Ecuaciones elípticas no-lineales con potencial de tipo absorbción a valor medida

Nicolas Saintier, U. de Buenos Aires

en collaboración con

Laurent Veron, U. de Tours (Francia)

Purpose of the talk.

I would like to present some existence results for an equation of the form

$$\begin{aligned} -\Delta u + g(u)\sigma &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where

- Ω is a smooth bounded open subset of $\mathbb{R}^N$,
- σ, μ are bounded measures in Ω with $\sigma \geq 0$
  the well-studied case is $\sigma = dx$,
- $g : \mathbb{R} \to \mathbb{R}$ is a continuous function of absorbing type i.e.

$$g(r)r \geq 0 \qquad \text{for any } r \in \mathbb{R}$$

typical example: $g(r) = |r|^{q-1}r$ with $q > 1$.

Plan of the talk.

1. Classical results in the linear case

$$\begin{aligned} -\Delta u &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

Plan of the talk.

1. Classical results in the linear case

$$\begin{aligned} -\Delta u &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

2. What happens when adding an absorbing nonlinearity ?

$$\begin{aligned} -\Delta u + g(u) &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

Plan of the talk.

1. Classical results in the linear case

$$\begin{aligned} -\Delta u &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

2. What happens when adding an absorbing nonlinearity ?

$$\begin{aligned} -\Delta u + g(u) &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

3. When $g(r) = |r|^{q-1}r$ with $q > 1$, there are two cases:
   1. if $q < \frac{N}{N-2}$, there exists a unique solution for any μ,
   2. if $q \geq \frac{N}{N-2}$, there exists a solution iff μ is not too concentrated (depending on q in a precise way.)

Plan of the talk.

1. Classical results in the linear case

$$\begin{aligned} -\Delta u &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

2. What happens when adding an absorbing nonlinearity ?

$$\begin{aligned} -\Delta u + g(u) &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

3. When $g(r) = |r|^{q-1}r$ with $q > 1$, there are two cases:
   1. if $q < \frac{N}{N-2}$, there exists a unique solution for any μ,
   2. if $q \geq \frac{N}{N-2}$, there exists a solution iff μ is not too concentrated (depending on q in a precise way.)
4. How do the previous alternative generalize when the nonlinearity is $g(u)\sigma$? What is “not too concentrated” ?

Classical results in the linear case.

Notion of solution

$u \in L^1(\Omega)$ is a very weak solution of

$$\begin{array}{rcll} -\Delta u & = & \mu & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array}$$

if for any $\zeta \in C_c^1(\overline{\Omega})$ such that $\Delta\zeta \in L^\infty$, there holds

$$-\int_\Omega u\Delta\zeta\,dx = \int_\Omega \zeta\,d\mu$$

Existence

For any μ bounded measure, there exists a unique solution u.
It can be obtained

- directly by

$$u(x) = \int_\Omega \mathbb{G}(x,y)\,d\mu(y) := \mathbb{G}[\mu](x)$$

where $\mathbb{G}$ is the Green function of Ω.

Existence

For any μ bounded measure, there exists a unique solution u.
It can be obtained

- directly by
$$u(x) = \int_\Omega \mathbb{G}(x,y)\, d\mu(y) := \mathbb{G}[\mu](x)$$
where $\mathbb{G}$ is the Green function of Ω.
- or a) first prove existence when $\mu \in L^1(\Omega)$ using the a priori estimate
$$\|u\|_1 \leq C\|\mu\|_1$$
Then b) when μ is a measure, regularize $\mu \Longrightarrow \mu_n, u_n$.
To prove that $u_n \to u$ weakly in L^1, use Dunford-Pettis thm - need that (u_n) is uniformly equiintegrable in L^1:
$$\int_\omega |u_n|\, dx < \varepsilon \qquad \text{if } |\omega| < \delta$$

There holds

$$\|u\|_{W^{1,q}} \leq c\|\mu\| =: c|\mu|(\Omega) \qquad q < \frac{N}{N-1}.$$

More precisely (sharp):

$$\|u\|_{L^{\frac{N}{N-2},\infty}} + \|\nabla u\|_{L^{\frac{N}{N-1},\infty}} \leq c\|\mu\|$$

Equation with absorption nonlinearity.

Notion of solution

$u \in L^1(\Omega)$ is a very weak solution of

$$\begin{aligned} -\Delta u + g(u) &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

if $g(u) \in L^1(\Omega, \rho dx)$, where $\rho(x) = dist(x, \partial\Omega)$, and

$$-\int_\Omega u\Delta\zeta \, dx + \int_\Omega g(u)\zeta \, dx = \int_\Omega \zeta \, d\mu$$

for any $\zeta \in C^1_c(\overline{\Omega})$ such that $\Delta\zeta \in L^\infty$.

Notice that $|\zeta/\rho| \leq C$ so that

$$g(u)\zeta = (\zeta/\rho)\rho g(u) \in L^1_\sigma$$

Existence when $\mu \in L^1$

Brezis and Strauss (1975) proved that:

1) for any $\mu \in L^1$ there exists a solution.

2) if moreover g is nondecreasing then this solution is unique and the map $\mu \to u$ is non-decreasing.

What happens when μ is a measure ?

Existence when μ is a bounded measure

Benilan and Brezis (80') noticed that the equation

$$\begin{aligned} -\Delta u + |u|^{q-1}u &= \delta_0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

(with $0 \in \Omega$) has no solution when $q \geq \frac{N}{N-2}$:

Existence when μ is a bounded measure

Benilan and Brezis (80') noticed that the equation

$$\begin{aligned} -\Delta u + |u|^{q-1}u &= \delta_0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

(with $0 \in \Omega$) has no solution when $q \geq \frac{N}{N-2}$: if u were a solution then taking $\phi_k(x) := \phi(kx)$ as test-function with $\phi \in C_c^\infty(B_0(1))$, $\phi(0) = 1$,

$$\begin{aligned} 1 &= -k^2 \int_{|x| \leq 1/k} u(x)\Delta\phi(kx)\,dx + \int_{|x| \leq 1/k} |u(x)|^{q-1}u(x)\phi(kx)\,dx \\ &\leq k^2 \|u\|_{L^q(B_0(1/k))} \|\Delta\phi(kx)\|_{L^{q'}(B_0(1/k))} + o(1) \\ &= o(1)k^{2-N/q')} + o(1) \\ &= o(1) \qquad \text{if } q \geq N/(N-2) \end{aligned}$$

Existence when μ is a bounded measure

Benilan and Brezis (80') noticed that the equation

$$\begin{aligned} -\Delta u + |u|^{q-1}u &= \delta_0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

(with $0 \in \Omega$) has no solution when $q \geq \frac{N}{N-2}$: if u were a solution then taking $\phi_k(x) := \phi(kx)$ as test-function with $\phi \in C_c^\infty(B_0(1))$, $\phi(0) = 1$,

$$\begin{aligned} 1 &= -k^2 \int_{|x| \leq 1/k} u(x)\Delta\phi(kx)\,dx + \int_{|x| \leq 1/k} |u(x)|^{q-1}u(x)\phi(kx)\,dx \\ &\leq k^2 \|u\|_{L^q(B_0(1/k))} \|\Delta\phi(kx)\|_{L^{q'}(B_0(1/k))} + o(1) \\ &= o(1)k^{2-N/q')} + o(1) \\ &= o(1) \qquad \text{if } q \geq N/(N-2) \end{aligned}$$

Contrary to the case $\mu \in L^1(\Omega)$,
there does not always exist a solution !

Existence when μ is a bounded measure

- To prove existence, one regularize μ and truncate g.
- Main difficulty: passing to the limit in $\int_\Omega g(u_n)\zeta\, dx$.

Existence when μ is a bounded measure

- To prove existence, one regularize μ and truncate g.
- Main difficulty: passing to the limit in $\int_\Omega g(u_n)\zeta\,dx$.
- Possible solution: use Vitali's theorem - need $\{g(u_n)\}$ to be uniformly equiintegrable in L^1:

$$\int_\omega g(u_n)\,dx \le \varepsilon \qquad \text{if } |\omega| \le \delta.$$

Existence when μ is a bounded measure

- To prove existence, one regularize μ and truncate g.
- Main difficulty: passing to the limit in $\int_\Omega g(u_n)\zeta\, dx$.
- Possible solution: use Vitali's theorem - need $\{g(u_n)\}$ to be uniformly equiintegrable in L^1:

$$\int_\omega g(u_n)\, dx \leq \varepsilon \qquad \text{if } |\omega| \leq \delta.$$

- If $|g(r)| \leq \tilde{g}(|r|)$ with $\tilde{g}$ continuous nondecreasing and
  1. either
$$\int^\infty \tilde{g}(t) t^{-1-\frac{N}{N-2}}\, dt < \infty,$$
  2. or
$$\int_\Omega \tilde{g}(\mathbb{G}[|\mu|])\rho\, dx < \infty$$

then there exists a solution. ($\mathbb{G}[|\mu|$ solution de $-\Delta u = |\mu|$ + Dirichlet B.C)

Consequence of the 1st existence condition

There exists a solution if

$$\int^{\infty} \tilde{g}(t) t^{-1-\frac{N}{N-2}}\, dt < \infty.$$

Consequence of the 1st existence condition

There exists a solution if

$$\int^{\infty} \tilde{g}(t)t^{-1-\frac{N}{N-2}}\,dt < \infty.$$

μ does not appear ! We get existence for any μ.

Consequence of the 1st existence condition

There exists a solution if

$$\int^{\infty} \tilde{g}(t) t^{-1-\frac{N}{N-2}}\, dt < \infty.$$

μ does not appear ! We get existence for any μ.
In particular when $g(r) = |r|^{q-1} r$,

If $q < \dfrac{N}{N-2}$, then there exists a solution for any μ.

Consequence of the 1st existence condition

There exists a solution if

$$\int^{\infty} \tilde{g}(t) t^{-1-\frac{N}{N-2}}\, dt < \infty.$$

μ does not appear ! We get existence for any μ.
In particular when $g(r) = |r|^{q-1} r$,

If $q < \dfrac{N}{N-2}$, then there exists a solution for any μ.

Optimal exponent (false when $q \geq \frac{N}{N-2}$: take μ a Dirac mass). So

If $q \geq \dfrac{N}{N-2}$, condition on μ giving existence of a solution ?

Let's examine the 2nd existence condition.

Consequence of the 2nd existence condition

Assume that $g(r) = |r|^{q-1}r$. There exists a solution if

$$\int_\Omega |\mathbb{G}[|\mu|]|^q \rho \, dx < \infty,$$

in particular if

$$\mathbb{G}[|\mu|] \in L^q(\Omega).$$

($\mathbb{G}[|\mu|$ solution de $-\Delta u = |\mu|$ + Dirichlet B.C)

Consequence of the 2nd existence condition

Assume that $g(r) = |r|^{q-1}r$. There exists a solution if

$$\int_\Omega |\mathbb{G}[|\mu|]|^q \rho \, dx < \infty,$$

in particular if

$$\mathbb{G}[|\mu|] \in L^q(\Omega).$$

($\mathbb{G}[|\mu|$ solution de $-\Delta u = |\mu|$ + Dirichlet B.C)

This holds if the measure μ is such that

$$|\mu| \in W^{-2,q}(\Omega).$$

Consequence of the 2nd existence condition

Assume that $g(r) = |r|^{q-1}r$. There exists a solution if

$$\int_\Omega |\mathbb{G}[|\mu|]|^q \rho\, dx < \infty,$$

in particular if

$$\mathbb{G}[|\mu|] \in L^q(\Omega).$$

($\mathbb{G}[|\mu|$ solution de $-\Delta u = |\mu|$ + Dirichlet B.C)

This holds if the measure μ is such that

$$|\mu| \in W^{-2,q}(\Omega).$$

What kind of measure can be approximated by measures in $W^{-2,q}(\Omega)$?

Consequence of the 2nd existence condition

Assume that $g(r) = |r|^{q-1}r$. There exists a solution if

$$\int_\Omega |\mathbb{G}[|\mu|]|^q \rho \, dx < \infty,$$

in particular if

$$\mathbb{G}[|\mu|] \in L^q(\Omega).$$

($\mathbb{G}[|\mu|$ solution de $-\Delta u = |\mu|$ + Dirichlet B.C)
This holds if the measure μ is such that

$$|\mu| \in W^{-2,q}(\Omega).$$

What kind of measure can be approximated by measures in $W^{-2,q}(\Omega)$?

$\Longrightarrow$ measures that are diffuse with respect to the $W^{2,q'}$-capacity.

The $W^{k,p}$-capacity of a compact set $K \subset \mathbb{R}^N$ is

$$c_{k,p}(K) = \inf\{\|\phi\|^p_{W^{k,p}} : \phi \in C_c^\infty(\mathbb{R}^N),\ \phi \geq 1_K \text{ on } K.\}.$$

The capacity of an open set U and then of an arbitrary set E are then defined by

$$c_{k,p}(U) = \sup\{c_{k,p}(K) : K \subset U \text{ compact}\},$$

$$c_{k,p}(E) = \inf\{c_{k,p}(U) : E \subset U,\ U \text{ open}\}.$$

Bessel capacity

The $W^{k,p}$-capacity of a compact set $K \subset \mathbb{R}^N$ is

$$c_{k,p}(K) = \inf \{ \|\phi\|^p_{W^{k,p}} : \phi \in C_c^\infty(\mathbb{R}^N),\ \phi \geq 1_K \text{ on } K. \}.$$

The capacity of an open set U and then of an arbitrary set E are then defined by

$$c_{k,p}(U) = \sup \{ c_{k,p}(K) : K \subset U \text{ compact} \},$$

$$c_{k,p}(E) = \inf \{ c_{k,p}(U) : E \subset U,\ U \text{ open} \}.$$

The same definition holds for Bessel potential spaces $H^{s,p}(\mathbb{R}^N)$:

$$H^{s,p}(\mathbb{R}^N) = \{ f = G_s * g,\ g \in L^p(\mathbb{R}^N) \}, \qquad G_s = \mathcal{F}^{-1}((1 + |\xi|^2)^{-s/2}).$$

They coincide with $W^{k,p}$ when $s = k$ (Calderon's theorem).

Diffuse measures

A measure μ is diffuse w.r.t $c_{s,p}$ if for any E,

$$c_{s,p}(E) = 0 \Longrightarrow |\mu|(E) = 0.$$

Diffuse measures

A measure μ is diffuse w.r.t $c_{s,p}$ if for any E,

$$c_{s,p}(E) = 0 \Longrightarrow |\mu|(E) = 0.$$

A measure μ s.t. $|\mu| \in H^{-s,p'}(\mathbb{R}^d)$ is $c_{s,p}$-diffuse.
Indeed if $u \geq 1_K$ then

$$|\mu|(K) \leq \int u\, d|\mu| = (|\mu|, u) \leq \||\mu|\|_{-s,p'} \|u\|_{s,p}.$$

Taking the inf on u gives

$$|\mu|(K) \leq \||\mu|\|_{-s,p'} c_{s,p}(K)^{1/p}.$$

A measure μ is diffuse w.r.t $c_{s,p}$ if for any E,

$$c_{s,p}(E) = 0 \Longrightarrow |\mu|(E) = 0.$$

A measure μ s.t. $|\mu| \in H^{-s,p'}(\mathbb{R}^d)$ is $c_{s,p}$-diffuse.
Indeed if $u \geq 1_K$ then

$$|\mu|(K) \leq \int u\, d|\mu| = (|\mu|, u) \leq |||\mu|||_{-s,p'} \|u\|_{s,p}.$$

Taking the inf on u gives

$$|\mu|(K) \leq |||\mu|||_{-s,p'} c_{s,p}(K)^{1/p}.$$

Feyel - De La Pradel (1977) showed that any $\mu \in \mathfrak{M}_b^+(\Omega)$ $c_{s,p}$-diffuse is limit of a nondecreasing sequence $\mu_n \in \mathfrak{M}_b^+(\Omega) \cap H^{-s,p'}(\Omega)$.

Baras and Pierre result

Recall that for any q the equation

$$\begin{aligned} -\Delta u + |u|^{q-1}u &= \mu \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

has a solution if $|\mu| \in H^{-2,q}$.

Baras and Pierre result

Recall that for any q the equation

$$\begin{aligned} -\Delta u + |u|^{q-1}u &= \mu \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

has a solution if $|\mu| \in H^{-2,q}$.

Baras and Pierre proved in 1984 that

(1) has a solution $\Longleftrightarrow$ μ is $c_{2,q'}$-diffuse.

Equation with measure-valued potential.

Position of the problem

We consider the equation

$$\begin{aligned} -\Delta u + |u|^{q-1} u \sigma &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

Position of the problem

We consider the equation

$$\begin{aligned} -\Delta u + |u|^{q-1} u \sigma &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

2 questions:

1) is there a critical exponent q_c such that

- if $q < q_c$ there is a solution for any μ
- if $q \geq q_c$, not all μ are good

How does q_c depend on σ ?

Position of the problem

We consider the equation

$$\begin{aligned} -\Delta u + |u|^{q-1}u\sigma &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

2 questions:

1) is there a critical exponent q_c such that

- if $q < q_c$ there is a solution for any μ
- if $q \geq q_c$, not all μ are good

How does q_c depend on σ ?

2) If $q \geq q_c$, is there a sufficient existence condition like μ being diffuse w.r.t. some capacity (depending on σ) ?

* Veron-Yarur (2012) studied the linear problem

$$\begin{aligned} -\Delta u + Vu &= 0 && \text{in } \Omega \\ u &= \mu && \text{in } \partial\Omega \end{aligned}$$

where $\mu \in \mathfrak{M}_b(\partial\Omega)$ and $0 \leq V \in L^\infty_{loc}(\Omega)$.

* Malusa y Orsina (1996) considered the linear problem

$$\begin{aligned} -\Delta u + u\sigma &= \mu && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega \end{aligned}$$

where $\mu \in \mathfrak{M}_b(\partial\Omega)$ and $\nu \in \mathfrak{M}(\partial\Omega)$ vanishes on polar sets. They proved existence, regularity and existence of a Green function.

* Triebel (book "Fractal and spectra") proved existence and uniqueness in some (sharp) Besov space for

$$\begin{aligned} -\Delta u + |u|^q\sigma &= \mu && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega \end{aligned}$$

where $0 < q < 1$, σ is d-regular for some $N-2 < d < N$ and $\mu \in L^p_\sigma(\Omega)$ for some $p > 1$.

Notion of solution

$u \in L^1(\Omega)$ is a very weak solution of

$$\begin{aligned} -\Delta u + g(u)\sigma &= \mu \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

if $\rho g(u) \in L^1_\sigma(\Omega)$, where $\rho(x) = dist(x, \partial\Omega)$, and for any $\zeta \in C^1_c(\overline{\Omega})$ such that $\Delta\zeta \in L^\infty$, there holds

$$-\int_\Omega u\Delta\zeta\,dx + \int_\Omega g(u)\zeta\,d\sigma = \int_\Omega \zeta\,d\mu.$$

Assumptions on σ

- If u is a solution then we expect $-\Delta u$ to be a bounded measure so that $u \in W^{1,q}(\Omega, q < N/(N-1)$.

Assumptions on σ

- If u is a solution then we expect $-\Delta u$ to be a bounded measure so that $u \in W^{1,q}(\Omega,\ q < N/(N-1)$.
- In general a function $v \in H^{s,p}$ has a precise representative which is defined (and coincide with u) $c_{s,p}$-q.e., i.e. everywhere but for a set E with $c_{s,p}(E) = 0$.

Assumptions on σ

- If u is a solution then we expect $-\Delta u$ to be a bounded measure so that $u \in W^{1,q}(\Omega, q < N/(N-1)$.
- In general a function $v \in H^{s,p}$ has a precise representative which is defined (and coincide with u) $c_{s,p}$-q.e., i.e. everywhere but for a set E with $c_{s,p}(E) = 0$.
- Then for $\int g(u)\zeta \, d\sigma$ to make sense we need $\sigma(E) = 0$
  $\Longrightarrow \sigma$ to be $c_{1,q}$-diffuse.

Assumptions on σ

- If u is a solution then we expect $-\Delta u$ to be a bounded measure so that $u \in W^{1,q}(\Omega$, $q < N/(N-1)$.
- In general a function $v \in H^{s,p}$ has a precise representative which is defined (and coincide with u) $c_{s,p}$-q.e., i.e. everywhere but for a set E with $c_{s,p}(E) = 0$.
- Then for $\int g(u)\zeta \, d\sigma$ to make sense we need $\sigma(E) = 0$ $\implies \sigma$ to be $c_{1,q}$-diffuse.
- $dim(E) \leq N - q$, so it would be OK if σ were not more concentrated than $\mathcal{H}^d$ with $d > N - q$. It thus seems to be a good idea to assume

$$\sigma(B_x(r)) \leq cr^d \qquad \text{for some } d > N - q.$$

$\implies$ Morrey space of measures.

Morrey space of measures

A measure μ belongs to the *Morrey space* $\mathcal{M}_p(\Omega)$, $1 \leq p \leq \infty$, if

$$|\mu|(B_x(r)) \leq C r^{N(1-1/p)} \qquad \text{for any } x \in \mathbb{R}^N,\ r > 0.$$

Morrey space of measures

A measure μ belongs to the *Morrey space* $\mathcal{M}_p(\Omega)$, $1 \leq p \leq \infty$, if

$$|\mu|(B_x(r)) \leq Cr^{N(1-1/p)} \qquad \text{for any } x \in \mathbb{R}^N,\ r > 0.$$

This definition is motivated by the case $\mu = f\,dx$ with $f \in L^p(\mathbb{R}^N)$:

$$|\mu|(B_x(r)) = \int_{B_x(r)} |f| \leq \|f\|_p |B_x(r)|^{1-1/p} \leq Cr^{n(1-1/p)}.$$

A measure μ belongs to the *Morrey space* $\mathcal{M}_p(\Omega)$, $1 \le p \le \infty$, if

$$|\mu|(B_x(r)) \le Cr^{N(1-1/p)} \qquad \text{for any } x \in \mathbb{R}^N,\ r > 0.$$

This definition is motivated by the case $\mu = f\,dx$ with $f \in L^p(\mathbb{R}^N)$:

$$|\mu|(B_x(r)) = \int_{B_x(r)} |f| \le \|f\|_p |B_x(r)|^{1-1/p} \le Cr^{n(1-1/p)}.$$

Note that $\mu \in \mathcal{M}_{\frac{N}{N-\theta}}(\Omega)$ if $|\mu|(B_x(r)) \le Cr^{\theta}$.

What's good about Morrey space (1)

Miyakawa (1990) studied these spaces. He showed in particular that

1. $\mathcal{M}_p$ is a Banach space for the norm

$$\|\sigma\|_p := \sup_{x,r} r^{-N(1-1/p)}|\sigma|(B_x(r)).$$

2. $\mathcal{M}_q(\Omega) \subset \mathcal{M}_p(\Omega)$ if $p \leq q$,
3. if $\mu \in \mathcal{M}_p$ with $p > N/2$ then $\mathbb{G}\mu \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ with

$$\|\mathbb{G}\mu\|_{C^{0,\alpha}} \leq c\|\mu\|_p.$$

Recall that $\mathbb{G}\mu(x) = \int_\Omega \mathbb{G}(x,y)d\mu(y)$ is the solution of

$$\begin{aligned} -\Delta u &= \mu \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

What's good about Morrey space (2)

We have result for the embedding

$$H^{s,p}(\Omega) \hookrightarrow L^q_\sigma(\Omega)$$

when σ belongs to some Morrey space (see Adams-Hedberg's book "Nonlinear potential theory")

What's good about Morrey space (2)

We have result for the embedding

$$H^{s,p}(\Omega) \hookrightarrow L^q_\sigma(\Omega)$$

when σ belongs to some Morrey space (see Adams-Hedberg's book "Nonlinear potential theory")

Let $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}(\Omega)$ i.e. $|\sigma|(B_x(r)) \leq cr^\theta$, $N > sp$, $1 < p < q < \frac{Np}{N-sp}$.

1)

$$H^{s,p}(\Omega) \hookrightarrow L^q_\sigma(\Omega) \qquad \Longleftrightarrow \qquad q \leq \frac{\theta p}{N - sp}$$

2) the embedding is compact iff $q < \frac{\theta p}{N-sp}$.

3) σ is $c_{s,p}$-diffuse.

Back to the problem

We consider the equation

$$\begin{aligned} -\Delta u + |u|^{q-1} u\sigma &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

We will assume that

- $\sigma \in M_{\frac{N}{N-\theta}}$ for some $N \geq \theta \geq N-2$ (i.e. $|\sigma|(B_x(r)) \leq cr^{\theta}$)
- g is continuous and absorbing:

$$g(r)r \geq 0 \qquad |r| \geq r_0$$

  for some r_0.

When needed we will shrink the range of θ and strenghten the assumptions on g.

Theorem

For any $\mu \in L^1_\rho(\Omega)$, there exists a solution u. If moreover g is nondecreasing, then if u' is a solution with right-hand side $\mu' \in L^1_\rho(\Omega)$,

$$-\int_\Omega |u-u'|\Delta\zeta dx + \int_\Omega |g(u)-g(u')|\zeta d\sigma \leq \int_\Omega |\mu-\mu'|dx,$$

and

$$-\int_\Omega (u-u')_+\Delta\zeta dx + \int_\Omega (g(u)-g(u'))_+\zeta d\sigma \leq \int_\Omega (\mu-\mu')_+dx$$

for all $\zeta \in W^{1,\infty}_0(\Omega)$ such that $\Delta\zeta \in L^\infty(\Omega)$ and $\zeta \geq 0$.

Theorem

For any $\mu \in L^1_\rho(\Omega)$, there exists a solution u. If moreover g is nondecreasing, then if u' is a solution with right-hand side $\mu' \in L^1_\rho(\Omega)$,

$$-\int_\Omega |u-u'|\Delta\zeta dx + \int_\Omega |g(u)-g(u')|\zeta d\sigma \le \int_\Omega |\mu-\mu'|dx,$$

and

$$-\int_\Omega (u-u')_+\Delta\zeta dx + \int_\Omega (g(u)-g(u'))_+\zeta d\sigma \le \int_\Omega (\mu-\mu')_+dx$$

for all $\zeta \in W_0^{1,\infty}(\Omega)$ such that $\Delta\zeta \in L^\infty(\Omega)$ and $\zeta \ge 0$.

Taking $\zeta = \mathbb{G}[1]$, we get uniqueness of the solution that we denote by u_μ, and that $\mu \mapsto u_\mu$ is non-decreasing.

Theorem

Let $N \geq 3$ and $\sigma \in \mathcal{M}^{+}_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N - \frac{N}{N-1}$. Assume moreover that g satisfies $|g(r)| \leq \tilde{g}(|r|)$ for $|r| \geq r_0$ where $\tilde{g}$ is continuous nondecreasing and

$$\int_{r_0}^{\infty} \tilde{g}(t) t^{-1-\frac{\theta}{N-2}} dt < \infty.$$

Then, for any bounded measure μ, there exists a solution u This solution is unique if moreover g is nondecreasing.

Theorem

Let $N \geq 3$ *and* $\sigma \in \mathcal{M}^{+}_{\frac{N}{N-\theta}}(\Omega)$ *with* $N \geq \theta > N - \frac{N}{N-1}$*. Assume moreover that* g *satisfies* $|g(r)| \leq \tilde{g}(|r|)$ *for* $|r| \geq r_0$ *where* $\tilde{g}$ *is continuous nondecreasing and*

$$\int_{r_0}^{\infty} \tilde{g}(t) t^{-1-\frac{\theta}{N-2}} dt < \infty.$$

Then, for any bounded measure μ*, there exists a solution* u *This solution is unique if moreover* g *is nondecreasing.*

Remark that

1) We recover Benilan-Brezis result when $\sigma = dx$.

(unconditional) Existence for measure data

Theorem

Let $N \geq 3$ and $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N - \frac{N}{N-1}$. Assume moreover that g satisfies $|g(r)| \leq \tilde{g}(|r|)$ for $|r| \geq r_0$ where $\tilde{g}$ is continuous nondecreasing and

$$\int_{r_0}^{\infty} \tilde{g}(t) t^{-1-\frac{\theta}{N-2}} dt < \infty.$$

Then, for any bounded measure μ, there exists a solution u This solution is unique if moreover g is nondecreasing.

Remark that

1) We recover Benilan-Brezis result when $\sigma = dx$.

2) when $g(r) = |r|^{q-1} r$ with $0 < q < \frac{\theta}{N-2}$, we obtain existence and uniqueness for any measure μ.

Theorem

Let $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ *with* $N \geq \theta > N - \frac{N}{N-1}$. *Assume that* g *is nondecreasing and vanishes at* 0.

If the bounded measure satisfies

$$\rho g(\mathbb{G}[|\mu|]) \in L^1_\sigma(\Omega),$$

then there exists a unique very weak solution.

Existence for measure data (2)

Theorem

Let $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ *with* $N \geq \theta > N - \frac{N}{N-1}$*. Assume that* g *is nondecreasing and vanishes at* 0*.*

If the bounded measure satisfies

$$\rho g(\mathbb{G}[|\mu|]) \in L^1_\sigma(\Omega),$$

then there exists a unique very weak solution.

If

- either μ and σ have disjoint support
- or $\mu \in \mathcal{M}_p(\Omega)$ with $p > N/2$

then $\mathbb{G}[\mu]$ is bounded on the support of σ so that $\mathbb{G}[\mu] \in L^1_\sigma$ and there exists a solution.

Consequence of the 2nd existence condition

Theorem

Let $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N - \frac{N}{N-1}$ and assume that g is continuous nondecreasig, $g(0) = 0$ and satisfies $|g(r)| \leq c(1 + |r|^q)$ (e.g. $g(r) = |r|^{q-1}r$)

Let $p > 1$ and $s \geq 0$ such that $\frac{\theta p}{N - sp} \geq q$.

If μ is diffuse w.r.t $c_{2-s,p'}$-capacity then there exists a unique solution.

Consequence of the 2nd existence condition

Theorem

Let $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N - \frac{N}{N-1}$ and assume that g is continuous nondecreasig, $g(0) = 0$ and satisfies $|g(r)| \leq c(1 + |r|^q)$ (e.g. $g(r) = |r|^{q-1}r$)

Let $p > 1$ and $s \geq 0$ such that $\frac{\theta p}{N - sp} \geq q$.

If μ is diffuse w.r.t $c_{2-s,p'}$-capacity then there exists a unique solution.

Consequence: take $p = q$, then

if μ is diffuse w.r.t $c_{2-\frac{N-\theta}{q},q'}$-capacity,
then there exists a unique solution.

We recover Baras-Pierre's sufficient condition when $\theta = N$.

An explicit example of admissible measure μ.

Proposition

Under the previous assumptions on σ and g, with $q \geq \frac{\theta}{N-2}$, if $\mu \in \mathcal{M}_{\frac{N}{N-\tilde{\theta}}}(\Omega)$ for some $\tilde{\theta} > \frac{(N-2)q-\theta}{q-1}$, then there exists a unique solution.

An explicit example of admissible measure μ.

Proposition

Under the previous assumptions on σ and g, with $q \geq \frac{\theta}{N-2}$, if $\mu \in \mathcal{M}_{\frac{N}{N-\tilde{\theta}}}(\Omega)$ for some $\tilde{\theta} > \frac{(N-2)q-\theta}{q-1}$, then there exists a unique solution.

It suffices to check that μ is diffuse w.r.t $c_{2-\frac{N-\theta}{q},q'}$.

In general, if $\mu \in \mathcal{M}_{\frac{N}{N-\tilde{\theta}}}(\Omega)$ with $\tilde{\theta} > N - sp$ then

$$|\mu|(K) \leq cste.c_{(s,p)}(K)^{\frac{1}{p}} \qquad K \text{ compact.}$$

Indeed $H^{s,p}(\Omega) \hookrightarrow L^1_{|\mu|}(\Omega)$. Thus for any $v \in H^{s,p}(\Omega)$, $v \geq 1$ on K,

$$|\mu|(K) \leq \int_K v d|\mu| \leq \|v\|_{L^1_{|\mu|}} \leq C\|v\|_{H^{s,p}}.$$

Some properties of good measures

Let $\sigma \in \mathcal{M}^+_{\frac{N}{N-\theta}}(\Omega)$ with $N \geq \theta > N - \frac{N}{N-1}$ and g be continuous nondecreasing with $g(0) = 0$.

(1) If $\{\mu_n\}_n \subset \mathfrak{M}_b^+(\Omega)$ is a non-decreasing sequence of good measures converging to some $\mu \in \mathfrak{M}_b^+(\Omega)$. Then μ is a good measure.

(2) If $\mu \in \mathfrak{M}_b^+(\Omega)$ is a good measure, any $\nu \in \mathfrak{M}_b^+(\Omega)$ such that $\nu \leq \mu$ is a good measure.

(3) Let $\mu, \mu' \in \mathfrak{M}_b^+(\Omega)$. If μ and $-\mu'$ are good measures, any $\nu \in \mathfrak{M}_b(\Omega)$ such that $-\mu' \leq \nu \leq \mu$ is a good measure.

Sketch of the proof of the previous Theorem

Let μ be diffuse w.r.t $c_{2-s,p'}$-capacity.

$$\Longrightarrow \mu^{\pm} \text{ are diffuse w.r.t } c_{2-s,p'}\text{-capacity.}$$

Sketch of the proof of the previous Theorem

Let μ be diffuse w.r.t $c_{2-s,p'}$-capacity.

$$\Longrightarrow \mu^{\pm} \text{ are diffuse w.r.t } c_{2-s,p'}\text{-capacity.}$$

Feyel - de La Pradel thm gives $\{\mu_n^{\pm}\}_n \subset \mathfrak{M}_b^+(\Omega)$ such that

$$\mu_n^{\pm} \in H^{s-2,p}(\Omega) \qquad \text{and} \qquad \mu_n^{\pm} \uparrow \mu^{\pm}.$$

Sketch of the proof of the previous Theorem

Let μ be diffuse w.r.t $c_{2-s,p'}$-capacity.

$$\Longrightarrow \mu^{\pm} \text{ are diffuse w.r.t } c_{2-s,p'}\text{-capacity.}$$

Feyel - de La Pradel thm gives $\{\mu_n^{\pm}\}_n \subset \mathfrak{M}_b^+(\Omega)$ such that

$$\mu_n^{\pm} \in H^{s-2,p}(\Omega) \qquad \text{and} \qquad \mu_n^{\pm} \uparrow \mu^{\pm}.$$

Then

$$\begin{aligned} \mu_n^{\pm} \in H^{s-2,p}(\Omega) \quad &\Longrightarrow \quad \mathbb{G}\mu_n^{\pm} \in H^{s,p}(\Omega) \hookrightarrow L_\sigma^{\frac{\theta p}{N-sp}}(\Omega) \hookrightarrow L_\sigma^q(\Omega) \\ &\Longrightarrow \quad \mu_n^{\pm} \text{ is good.} \end{aligned}$$

Sketch of the proof of the previous Theorem

Let μ be diffuse w.r.t $c_{2-s,p'}$-capacity.

$$\Longrightarrow \mu^{\pm} \text{ are diffuse w.r.t } c_{2-s,p'}\text{-capacity.}$$

Feyel - de La Pradel thm gives $\{\mu_n^{\pm}\}_n \subset \mathfrak{M}_b^+(\Omega)$ such that

$$\mu_n^{\pm} \in H^{s-2,p}(\Omega) \qquad \text{and} \qquad \mu_n^{\pm} \uparrow \mu^{\pm}.$$

Then

$$\begin{aligned} \mu_n^{\pm} \in H^{s-2,p}(\Omega) \quad &\Longrightarrow \quad \mathbb{G}\mu_n^{\pm} \in H^{s,p}(\Omega) \hookrightarrow L_{\sigma}^{\frac{\theta p}{N-sp}}(\Omega) \hookrightarrow L_{\sigma}^{q}(\Omega) \\ &\Longrightarrow \quad \mu_n^{\pm} \text{ is good.} \end{aligned}$$

From the previous results

$$\mu^{\pm} \text{ is good.}$$

Since $-\mu^- \leq \mu \leq \mu^+$, we deduce that μ is good.

Conclusions

We obtained sufficient existence conditions for

$$\begin{aligned} -\Delta u + g(u)\sigma &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

generalizing those known when $\sigma = dx$ under the main assumptions that g is absorbing and

$$|\sigma|(B_x(r)) \leq cr^{\theta} \qquad \theta > N - \frac{N}{N-1}.$$

In particular if $g(r) = |r|^{q-1}r$,

(1) if $q < \frac{\theta}{N-2}$ there is a solution for any μ

(2) if $q \geq \frac{\theta}{N-2}$, there exists a solution for any $c_{2-\frac{N-\theta}{q},q'}$-diffuse μ.

Muchas gracias !

Current work:

find a necessary existence condition on μ.

Towards a sufficient condition of existence

Assume that

$$g(r) = |r|^{q-1}r$$

Recall the 2nd existence condition:

if $\int_\Omega \mathbb{G}[|\mu|]^q \, d\sigma < \infty$ then there exists a solution.

Towards a sufficient condition of existence

Assume that

$$g(r) = |r|^{q-1} r$$

Recall the 2nd existence condition:

$$\text{if } \int_\Omega \mathbb{G}[|\mu|]^q \, d\sigma < \infty \text{ then there exists a solution.}$$

It is then "natural" to consider the capacity

$$c_q^\sigma(K) = \inf \left\{ \int_\Omega |v|^{q'} \, d\sigma : v \in L_\sigma^{q'}(\Omega),\ v \geq 0,\ \mathbb{G}[v\sigma] \geq 1 \text{ on } K \right\}.$$

Towards a sufficient condition of existence

σ is θ-regular if

$$\frac{1}{c}r^\theta \leq \sigma(B_r(x)) \leq cr^\theta \qquad x \in \mathrm{supp}(\sigma),\ 0 < r < 1.$$

in other words σ is θ-regular iff it is equivalent to $\mathcal{H}^\theta_{|\mathrm{supp}(\sigma)}$.

Towards a sufficient condition of existence

σ is θ-regular if

$$\frac{1}{c} r^{\theta} \leq \sigma(B_r(x)) \leq c r^{\theta} \qquad x \in \mathrm{supp}(\sigma),\ 0 < r < 1.$$

in other words σ is θ-regular iff it is equivalent to $\mathcal{H}^{\theta}_{|\mathrm{supp}(\sigma)}$.

Theorem

Let $q > 1$ and $\sigma \in \mathcal{M}_{\frac{N}{N-\theta}}$ be θ-regular with $N \geq \theta > N-2$. If the bounded measure μ is such that there exists a solution then μ is diffuse wr.t. c_q^{σ} capacity.

Moreover

$$c_q^{\sigma}(K) \sim c_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(K) := \inf\left\{ \|\zeta\|^{q'}_{B_{q',\infty}^{2-\frac{N-\theta}{q}}} : \zeta \in B_{q',\infty}^{2-\frac{N-\theta}{q},\Gamma}(\Omega),\ \zeta \geq 1_K \right\},$$