Sign-changing solutions for the fractional Schrödinger equation

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The problem

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - f(u) = 0 \quad \text{in } \mathbb{R}^N$$

where

- $0 < s < 1$,
- $1 < p < \frac{N+2s}{N-2s}$,
- $N > 2s$, $f(t) = |t|^{p-1}t$
- $V \in L^\infty(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} V > 0$.

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- $V \in L^\infty(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} V > 0$.
- $(-\Delta)^s$ is the fractional Laplace operator

$$(-\Delta)^s u(x) = c(n,s)\text{p.v.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy$$

- The fractional nonlinear Schrödinger equation

$$i\psi_t = \varepsilon^{2s}(-\Delta)^s\psi + W(x)\psi - |\psi|^{p-1}\psi$$

- We look for standing-wave solutions,

$$\psi(x,t) = u(x)e^{iEt}$$

with u a real-valued function.

- If we consider $V(x) = W(x) + E$,

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N$$

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- Long-Lv 2017 sign-changing solutions to

$$\varepsilon^{2s}(-\Delta)^s u + Vu - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N$$

with $V(x) = V(|x|)$.

Goal

We look for a sign-changing solution to the equation

$$\varepsilon^{2s}(-\Delta)^s u + Vu - f(u) = 0 \quad \text{in } \mathbb{R}^N$$

with positive spikes and negative spikes, both concentrating at a local minimum of $V(x)$, as $\varepsilon \to 0$.

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After absorbing ε, the equation becomes

$$(-\Delta)^s v + V(\varepsilon x)v - f(v) = 0 \quad \text{in } \mathbb{R}^N$$

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- $u_\varepsilon \approx w_{\lambda_1} - w_{\lambda_2} \quad , \quad \lambda_i = V(Q_i)$

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- Denote

$$\bar{w}_{\lambda_j}(x) := w_{\lambda_j}\left(\frac{x - Q_j}{\varepsilon}\right) = \lambda_j^{\frac{1}{p-1}} w\left(\lambda_j^{\frac{1}{2s}}\left(\frac{x - Q_j}{\varepsilon}\right)\right)$$

Main result

Theorem

Assume $(V_0) - (V_2)$

Then, for $\varepsilon > 0$ *small enough, there exists* $u_\varepsilon \in H^{2s}(\mathbb{R}^N)$ *solution to*

$$\varepsilon^{2s}(-\Delta)^s u + Vu - f(u) = 0 \quad \text{in } \mathbb{R}^N$$

$$u_\varepsilon(x) = \sum_{i=1}^{h} \bar{w}_{\lambda_i^\varepsilon}(x) - \sum_{i=h+1}^{2h} \bar{w}_{\lambda_i^\varepsilon}(x) + \bar{\varphi}_\varepsilon(x)$$

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where

- $\bar{\varphi}_\varepsilon \in H^{2s}(\mathbb{R}^N)$, *with* $\bar{\varphi}_\varepsilon \to 0$ *in* $H^{2s}(\mathbb{R}^N)$,*as* $\varepsilon \to 0$.
- $\lambda_i^\varepsilon = V(Q_i^\varepsilon)$, *with* $Q_i^\varepsilon \in \Omega$
- $V(Q_i^\varepsilon) \to \min_\Omega V$, *as* $\varepsilon \to 0$.

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- A minimization argument

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$$w_i(x) = \lambda_i^{\frac{1}{p-1}} w(\lambda_i^{\frac{1}{2s}}(x - q_i))$$

$$W(x) = \sum_{i=1}^{2h} \tau_i w_i(x)$$

$\tau_i \in \{-1, 1\}$ for every $i = 1, \dots, \ell$

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- We project the equation and look a solution ϕ such that

$$\begin{cases} (-\Delta)^s (W + \phi) + V(\varepsilon x)(W + \phi) - f(W + \phi) = \sum_{i=1}^{\ell} \sum_{l=1}^{N} c_{il} Z_{il} & \text{in } \mathbb{R}^N \\ \langle Z_{il}, \phi \rangle = 0 & \text{for all } i, l, \end{cases}$$

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where

$$L_\varepsilon(\phi) := (-\Delta)^s \phi + V(\varepsilon x)\phi - f'(W)\phi,$$

$$N_\varepsilon(\phi) := f(W + \phi) - f(W) - f'(W)\phi$$

$$E_\varepsilon := \sum_{i=1}^{\ell} \tau_i (V(Q_i) - V(\varepsilon x)) w_i + f(W) - \sum_{i=1}^{\ell} \tau_i f(w_i).$$

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- Banach fixed point Theorem

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$$J_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^N} v(-\Delta)^s v + V(\varepsilon x)v^2\,dx - \int_{\mathbb{R}^N} F(v)\,dx$$

with $F(t) = \frac{1}{p+1}|t|^{p+1}$.

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- $\mathcal{J}_\varepsilon(\mathbf{q})$

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$$\gamma_{ij} = V(Q_i)^{\frac{1}{p-1}-\frac{N+2s}{2s}} V(Q_j)^{\frac{p}{p-1}}, \quad \text{for } i \neq j,$$

and

$$\gamma = \frac{\gamma_0}{2} \int_{\mathbb{R}^N} w^p \, dx$$

$$I_w = \frac{1}{2} \int_{\mathbb{R}^N} (w(-\Delta)^s w + w^2) \, dx - \int_{\mathbb{R}^N} F(w) \, dx$$

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$$\begin{aligned} &\mu_2 + 2hI_w V(Q_0)^{\frac{p+1}{p-1}-\frac{N}{2s}} + O\left(\varepsilon^{\min\{2s, \frac{2N}{N+2s}\}}\right) \\ &\leq I_w \sum_{i=1}^{\ell} V(Q_{i\varepsilon})^{\frac{p+1}{p-1}-\frac{N}{2s}} + O\left(\varepsilon^{\min\{2s, \frac{2N}{N+2s}\}}\right) = \mathcal{J}_\varepsilon(\mathbf{q}_\varepsilon) \\ &\leq 2hI_w V(Q_0)^{\frac{p+1}{p-1}-\frac{N}{2s}} + C\varepsilon^{\min\{2s, \frac{2N}{N+2s}\}} \end{aligned}$$

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$\square$

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- Principal reference: D'Aprile-Pistoia 2009
- min-max argument

Hay brownie casero de premio!