf{D}(R^n)$, $g_i \in \mathbf{D}(R^m)$, and the λ_i are constants, $i = 1, 2, \dots k$. The sets $\mathbf{D}(R^n)$, $\mathbf{D}(R^m)$, $\mathbf{D}(R^{n+m})$ are dense in all the L^p spaces. We define the following product operators:

$$\widetilde{T}h(x, y) = (\widetilde{T}h)(x, y) = [Th(., y)](x) = \sum \lambda_i g_i(y) \, . \, [Tf_i](x).$$

$$\widetilde{S}h(x, y) = (\widetilde{S}h)(x, y) = [Sh(x, .)](y) = \sum \lambda_i f_i(x) \, . \, [T_i g](y).$$

$$\widetilde{S}\,\widetilde{T}h(x, y) = \widetilde{T}\,\widetilde{S}h(x, y) = [\widetilde{T}\,\widetilde{S}h](x, y) = \sum \lambda_i [Tf_i](x) \, . \, [Sg_i](y).$$

PROPOSITION 4. a) *If T is of the type* p, *or if T satisfies condition* (p), *then $\widetilde{T}$ is also of the type* (p), *or $\widetilde{T}$ satisfies condition* (p), *respectively.*

b) *If T and S are of the type* p, *then $\widetilde{T}\,\widetilde{S}$ is also of the tpye* p.

c) *If T satisfies condition* (p), *and S is of the type* p, *then $\widetilde{T}\,\widetilde{S}$ satisfies condition* (p).

d) *If T and S satisfy condition* (p), *then for every $\alpha <$ p, and* $E \subset R^{n+m}$,

$$\int_E | \widetilde{T}\,\widetilde{S}h |^p \, dxdy \leq O_{\alpha, p} \cdot | E |^{1-\alpha/p} \Big\{ \int_{R^n} \int_{R^m} | \widetilde{S}h(x, y) |^p \, dxdy \Big\}^{\alpha/p}.$$

Proof. a) and b) are immediate consequences of Fubini's theorem. For instance,

$$\iint\limits_{R^{n+m}} |\tilde{T}\tilde{S}h|^p \, dxdy = \int\limits_{R^m} dy \int\limits_{R^n} |\tilde{T}(\tilde{S}h(x, y))|^p \, dx \leq$$

$$O_p \int\limits_{R^m} dy \int\limits_{R^n} |\tilde{S}h(x, y)|^p \, dx = O_p \int\limits_{R^n} dx \int\limits_{R^m} |\tilde{S}h(x, y)|^p \, dy$$

$$\leq O_p' \int\limits_{R^n} dx \int\limits_{R^m} |h(x, y)|^p \, dy = O_p' \iint\limits_{R^{n+m}} |h(x, y)|^p \, dxdy.$$

In order to prove c) and d), it is sufficient to observe that condition (p) is equivalent to (3), and that (if $E \subset R^n \times R^m$),

$$\int\limits_{E} |\tilde{T}\tilde{S}h|^\alpha \, dxdy = \int\limits_{R^m} dy \int\limits_{R^n} \varphi_E |\tilde{T}\tilde{S}h|^\alpha \, dx \leq$$

$$O_p \int\limits_{R^m} dy \, . \, |E_y|^{1-\alpha/p} \Big\{ \int\limits_{R^n} |\tilde{S}h(x, y)|^p \, dx \Big\}^{\alpha/p} \leq$$

$$O_p \Big\{ \int\limits_{R^m} |E_y| \, dy \Big\}^{\frac{p-\alpha}{p}} \Big\{ \int\limits_{R^m} dy \int\limits_{R^n} |\tilde{S}h(x, y)|^p \, dx \Big\}^{\alpha/p}$$

$$= O_p \, . \, |E|^{1-\alpha/p} \Big\{ \iint\limits_{R^n \times R^m} |\tilde{S}h(x, y)|^p \, dx \Big\}^{\alpha/p},$$

where E_a denotes the intersection of the set E with the line $y = a$.

Let us consider now instead of the space R^1 the unit circunference C^1, and instead of R^n the torus $C^n = C^1 \times C^1 \times \ldots \times C^1$. In this case C^n has finite measure and all the functions $f(x)$ defined on C^n will be assumed to be periodic functions. All the precedent results apply to C^n. If T is defined on $\mathbf{D}(C^n)$ and satisfies condition (p), then, since C^n is of finite measure

$$\int\limits_{C^n} |Tf|^\alpha \, dt \leq O_\alpha \Big\{ \int\limits_{C^n} |f(t)|^p \, dt \Big\}^{\alpha/p},$$

for $\alpha < p$.

Moreover, from the last inequality of the proof of Proposition 4, and from Theorem 2 of the precedent paper, it follows immediately that ;

Proposition 4 a). *Let* T *and* S *be defined on* C^n *and* C^m, *respectively. If* T *and* S *satisfy condition* (p) *and condition* (p_1), $p < p_1$, *then for every* $\alpha < p$,

$$| \iint\limits_{C^n C^m} | \widetilde{T}\widetilde{S}\, h(t, s)|^{\alpha}\, dsdt \leq O_{\alpha} \{ \iint\limits_{C^{n+m}} |h|^{p}(1 + \log^{+} |h|)\, dsdt \}^{\alpha/p} + O_{\alpha}'.$$

Definition 3. *We shall write* $\widetilde{M} <_y <_x \widetilde{T}$, *if for each* h $(x, y) \in \mathbf{D}$ $(R^n \times R^m)$ *and for each point* $(x, y) \in R^{n+m}$, *there is a cube* $Q(x) \subset R^n$ *with center at* x. *and a cube* $Q(y) \subset R^m$ *with center at* y, *such that*

$$| \widetilde{M}\, h(x, y)| \leq \frac{O_1}{|Q(x)|\,|Q(y)|} \iint\limits_{Q(x) \times Q(y)} | \widetilde{T}\, h(t, s)|\, dtds.$$

We shall write $\widetilde{M} << _x <_y \widetilde{T}$, *if*

$$| \widetilde{M}\, h(x, y)| \leq \frac{O_1}{|Q(x)|\,|Q(y)|} \iint\limits_{Q(x) \times Q(y)} | \widetilde{T}(\varphi_{Q(x)} \cdot h)(t, s)|\, dtds,$$

where $\varphi_{Q(x)}(t, s)$ *is the characteristic function of the cube* $Q(x) \subset R^n \subset R^{n+m}$. *We shall write* $\widetilde{M} << _x << _y \widetilde{T}$, *if*

$$| \widetilde{M}\, h(x, y)| \leq \frac{O_1}{|Q(x)|\,|Q(y)|} \iint\limits_{Q(x) \times Q(y)} | \widetilde{T}(\varphi_{Q(x)} \cdot \varphi_{Q(y)} \cdot h)(t, s)|\, dtds.$$

Theorem 2. a) *If* $\widetilde{M} <_x <_y \widetilde{S}\widetilde{T}$, *and if* T *and* S *are of the type* $p > 1$, *then* $\widetilde{M}$ *is also of the type* p.

b) *If* $\widetilde{M} <_x << _y \widetilde{T}\widetilde{S}$, *if* S *is of the type* p, *and if* T *is of the type* p_1, $p_1 > p > 1$, *then* $\widetilde{M}$ *is of the type* p_1

c) *If* $\widetilde{M} << _x << _y \widetilde{T}\widetilde{S}$, *and if* T *and* S *are of the type* p, *then* M *is of the type* p_1, *for every* $p_1 > p$.

PROOF. *a*) Let Λ^x be the Hardy-Littlewood maximal operator acting on the space $R^n = \{x\}$, and Λ^y the same operator acting on $R^m = \{y\}$. By the assumption, $\widetilde{M} <_x <_y \widetilde{T}\widetilde{S}$, hence $|\widetilde{M}h| \leq |\widetilde{\Lambda}^x\widetilde{\Lambda}^y\widetilde{S}\widetilde{T}h|$. By Proposition 2, *c* and Proposition 4*a*)

$$\iint\limits_{R^{n+m}} |\widetilde{M}h|^p \, ds dt \leq O_p \iint\limits_{R^{n+m}} |\widetilde{T}\widetilde{S}h|^p \, ds dt \leq O_p' \iint\limits_{R^{n+m}} |h(s, t)|^p \, ds dt,$$

so that $\widetilde{M}$ is of the type p.

b) By assumption, for each $h(x, y)$ and each $(x, y) \in R^{n+m}$, there is a cube $Q(x) \subset R^n$, and a cube $Q(y) \subset R^m$, such that

$$|\widetilde{M}h(x, y)| \leq \frac{O_1}{|Q(x)||(Q(y)|} \iint\limits_{Q(x) \times Q(y)} |\widetilde{T}\widetilde{S}(\varphi_{Q(y)}h)(t, s) \, dt ds$$

$$\leq \frac{O_1}{|Q(x)|} \int\limits_{Q(x)} \left\{ \frac{1}{|Q(y)|} \int\limits_{Q(y)} |\widetilde{S}\widetilde{T}(\varphi_{Q(y)}h)(t, s)| \, ds \right\} dt$$

$$\leq \frac{O_1}{|Q(x)|} \int\limits_{Q(x)} \left\{ \frac{1}{Q(y)} \int\limits_{Q(y)} |\widetilde{S}\widetilde{T}(\varphi_{Q(y)}h)(t, s)|^p \, ds \right\}^{1/p} dt.$$

Taking in account that S satisfies condition (p), by Proposition 4, *a*) and by Proposition 2, *b*)

$$|\widetilde{M}h(x, y)| \leq$$

$$O_1 |Q(x)|^{-1} \int\limits_{Q(x)} \{ |Q(y)|^{-1} \int\limits_{R^m} (\varphi_{Q(y)}(s) |\widetilde{T}h(t, s)|^p \, ds \}^{1/p} dt$$

$$\leq O_1 \widetilde{\Lambda}^x (\{ \widetilde{\Lambda}^y (|Th|^p) \}^{1/p}) \leq O_1 \{ \widetilde{\Lambda}^x (\widetilde{\Lambda}^y (|Th|^p)) \}^{1/p},$$

Hence, by Proposition 2, *c*), by Proposition 4, *a*), and by the assumption,

$$\iint\limits_{R^{n+m}} |\widetilde{M}h|^{p_1} \, dx dy \leq O_1^{p_1} \iint\limits_{R^{n+m}} \{ \widetilde{\Lambda}^x (\widetilde{\Lambda}^y (|Th|^p)) \}^{p_1/p} \, dx dy$$

$$\leq O_{p_1} \iint\limits_{R^{n+m}} \{ \Lambda^y (|Th|^p) \}^{p_1/p} \, dx dy$$

$$\leq O_{p_1} \iint_{R^{n+m}} |Th|^{p_1}\, dxdy \leq O_{p_1} \iint_{R^{n+m}} |h|^{p_1}\, dxdy,$$

and this shows that $\widetilde{M}$ is of the type p_1, for every $p_1 > p$,

c) As in *b*), we shall have

$$|\widetilde{M} h(x, y)| \leq \frac{1}{|Q(x)|} \int_{Q(x)} \left\{ \frac{1}{|Q(y)|} \int_{Q(y)} |\widetilde{T}(\varphi_{Q(x)} h)(t, s)|^p \, ds \right\}^{1/p} dt$$

$$\leq \left\{ \frac{1}{|Q(x)|} \int_{Q(y)} \left\{ \frac{1}{|Q(y)|} \int_{Q(y)} |\widetilde{T}(\varphi_{Q(x)} h)(t, s)|^p \, ds \right\} dt \right\}^{1/p}$$

$$\leq \left\{ \frac{1}{|Q(y)|} \int_{Q(y)} \left\{ \frac{1}{|Q(x)|} \int_{R^n} |\widetilde{T}(\varphi_{Q(x)} h)(t, s)|^p \, dt \right\} ds \right\}^{1/p}$$

$$\leq \left\{ \frac{1}{|Q(y)|} \int_{Q(y)} \left\{ \frac{1}{|Q(x)|} \int_{R^n} \varphi_{Q(x)} |h(t, s)|^p \, dt \right\} ds \right\}^{1/p}$$

$$\leq \{ \widetilde{\Lambda}^y \widetilde{\Lambda}^x (|h|^p) \}^{1/p}$$

Hence

$$\iint_{R^{n+m}} |\widetilde{M} h|^{p_1}\, dxdy \leq \iint_{R^{n+m}} \{ \widetilde{\Lambda}^y \widetilde{\Lambda}^x (|h|^p) \}^{p_1/p}\, dxdy$$

$$\leq \iint_{R^{n+m}} |h(x, y)|^{p_1}\, dxdy,$$

and this proves Theorem 2.

THEOREM 2*a*. *Let* T *and* S *be operators acting on the spaces* C^n *and* C^m, *respectively, and satisfying conditions (1) and* (p), $p > 1$.

a) *If* $\alpha < 1$ *and* $|\widetilde{M}|^\alpha <_x <_y |\widetilde{S}\widetilde{T}|^\alpha$, then

$$\iint_{C^{n+m}} |Mh|^\beta\, dxdy \leq O_\beta \left\{ \iint_{C^{n+m}} |h|(1 + \log^+|h|) \right\}^\beta + O_\beta', \tag{8}$$

for every β *such that* $\alpha < \beta < 1$.

b) *If* $\alpha < 1$, *and* $|\widetilde{M}|^\alpha <_x << _y |\widetilde{T}\widetilde{S}|^\alpha$, *then* (8) *is true for any* β, $\alpha < \beta < 1$.

c) If $\alpha < 1$, *and* $|\widetilde{M}|^\alpha << _x << _y |\widetilde{T}\widetilde{S}|^\alpha$, *then* (8) *is true for* $\alpha < \beta < 1$.

PROOF. *a*) If $|M|^\alpha <_x <_y |\widetilde{T}\widetilde{S}|^\alpha$, then $|\widetilde{M} h|^\alpha \le \widetilde{\Lambda}^x \widetilde{\Lambda}^y (|\widetilde{T}\widetilde{S} h|^\alpha)$. By Proposition 2 and Proposition 4

$$\iint_{C^{n+m}} |\widetilde{M} h|^\beta \, dt\, ds \le O_\beta \iint_{C^{n+m}} \{\widetilde{\Lambda}_x \widetilde{\Lambda}_y (|\widetilde{T}\widetilde{S} h|^\alpha)\}^{\beta/\alpha} \, dt\, ds \le O_\beta \iint_{C^{n+m}} |\widetilde{T}\widetilde{S} h|^\beta \, dt\, ds.$$

and by Proposition 4 *a* it follows (8).

b) We have

$$|\widetilde{M} h(x, y)|^\alpha \le \frac{1}{|Q(x)|} \int_{Q(x)} \Big\{ \frac{1}{|Q(y)|} \int_{Q(y)} |\widetilde{T}\widetilde{S}(\varphi_{Q(y)} h)|^\alpha \, ds \,\Big|\, dt$$

$$\le \frac{O_\alpha}{|Q(x)|} \int_{Q(x)} \frac{1}{|Q(y)|} \Big\{ \int_{C^m} \varphi_{Q(y)}(s) \, |\widetilde{T} h| \, ds \Big\}^\alpha dt$$

$$\le O^\alpha \widetilde{\Lambda}^x (\{\widetilde{\Lambda}^y (\; T h|)\}^\alpha,$$

$$\iint_{C^{n+m}} |\widetilde{M} h(x, y)|^\beta \, dx\, dy \le O_\beta \iint_{C^{n+m}} [\widetilde{\Lambda}^x (\{\widetilde{\Lambda}^y (|T h|)\}^\alpha)]^{\beta/\alpha} \, dz\, dy$$

$$\le O_\beta \iint_{C^{n+m}} \{\widetilde{\Lambda}^y (|T h|)\}^\beta \, dx\, dy \le O_\beta' \Big\{ \iint_{C^{n+m}} |\widetilde{T} h| \, dx\, dy \Big\}^\beta,$$

and by Proposition 4, and by the Theorem 2 of the precedent paper, we obtain (8).

The part *c*) of the theorem is proved in a similar way.

4. **A maximal theorem for product operators..** — Let $T_\varepsilon \, (\varepsilon > 0)$ be a set of linear operators defined on $\mathbf{D}(R^n)$ (or on $\mathbf{D}(C^n)$) and satisfying the following condition:

For each $\varepsilon > 0$, for each $f \in \mathbf{D}(R^n)$, and for each $x_0 \in R^n$, there is a cube $Q = Q(x_0, \varepsilon)$ with center at x_0 and such that for every point $x_1 \in {}^1/_2\, Q$, it is true that

$$\text{(E)} \quad |T_\varepsilon f(x_0)| \le O_1 \{ |Tf(x_1)| + |T(\varphi_{Q(x_0)} f)(x_1)| + N_1 f(x_0) \},$$

where O_1 does not depend on ε and f. The operator N_1 has not to be linear but only monotonic: $f| \le |f'|$ implies $N_1(|f|) \le N_1 |f'|)$.

Similarly let S_ε $(\varepsilon > 0)$ be a set of operator on $\mathbf{D}(R^m)$ (or $\mathbf{D}(C^m)$) satisfying

$$\text{(E}'\text{)} \quad |S_\varepsilon g(y_0)| \leq O_1 \{ |Sg(y_1)| + |S(\varphi_{Q(y_0)} g)(y_1)| + N_2 g(y_0) \}$$

The operators T_ε, S_ε define on $\mathbf{D}(R^n \times R^m)$ a set of operators

$$\widetilde{T}_\varepsilon \widetilde{S}_\varepsilon = \widetilde{T}_\varepsilon \widetilde{S}_\varepsilon .$$

Let $\widetilde{M} h(x, y)$ be the maximal operator defined on $\mathbf{D}(R^n \times R^m)$ by

$$\widetilde{M} h(x, y) = \sup_{\varepsilon > 0,\, \varepsilon > 0} | \widetilde{T}_\varepsilon \widetilde{S}_\varepsilon (x, y) . \tag{M}$$

THEOREM 3. *If the operators* T_ε, S_ε, N_i, S, *and* T, *commute, then :*

a) *If the operators* N_1, N_2, S, T *are of the type* p. *for every* $\mathrm{p} > 1$, *then* $\widetilde{\mathrm{M}}$ *is also of the type* p, *for every* $\mathrm{p} > 1$.

b) *If the above operators are defined on the space* C^n *and* C^m, *and if* N_1, N_2, S *and* T *satisfy the condition* (1) *and the condition* (p), $\mathrm{p} > 1$, *then*

$$\text{(F)} \quad \iint_{C^{n+m}} | \widetilde{\mathrm{M}} h(x, y)|^\alpha \, dx dy \leq O_\alpha \Big\{ \iint_{C^{n+m}} |h| (1 + \log^+ |h| \Big\}^\alpha + O_\alpha{}^1$$

for every $\alpha < 1$.

PROOF. Let us define $M^x f(x) = \sup_{\varepsilon > 0} | T_\varepsilon f(x)|$, $M^y g(y) = \sup_{\varepsilon > 0} S_\varepsilon g(y)|$.

Then by the assumption, for every $(x_0, y_0) \in R^{n+m}$, and for any $(x_1, y_1) \in {}^1\!/_2\, Q(x_0) \times {}^1\!/_2\, Q(y_0)$ (we may assume that $O_1 = 1$) :

$$| \widetilde{T}_\varepsilon \widetilde{T}_\varepsilon h(x_0, y_0)| \leq$$

$$| \widetilde{T} \widetilde{S}_\varepsilon h(x_1, y_0)| + | \widetilde{T}(\varphi_{Q(x_0)} \cdot \widetilde{S}_\varepsilon h)(x_1, y_0)| + \widetilde{N}_1 \widetilde{S}_\varepsilon h(x_0, y_0)$$

$$\leq | \widetilde{T} S_\varepsilon h(x_1, y_0) + | \widetilde{T}(\varphi_{Q(x_0)} \cdot \widetilde{S_\varepsilon h})(x_1, y_0)| + \widetilde{N}_1 \widetilde{M}^y h(x_0, y_0),$$

$$| \widetilde{T} \widetilde{S}_\varepsilon h(x_1, y_0)| = | \widetilde{S}_\varepsilon \widetilde{T} h(x_1, y_0)| \leq | \widetilde{S} \widetilde{T} h(x_1, y_1(| + | \widetilde{S}(\varphi_{Q(y_0)} \widetilde{T} h)(x_1, y_1)$$
$$+ \widetilde{N}_2 \widetilde{T} h(x_1, y_0),$$

$$| \widetilde{T}(\varphi_{Q(x_0)} \widetilde{S}_\varepsilon h)(x_1, y_0)| < | \widetilde{S}(\varphi_{Q(x_0)} \widetilde{T} h)(x_1, y_1)| + | \widetilde{S} \widetilde{T}(\varphi_{Q(x_0)} \varphi_{Q(y_0)} h(x_1, y_1)$$
$$+ N_2 T(\varphi_{Q(x_0)} h)(x_1, y_1).$$

Taking ε, η, such that $|\widetilde{M}h(x_0, y_0)| < 2\,|\widetilde{T}_\varepsilon \widetilde{S}_\eta h(x_0, y_0)|$, and raising to the α-power, we obtain :

$$|\widetilde{M}h(x_0, y_0)|^\alpha \leq O_\alpha \{|\widetilde{T}\widetilde{S}h(x_1, y_1)|^\alpha + |\widetilde{T}\widetilde{S}(\varphi_{Q(y_0)}h)(x_1, y_1)|^\alpha +$$

$$|\widetilde{T}(\varphi_{Q(x_0)}\widetilde{S}h)(x_1, y_1)|^\alpha + |\widetilde{T}\widetilde{S}(\varphi_{Q(x_0)}\varphi_{Q(y_0)}h)(x_1, y_1)|^\alpha$$

$$+ |\widetilde{N}_1\widetilde{M}^y h(x_0, y_0)|^\alpha + |\widetilde{N}_2\widetilde{T}h(x_1, y_0)|^\alpha$$

$$+ |\widetilde{N}_2\widetilde{T}(\varphi_{Q(x_0)}h)(x_1, y_0)|^\alpha\}.$$

for every ≤ 1, and $(x_1, y_1) \in {}^1/_2\,Q(x_0) \times {}^1/_2\,Q(y_0)$.

Integrating over the set ${}^1/_2\,Q(x_0) \times {}^1/_2\,Q(y_0)$, in x_1, y_1, we shall obtain that

$$\widetilde{M} \leq \widetilde{M}_1 + \widetilde{M}_2 + \widetilde{M}_3 + \ldots + \widetilde{M}_7,$$

where

$$|\widetilde{M}_1|^\alpha <_x <_y |\widetilde{T}\widetilde{S}|^\alpha, \qquad |\widetilde{M}_2|^\alpha <_x << _y |\widetilde{T}\widetilde{S}|^\alpha,$$

$$|\widetilde{M}_3|^\alpha << _x <_y |\widetilde{T}\widetilde{S}|^\alpha, \qquad |\widetilde{M}_4|^\alpha << _x << _y |\widetilde{T}\widetilde{S}|^\alpha,$$

$$|\widetilde{M}_5|^\alpha \leq |\widetilde{N}\widetilde{M}^y|^\alpha, \qquad |\widetilde{M}_6|^\alpha <_x |\widetilde{N}_2\widetilde{T}|^\alpha,$$

$$|\widetilde{M}_7| << _x |\widetilde{N}_2\widetilde{T}|^\alpha.$$

Hence by theorem 2, and by theorem $2a$, this proves theorem 3.

From Theorem 3, and Proposition $3a$, and Corolary $3a$, we obtain the

THEOREM $3a$. *Let* $K_1(z)$, $K_2(u)$ *be two kernels of the form* $(5a)$, *defined in the space* R^n *and* R^m (*or* C^n *and* C^m) *respectively, and*

$$\widetilde{H}_{\varepsilon\eta}\,h(z, w) = \iint\limits_{\substack{|z-u|<\varepsilon \\ |w-v|<\eta}} K_1(z-u)\;K_2(u-v)\;h(u, v)\,du\,dv$$

$$= \widetilde{H}_\varepsilon \widetilde{H}_\eta h(z, w),$$

the double Hilbert transform on $R^n \times R^m$, *and*

$$\widetilde{M}h(z, w) = \sup_{\varepsilon, \eta > 0} |\widetilde{H}_{\varepsilon\eta}\,h(z, w)|$$

Taking ε, η, such that $|\widetilde{M}\,h(x_0, y_0)| < 2\,|\widetilde{T}_\varepsilon\,\widetilde{S}_\eta\,h(x_0, y_0)|$, and raising to the α-power, we obtain :

$$
\begin{aligned}
|\widetilde{M}\,h(x_0, y_0)|^\alpha \leq O_\alpha \{ & |\widetilde{T}\widetilde{S}\,h(x_1, y_1)|^\alpha + |\widetilde{T}\widetilde{S}(\varphi_{Q(y_0)}\,h)(x_1, y_1)|^\alpha + \\
& |\widetilde{T}(\varphi_{Q(x_0)}\widetilde{S}\,h)(x_1, y_1)|^\alpha + |\widetilde{T}\widetilde{S}(\varphi_{Q(x_0)}\,\varphi_{Q(y_0)}\,h)(x_1, y_1)|^\alpha \\
& + |\widetilde{N}_1\widetilde{M}^y h(x_0, y_0)|^\alpha + |\widetilde{N}_2\widetilde{T}\,h(x_1, y_0)|^\alpha \\
& + |\widetilde{N}_2\widetilde{T}(\varphi_{Q(x_0)}\,h)(x_1, y_0)|^\alpha \},
\end{aligned}
$$

for every ≤ 1, and $(x_1, y_1) \in {}^1/_2\,Q(x_0) \times {}^1/_2\,Q(y_0)$.

Integrating over the set ${}^1/_2\,Q(x_0) \times {}^1/_2\,Q(y_0)$, in x_1, y_1, we shall obtain that

$$\widetilde{M} \leq \widetilde{M}_1 + \widetilde{M}_2 + \widetilde{M}_3 + \ldots + \widetilde{M}_7,$$

where

$$
\begin{aligned}
& |\widetilde{M}_1|^\alpha <_x <_y |\widetilde{T}\widetilde{S}|^\alpha, && |\widetilde{M}_2|^\alpha <_x \ll_y |\widetilde{T}\widetilde{S}|^\alpha, \\
& |\widetilde{M}_3|^\alpha \ll_x <_y |\widetilde{T}\widetilde{S}|^\alpha, && |\widetilde{M}_4|^\alpha \ll_x \ll_y |\widetilde{T}\widetilde{S}|^\alpha, \\
& |\widetilde{M}_5|^\alpha \leq |\widetilde{N}\,\widetilde{M}^y|^\alpha, && |\widetilde{M}_6|^\alpha <_x |\widetilde{N}_2\widetilde{T}|^\alpha, \\
& |\widetilde{M}_7| \ll_x |\widetilde{N}_2\widetilde{T}|^\alpha.
\end{aligned}
$$

Hence by theorem 2, and by theorem $2a$, this proves theorem 3.

From Theorem 3, and Proposition $3a$, and Corolary $3a$, we obtain the

THEOREM $3a$. *Let* $\mathrm{K}_1(\mathrm{z})$, $\mathrm{K}_2(u)$ *be two kernels of the form* $(5\,a)$, *defined in the space* R^n *and* R^m (*or* C^n *and* C^m) *respectively, and*

$$
\begin{aligned}
\widetilde{H}_{\varepsilon,\eta}\,h(z, w) &= \iint\limits_{\substack{|z-u| < \varepsilon \\ |w-v| < \eta}} K_1(z-u)\;K_2(w-v)\;h(u, v)\,du\,dv \\
&= \widetilde{H}_\varepsilon\,\widetilde{H}_\eta\,h(z, w),
\end{aligned}
$$

the double Hilbert transform on $\mathrm{R}^n \times \mathrm{R}^m$, *and*

$$\widetilde{M}\,h(z, w) = \sup_{\varepsilon,\eta > 0} |\widetilde{H}_{\varepsilon\eta}\,h(z, w)|$$

Then $\widetilde{\mathrm{M}}$ *is of the type* p *for every* $\mathrm{p} > 1$, *and* $\widetilde{\mathrm{M}}$ *satisfies the inequality* (F).

(Of course it is necessary not to confuse the double Hilbert transform $\widetilde{H}_{\varepsilon,\varepsilon}$ with the simple $(n+m)$-dimensional Hilbert transform).

In the case $n = m = 1$, $K_1(x) = K_2(x) = x^{-1}$, theorem $3a$ was proved by S. Sokolowsky and Zygmund ([4], [5]) using the theory of double trigonometrical series and analytic functions.

BIBLIOGRAPHY

1. A. Zygmund, *Trigonometrical Series*, Monogr. Math. Warsaw, 1935.
2. M. Cotlar, *A general interpolation theorem*, This journal, I (1955).
3. A. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Mathematica, 88 (1952).
4. K. Sokol-Sokolowsky, *On trigonometric series conjugate to Fourier series of two variables*, Fundamenta Matematicae, 33 (1939-46).
5. A. Zygmund, *On the boundary values of functions of several complex variables*, Fundamenta Matematicae, 36 (1949).

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ALGUNAS GENERALIZACIONES DEL TEOREMA MAXIMAL DE HARDY y LITTLEWOOD

Por MISCHA COTLAR

Si $T = Tf$ es un operador del tipo p, o del m-tipo p (ver las definiciones en el trabajo que precede), y si $M = Mf$ es otro operador tal que $|Mf(P)| \leq |Tf(P)|$ para toda f, entonces es evidente que Mf es también del mismo tipo que Tf. Sin embargo el asunto ya no es tan inmediato si la relación ordinaria $\leq$ se reemplaza por otras más generales.

En este trabajo consideramos dos relaciones: $M \prec T$ y $M << T$ que generalizan la relación ordinaria $\leq$, y que expresan que M es $\leq T$ «localmente y en media». El teorema maximal de Hardy-Littewood (ver [1], p. 244, en la bibliografía que precede) corresponde al caso $T = I =$ operador identidad y $M \prec T$.

Estudiamos también el caso de operadores productos $M_x M_x f f(x, y)$. En este caso la situación permanece la misma si $p > 1$, pero es necesario introducir modificaciones si $p = 1$. Finalmente damos un teorema maximal general para transformadas dobles.

Como damos demostraciomes directas, para la lectura no es necesario el conocimiento del teorema de Hardy-Littlewood.

A unified theory of Hilbert transforms and ergodic theorems*

BY MISCHA COTLAR

The analogy between the theory of Hilbert transforms and the ergodic theorems, and in particular the differentiation theory, has been repeatedly stressed by several authors, particularly by Lusin and Zygmund. However, the two theories have been treated by entirely different methods, the proofs for the Hilbert transforms being considerably more complex then those of the ergodic theorems. This is due to the fact that the ergodic theory deals with positive operators, while the Hilbert transforms are non-positive operators.

The aim of this paper is to give a general theory which contains the theory of Hilbert transforms and ergodic theorems as special cases. Let $R^n = \{x\}$ be the n-dimensional euclidean space, $K(x)$ an integrable function, and let $K_i(x) = 2^{-ni} K(2^{-i}x)$, $i = \pm 1, \pm 2, \pm 3, \ldots$ If $\Omega = \{P\}$ is an abstract space with a measure μ and a n-dimensional continuous group $\{\sigma_x\}$, $x \in R^n$, of measure preserving transformations, we define for each $m = 1, 2, \ldots$ the operator

$$H_m f(\mathrm{P}) = \sum_{i=-m}^{m} \int_{R^n} f(\sigma_x P) K_i(x)\, dx.$$

We prove that, under certain assumptions on the kernel $K(x)$, $H_m f(P)$ converges pointwise to a limit $Hf(P)$, almost everywhere, for every $f \in L^p(\Omega, \mu)$, and every $p \geq 1$. If $p > 1$, $H_m f$ also converges in the p^th^ mean to Hf, and the maximal operator $Mf = \sup |H_m f|$ is bounded in $L^p(\Omega, \mu)$.

If $\Omega = R^n$, $\sigma_x t = x + t$, and $K(x) = \omega(x)/|x|^n$ for $1 \leq |x| \leq 2$ and zero otherwise ($\omega(x)$ does not depend on $|x|$), then $Hf = \lim H_m f$ is the n-dimensional Hilbert transform and we obtain the classical results of Lusin-Riesz-Kolmogoroff [1] concerning the ordinary Hilbert transform, as well as the recent ones of Zygmund-Calderón [2]

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concerning the generalized n-dimensional Hilbert transforms. If Ω is a general measure space, and $K(x) = -1$ if $|x| < 1$, $K(x) = +1$ if $1 \leq |x| \leq 2$, then the $H_m f$ reduce to the ergodic operators, and we obtain the ergodic theorems of von Neumann, Birkhoff and Wiener [3]. Besides, we give similar theorems for the double operators $H_{mn} f(P, Q) = H_m H_n f(P, Q)$, which contain as special cases the n-parametric ergodic theorems due to Zygmund and Dunford [4] and the theorems concerning the double Hilbert transform due to Zygmund and Sokolowsky [5]. Moreover, these theorems extend, in particular, the Zygmund-Sokolowsky results for n-dimensional kernels.

In the proof of the above general theorems we use only measure-theoretical methods. In fact, our proofs are based on three general theorems concerning operators in L^p – spaces, which have been given in the three precedent papers ([6], [7], [8]).

If we are interested in the case $\Omega = R^n$ only, the theory simplifies considerably, and as was shown is the examples of the just mentioned papers, yields in particular a new and simplified treatement of the theory of Hilbert transforms.

On the other hand, we show that using the Tauberian theorems for topological groups, the precedent theory can be extended for kernels $K(x)$ defined in locally compact abelian groups. Among other things this permits the unification of the «discrete» and «continuous» theories, such as the M. Riesz theory of discrete Hilbert transforms of sequences and the ordinary Hilbert transforms of functions. As another byproduct we obtain also an extension of the ergodic theorems to the general groups due to Calderón [9]. From our point of view, the Hilbert transforms and ergodic theorems may be considered as special cases of the tauberian theorems.

Of course, in the present paper are extended to the general operators H_m only the first most basic facts from the theory of Hilbert transforms or ergodic theory. It seems to us that the extension of the more profound features of these theories could open an interesting field of research. (Cfr. § 6, C) and D)).

1. Introduction. In this section we will recall the fundamental facts about Hilbert transforms and ergodic theorems, and explain the main idea of the paper.

A) The fundamental theorem of the theory of integration asserts that the indefinite integral of an integrable function is differentiable almost everywhere.

If $f(x)$ is integrable on every finite interval of the line $(-\infty, \infty)$, and if for each $\varepsilon > 0$ we define the transform $D_\varepsilon f$ by

$$D_\varepsilon f = D_\varepsilon f(x) = [D_\varepsilon f](x) = \frac{1}{\varepsilon} \int_0^\varepsilon f(x+t)\,dt = \frac{1}{\varepsilon} \int_x^{x+\varepsilon} f(t)\,dt,$$

then this theorem asserts that

$$\lim_{\varepsilon \to 0} D_\varepsilon f(x) = f(x), \text{ for almost all } x.$$

If $p > 1$, then it is also true that

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} | D_\varepsilon f(x) - f(x) |^p \, dx = 0, \quad \text{for } f \in L^p.$$

Moreover, if Λf is defined by

$$\Lambda f(x) = \sup_{\varepsilon > 0} | D_\varepsilon f(x) |,$$

then the Hardy-Littlewood maximal theorem asserts that (cfr. [**8**])

$$\int_{-\infty}^{\infty} | \Lambda f(x) |^p \, dx \leq O_p \int_{-\infty}^{\infty} | f(x) |^p \, dx \qquad (p > 1).$$

Here O_p depends on p alone. The precedent inequality is not true if $p = 1$. In this case, we have instead the following inequalities:

$$\int_{-A}^{A} | \Lambda f(x) |^\alpha \, dx \leq O_\alpha \,.\, A^{1-\alpha} \left\{ \int_{-\infty}^{\infty} | f(x) | \, dx \right\}^\alpha, \text{ if } 0 < \alpha < 1,$$

$$\int_{-A}^{A} | \Lambda f(x) | \, dx \leq O_A \left\{ \int_{-\infty}^{\infty} |f| (1 + \log^+ |f|) \, dx \right\} + O_A'.$$

It was observed by N. Wiener [**3**] that the above theorems concerning the operators D_ε are particular cases of theorems for a wider class of operators arising in the ergodic theory. In fact, if for each real number t, we define the translation $\sigma_t x = x + t$, then each σ_t is a measure-perserving transformation on $R^1 = (-\infty, \infty)$, in the sense that for each measurable set $E \subset R^1$, $\sigma_t E$ is also measurable and of the same measure as E, and $D_\varepsilon f$ may be written in the following form

$$D_\varepsilon f(x) = \frac{1}{\varepsilon} \int_0^\varepsilon f(\sigma_t x)\, dt.$$

The ergodic theory considers, more generally, an abstract measure space $\Omega = \{P\}$ and a group of measure-perserving transformations $\{\sigma_t P\}$, $(-\infty < t < \infty)$ of Ω. $D_N f$ is now defined similarly by

$$D_N f(P) = \frac{1}{N} \int_0^N f(\sigma_t P)\, dt.$$

The ergodic theorem asserts that the above properties of the operators $D_\varepsilon f(x)$ hold for these more general operators $D_N f(P)$. The pointwise and mean convergence theorems correspond here, respectively, to the Birkhoff and von Neumann ergodic theorems, and the Hardy-Littlewood's maximal theorem to the Wiener dominated theorem. The only unimportant difference is that in the case of the ergodic theorems the limit is taken as $N \to \infty$, instead of $\varepsilon \to 0$.

It is important to note that the operators D_ε, or D_N, are positive operators, in the sense that $D_N f(x) \geq 0$ almost everywhere if $f(x) \geq 0$ for almost all x. This property of D_N is essential in the current proofs of the ergodic theorems, — those proofs do not apply to non-positive operators.

B) An important case of non positive operators, for which the above theorems still hold is exhibited by the theory of Hilbert transforms. The (ordinary) Hilbert transform Hf of the function $f(x)$, $(-\infty < x < \infty)$, is defined by

$$(I) \qquad Hf(x) = \int_{-\infty}^{\infty} \frac{f(t)}{x-t}\, dt.$$

Hf is understood as the limit, as $\varepsilon \to 0$, of $H_\varepsilon f$, where $H_\varepsilon f$ is defined for each $\varepsilon > 0$ by

$$(Ia) \qquad H_\varepsilon f(x) = \int_{|t-x| > \varepsilon} \frac{f(t)}{x-t}\, dt = \int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty} \frac{f(t)}{x-t}\, dt.$$

Lusin, Privaloff and Plessner proved the pointwise convergence of $H_\varepsilon f$: for every $f \in L^p$, $p \geq 1$, the limit

$$(II) \qquad \lim_{\varepsilon \to 0} H_\varepsilon f(x) = Hf(x)$$

exists for almost all x. The limit function $Hf(x)$ is then taken as the definition of the singular integral (I).

While the function $Hf(x)$ exists, it may not be integrable. M. Riesz has shown that if $f \in L^p$, and $p > 1$ then also $Hf \in L^p$ and $H_\epsilon f$ converges to Hf in the p$^{\text{th}}$-mean, i.e.

$$\text{(III)} \quad \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} | Hf(x) - H_\epsilon f(x) |^p \, dx = 0, \qquad \text{for } f \in L^p,\ p > 1.$$

Moreover, the following inequality of M. Riesz

$$\text{(IV)} \qquad \int_{-\infty}^{\infty} | Hf(x) |^p \, dx \leq \mathrm{O}_p \int_{-\infty}^{\infty} | f(x) |^p \, dx, \qquad (p > 1),$$

holds for any $f \in L^p$, where O_p dependes on p alone.

Kolmogoroff proved that for $f \in L^1$ the limit operator satisfies the inequality

$$\text{(V)} \int_{-A}^{A} | H f(x) |^\alpha \, dx \leq O_\alpha A^{1-\alpha} \left\{ \int_{-\infty}^{\infty} | f(x) | \, dx \right\}^\alpha, \quad \text{if} \quad 0 < \alpha < 1.$$

This inequality was completed by Zygmund as follows:

$$\text{(VI)} \qquad \int_{-A}^{A} | H f(x) | \, dx \leq O_A \int_{-\infty}^{\infty} |f| (1 + \log^+ |f|) \, dx + O'_A .$$

Finally Zygmund [**1**] proved the maximal theorems for H_ϵ: if $Mf(x)$ is defined by

$$\text{(VII)} \qquad Mf(x) = MHf(x) = \sup_{0 < \epsilon < 1} | H_\epsilon f(x) |,$$

then

$$\text{(VII } a) \quad \int_{-\infty}^{\infty} | Mf(x) |^p \, dx \leq O_p \int_{-\infty}^{\infty} | f(x) |^p \, dx \qquad (\text{if } p > 1),$$

$$\text{(VII} b) \quad \int_{-A}^{A} | Mf |^\alpha \, dx \leq O_\alpha A^{1-\alpha} \left\{ \int_{-\infty}^{\infty} |f| \, dx \right\}^\alpha \quad (\text{if } 0 < \alpha < 1),$$

$$\text{(VII } c) \quad \int_{-A}^{A} | Mf | \, dx \leq O_A \int_{-\infty}^{\infty} |f| (1 + \log^+ |f|) \, dx + O'_A .$$

Thus, though the operators H_ϵ are not positive they possess the same properties as the positive operators D_ϵ or D_N. According to

Lusin, the pointwise convergence of the operator D_ε expresses the « fundamental differential property of the first order » of integrable functions, which is due to the « smallness » of $|f(x+\varepsilon)-f(x)|$ as $\varepsilon \to 0$. Lusin considers the corresponding property (II) of the operators H_ε as the « fundamental differential property of the second order » of integrable functions. which holds no longer because of the smalness of $|f(x+\varepsilon)-f(x)|$, but because of the cancellation of the positive and negative values of $f(x+\varepsilon)-f(x)$.

C) Though the properties of the Hilbert transforms expresses fundamental facts of real variable theory the proofs of Lusin, Riesz and Kolmogoroff were based entirely on the theory of analytic functions. For this reason, and also with a view to further generalizations, Lusin proposed the problem of giving a direct proof by real variables methods. This problem was solved by Besicovitch in two important papers [**10**] and [**11**]. The method of Besicovitch were perfected by Titchmarch [**12**], and Pollard [**13**] extended Besicovitch's results to Hilbert-Stieljes transforms.

The methods of Besicovitch and Titchmarch were developed by Zygmund and Calderón [**2**] who extended the above theorems of the Hilbert transform to a very wide class of n-dimensional transforms which arise in potential theory (cfr. Mijlin [**14**]). Their generalization is the following. Letting $K(x)=x^{-1}$, (I) may be written as a convolution whith the kernel $K(x)$:

$$Hf(x)=f*K(x)=\int_{-\infty}^{\infty} K(x-t)f(t)\,dt.$$

We note that the kernel $K(x)=x^{-1}$ is not integrable near the points $x=0$, $x=\infty$, and that $n=1$ is the only number such that x^{-n} is simultaneously not integrable at O and ∞. Moreover as we have observed already, the property $K(x)+K(-x)=0$ is essential for the existence of $Hf(x)$ because of the interference of the positive and negative values.

Hence if we want to consider in the plane $R^2=\{z\}$, $z=\{x,y\}$, an analogue of the kernel $K(x)=x^{-1}$, it is natural to take the kernels $K(z)$ of the form

$$K(z)=\frac{\omega(\theta)}{|z|^2}, \qquad z=|z|e^{i\theta},$$

where the function $\omega(\theta)$ does not depend on $|z|$ and satisfies

$$\int_0^{2\pi} \omega(\theta)\, d\theta = 0.$$

The last condition corresponds to the property $K(x) + K(-x) = 0$ of $K(x) = x^{-1}$, and $|z|^2$ in the denominator of $K(z)$ corresponds to the fact that $n = 2$ is the only number such that $|z|^n$ is not integrable, both at $z = 0$ and $z = \infty$, over the plane R^2. To each kernel $K(z)$, $z \in R^2$, with the above properties, corresponds a two dimensional Hilbert transform

$$\text{(I}'\text{)} \qquad Hf(z) = f * K(z) = \iint_{R^2} K(z-u)\, f(u)\, du,$$

defined as the limit of

$$\text{(I}\,a'\text{)} \qquad H_\varepsilon f(z) = \iint_{|u-z|>\varepsilon} K(z-u)\, f(u)\, du,$$

as $\varepsilon \to 0$.

Similar definitions are made in the n-dimensional case, $n \geq 2$.

Under very general assumptions on the function $\omega(\theta)$, Zygmund and Calderón prove that the theorems (II)-(IV) and (VII a) hold for the n-dimensional Hilbert transforms (I′), thus obtaining a generalization of the classical theory to n-dimensional kernels $K(z)$.

b) In [2] Zygmund and Calderón *do* not consider the double transforms, or the cartesian product of transforms of type (I′). In the classical case, that is the 1-dimensional case of the kernel $K(x) = x^{-1}$, the double Hilbert transform is defined by

$$\text{(I}\,b\text{)} \qquad Hf(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(s, t)\, ds\, dt}{(x-s)\,(y-t)}$$

$$= \iint_{R^2} K(x-s)\, K(y-t)\, f(s, t)\, ds\, dt$$

$$= \lim_{\varepsilon, \varepsilon' \to 0} H_{\varepsilon\varepsilon'} f(x, y), \qquad (K(x) = x^{-1}),$$

$$\text{(I}'\,b'\text{)} \qquad H_{\varepsilon\varepsilon'} f(x, y) = \iint_{\substack{|x-s|>\varepsilon \\ |y-t|>\varepsilon'}} f(s,t)\, K(x-s)\, K(y-t)\, ds\, dt.$$

Of course it is important not to confuse the double transform (I*b*) of 1-dimensional kernels $K(x) = x^{-1}$, with the simple transforms (I′) of 2-dimensional kernels $K(z)$. There is only one double transform (I*b*) but an infinity of 2-dimensional transforms (I′). While in