A free boundary problem with gradient constraint and Tug-of-War games

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A free boundary problem

In this talk we deal with the limit as $p \to \infty$ of solutions to

$$\begin{cases} -\Delta_p u_p(x) + \chi_{\{u_p>0\}}(x) = 0 & \text{in } \Omega \\ u_p(x) = g(x) & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ stands for the p-Laplace operator, g is a continuous boundary data and $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain.

In this context, $\partial\{u > 0\} \cap \Omega$ is the *free boundary* of the problem.

A free boundary problem

The unique weak solution is the minimizer the following functional

$$\mathfrak{J}_p[v] = \int_\Omega \left(\frac{1}{p} |\nabla v(x)|^p + v\chi_{\{v>0\}}(x) \right) dx$$

in

$$\mathbb{K} = \left\{ v \in W^{1,p}(\Omega) \text{ and } v = g \text{ on } \partial\Omega \right\}.$$

Gradient constraint

Recently, motivated by game theory ("Tug of-war games"), Juutinen-Parvianen-R. studied the limit as $p \to \infty$ in the following problem

$$\begin{cases} -\Delta_p u_p(x) & = & -f(x) & \text{in} & \Omega \\ u_p(x) & = & g(x) & \text{on} & \partial\Omega \end{cases}$$

with a forcing term $f \geq 0$ and a continuous boundary datum g.

Gradient constraint

It hold that that , $\{u_p\}_{p\geq 2}$ converges, up to a subsequence, to a limiting function u_∞, which is a solution to the following problem in the viscosity sense

$$\begin{cases} \max\left\{-\Delta_\infty u_\infty(x), -|\nabla u_\infty(x)| + \chi_{\{f>0\}}(x)\right\} & = & 0 & \text{in} & \Omega \\ u_\infty(x) & = & g(x) & \text{on} & \partial\Omega, \end{cases}$$

where

$$\Delta_\infty u(x) = \nabla u(x)^T D^2 u(x) \cdot \nabla u(x)$$

is the $\infty-$*Laplacian*.

Gradient constraint

Such limit problems are known as problems with *Gradient constraint*. By considering solutions to

$$\min\{\Delta_\infty u, |\nabla u| - \epsilon\} = 0 \quad \text{and} \quad \max\{\Delta_\infty u, \epsilon - |\nabla u|\} = 0$$

Jensen provides a mechanism to obtain solutions of the infinity-Laplace equation $-\Delta_\infty u = 0$ via an approximation procedure.

He proved uniqueness for the infinity-Laplace equation by first showing that it holds for the approximating equations and then sending $\epsilon \to 0$.

Gradient constraint

$$\begin{cases} \min\{\Delta_\infty u_\infty(x), |\nabla u_\infty(x)| - h(x)\} & = & 0 & \text{in} & \Omega \\ u_\infty(x) & = & g(x) & \text{on} & \partial\Omega, \end{cases}$$

We highlight that, in general, the uniqueness of solutions to these problems is an easy task if h is a continuous function and strictly positive everywhere. Nevertheless, the case $h \geq 0$ is difficult.

This resembles the scenario that holds for the infinity Poisson equation

$$-\Delta_\infty u = h,$$

where the uniqueness is known to hold if $h > 0$ or $h \equiv 0$, and the case $h \geq 0$ is an open problem.

Gradient constraint

For the problem

$$\begin{cases} \max\{-\Delta_\infty u_\infty(x), -|\nabla u_\infty(x)| + \chi_D(x)\} & = & 0 & \text{in} & \Omega \\ u_\infty(x) & = & g(x) & \text{on} & \partial\Omega, \end{cases}$$

Juutinen-Parviainen-R proved uniqueness under the topological condition $\overline{D} = \overline{D^\circ}$ on the set $D \subset \mathbb{R}^N$.

Furthermore, they show counterexamples where the uniqueness fails.

Finally, from a regularity viewpoint, Juutinen-Parviainen-R. also establishes that viscosity solutions are Lipschitz continuous.

Limit free boundary problem.

In our next result, we show existence and regularity of limit solutions. We will assume in this limit procedure that the boundary datum g is a fixed Lipschitz function.

Theorem (Limiting problem)

Up to a subsequence, $u_p \to u_\infty$ uniformly in $\overline{\Omega}$. This limit fulfils in the viscosity sense

$$\begin{cases} \max\left\{-\Delta_\infty u_\infty,\ -|\nabla u_\infty| + \chi_{\{u_\infty>0\}}\right\} = 0 & \text{in } \Omega \cap \{u_\infty \geq 0\} \\ u_\infty = g & \text{on } \partial\Omega. \end{cases}$$

Finally, u_∞ is a Lipschitz continuous function with

$$[u_\infty]_{Lip(\overline{\Omega})} \leq \mathfrak{C}(N) \max\left\{1, [g]_{Lip(\partial\Omega)}\right\}.$$

Limit free boundary problem. Viscosity solutions

Definition An upper semi-continuous (resp. lower semi-continuous) function u is a viscosity subsolution (resp. supersolution) if, whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u - \varphi$ has a strict local maximum (resp. minimum) at x_0, then

$$\max\{-\Delta_\infty \varphi(x), \chi_{\{u \geq 0\}}(x_0) - |\nabla \varphi(x_0)|\} \leq 0$$

respectively

$$\max\{-\Delta_\infty \varphi(x), \chi_{\{u > 0\}}(x_0) - |\nabla \varphi(x_0)|\} \geq 0.$$

Limit free boundary problem.

Notice that the limit equation can be written as a fully non-linear second order operator as follows

$$\begin{array}{rcl} F_\infty : \mathbb{R} \times \mathbb{R}^N \times \mathrm{Sym}(N) & \longrightarrow & \mathbb{R} \\ (s, \xi, X) & \longrightarrow & \max\left\{-\xi^T X\xi, -|\xi| + \chi_{\{s>0\}}\right\}, \end{array}$$

which is non-decreasing in s. Moreover, F_∞ is a degenerate elliptic operator in the sense that

$$F_\infty(s, \xi, X) \le F_\infty(s, \xi, Y) \quad \text{whenever} \quad Y \le X.$$

Nevertheless, F_∞ is not in the framework of the general theory (Crandall-Ishii-Lions).

Limit free boundary problem.

Then, to prove uniqueness of limit solutions becomes a non-trivial task. We overcome such difficulty by using ideas from Juutinen-Parviainen-R. and show that solutions to the limit problem are unique.

Theorem (Uniqueness)

There is a unique viscosity solution to the limit problem. Moreover, a comparison principle holds, i.e., if

$$g_1 \leq g_2$$

on $\partial\Omega$ then the corresponding solutions u_∞^1 and u_∞^2 verify

$$u_\infty^1 \leq u_\infty^2$$

in Ω.

Limit free boundary problem.

We have a sharp lower control on how limit solutions detach from their free boundaries and a convergence result.

Theorem (Linear growth for limit solutions)

Let u_∞ be a uniform limit of u_p and $\Omega' \Subset \Omega$. Then, for any $x_0 \in \partial\{u_\infty > 0\} \cap \Omega'$ and any $0 < r \ll 1$, the following estimate holds:

$$\sup_{B_r(x_0)} u_\infty(x) \geq r.$$

We also have a result on convergence of the free boundaries

$$\partial\{u_p > 0\} \to \partial\{u_\infty > 0\} \quad as \quad p \to \infty,$$

in the sense of the Hausdorff distance.

Limit free boundary problem. Game theory

Our original main motivation to consider this problem comes from its connection to modern game theory.

Recently, Peres-Schram-Sheffield-Wilson introduced a two player random turn game called “Tug-of-war”, and showed that, as the “step size” converges to zero, the value functions of this game converge to the unique viscosity solution of the infinity-Laplace equation $-\Delta_\infty u = 0$.

Let us describe this game.

Tug-of-War games

Rules

- Two-person, zero-sum game: two players are in contest and the total earnings of one are the losses of the other.
- Player I, plays trying to maximize his expected outcome.
- Player II is trying to minimize Player I's outcome.
- $\Omega \subset \mathbb{R}^n$, bounded domain with $\partial\Omega \subset \Gamma$.
- $F : \Gamma \to \mathbb{R}$ be a Lipschitz continuous final payoff function.
- Starting point $x_0 \in \Omega$. A coin is tossed and the winner chooses a new position $x_1 \in B_\epsilon(x_0)$.
- At each turn, the coin is tossed again, and the winner chooses a new game state $x_k \in B_\epsilon(x_{k-1})$.
- Game ends when $x_\tau \in \Gamma$, and Player I earns $F(x_\tau)$ (Player II earns $-F(x_\tau)$)

Remark

The sequence $\{x_0, x_1, \cdots, x_N\}$ has some probability, which depends on

- The starting point x_0.
- The strategies of players, S_I and S_{II}.

Expected result

Taking into account the probability defined by the initial value and the strategies:

$$\mathbb{E}^{x_0}_{S_I, S_{II}}(F(x_N))$$

"Smart" players

- Player I tries to choose at each step a strategy which maximizes the result.
- Player II tries to choose at each step a strategy which minimizes the result.

Extremal cases

- $$u_I(x) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^x_{S_I,S_{II}}(F(x_N))$$
- $$u_{II}(x) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}^x_{S_I,S_{II}}(F(x_N))$$

Definition

The game has a value $\Leftrightarrow$ $u_I = u_{II}$.

Theorem

Peres-Schram-Sheffield-Wilson (2008). Under very general hypotheses, the game has a value.

Dynamic Programming Principle

Main Property (Dynamic Programming Principle)

$$u(x) = \frac{1}{2}\left\{ \sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right\}.$$

Denote u the value of the game, and assume that there are x_M, x_m such that:

- $u(x_M) = \max_{|x-y|\leq\epsilon} u(y)$.
- $u(x_m) = \min_{|x-y|\leq\epsilon} u(y)$.

Dynamic Programming Principle

$$u(x) = \frac{1}{2}\left\{\max_{B_\epsilon(x)} u + \min_{B_\epsilon(x)} u\right\} = \frac{1}{2}\left\{u(x_M) + u(x_m)\right\}$$

that is,

$$0 = \left\{u(x_M) + u(x_m) - 2u(x)\right\}$$

Idea

- $x_M \approx x + \epsilon \frac{\nabla u(x)}{|\nabla u(x)|}$
- $x_m \approx x - \epsilon \frac{\nabla u(x)}{|\nabla u(x)|}$
- $\frac{u(x+\epsilon\vec{v}) + u(x-\epsilon\vec{v}) - 2u(x)}{\epsilon^2}$ $\equiv$ discretization of the second derivative in the direction of $\vec{v}$

Therefore

Dynamic programming principle $\approx$ discretization of the second derivative in the direction of the gradient.

Remark

Second derivative in the direction of the gradient
$\equiv$ ∞-Laplacian.

Theorem by Peres-Schramm-Sheffield-Wilson.

Existence and uniqueness of the limit of the values of ϵ-Tug-of-war games as $\epsilon \to 0$
$\Rightarrow$
Alternative proof of existence and uniqueness for the problem

$$\begin{cases} \Delta_\infty u = 0, & \Omega, \\ u|_{\partial\Omega} = F & \partial\Omega. \end{cases}$$

Limit free boundary problem. Game theory

Now, we want to obtain a game approximation for a free boundary problem that involves the set where the solution is positive, $\{u > 0\}$.

This task involves the following difficulty, if one tries to play with a rule of the form ”one player sells the turn when the expected payoff is positive”, then the value of the game will not be well defined since this rule is an anticipating strategy (the player needs to see the future in order to decide whet he is going to play).

Limit free boundary problem. Game theory

Here we define and study a variant of the Tug-of-war game, that we call *Pay or Leave Tug-of-War*.

In our game, one of the players decide to play the usual Tug-of-war or to pass the turn to the other player who decides to end the game immediately (and get 0 as final pay-off) or move and pay ϵ (which is the step size).

We show that the value functions of this new game, namely u^ε, satisfy a Dynamic Programming Principle (DPP) given by

$$u^\varepsilon(x) = \min\left\{ \frac{1}{2}\left(\sup_{y\in B_\epsilon(x)} u^\varepsilon(y) + \inf_{y\in B_\epsilon(x)} u^\varepsilon(y) \right); \right.$$
$$\left. \max\left\{ 0; \sup_{y\in B_\epsilon(x)} u^\varepsilon(y) - \epsilon \right\} \right\}.$$

Limit free boundary problem. Game theory

Moreover, we show that the sequence u^{ε} converge and the corresponding limit is a viscosity solution to the free boundary problem. Therefore, besides its own interest, the game-theoretic scheme provides an alternative mechanism to prove the existence of a viscosity solution.

Theorem

Let u^{ε} be the value functions of the game previously described. Then, it holds that

$$u^{\epsilon} \to u \qquad \textit{uniformly in} \quad \overline{\Omega},$$

being u the unique viscosity solution to the limit equation.

References

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