SOLUCIONES MULTIPICOS PARA LA ECUACIÓN DE YAMABE EN UNA VARIEDAD PRODUCTO

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The Yamabe's Problem

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The Yamabe problem lies in finding for any closed Riemannian manifold (M, g) of dimension $n \geq 3$ a conformal metric $\tilde{g}$ of constant scalar curvature.

Recall that two metrics $\tilde{g}$ and g are said to be conformal if $\tilde{g} = u\, g$ for some smooth positive function u.
The Yamabe problem can be reduced to the solvability of a certain semilinear elliptic equation.

The Yamabe equation

Let $\tilde{g} \in [g]$. Let S_g and $S_{\tilde{g}}$ denote the scalar curvatures of (M, g) and $(M, \tilde{g})$ respectively. The relation between S_g and $S_{\tilde{g}}$ is simplified if we put $\tilde{g} = u^{p-2} g$ with $p = \frac{2n}{n-2}$:

$$S_{\tilde{g}} = u^{-(p-1)} \left(S_g \, u - 4 \, \frac{n-1}{n-2} \, \Delta_g u \right)$$

where Δ_g is the Laplace-Beltrami operator of the metric g. Hence $\tilde{g}$ has constant scalar curvature λ if and only if u satisfies the *Yamabe equation*:

$$-\, a \, \Delta_g u + S_g \, u = \lambda \, u^{p-1} \tag{1}$$

where $a = a_n = 4\frac{n-1}{n-2}$.

In fact, the way to prove that the equation

$$-a\,\Delta_g u + S_g\, u = \lambda\, u^q$$

has a solution depends strongly on q.
When $q = 1$, the equation is just the linear eigenvalue problem for $-a\Delta_g + S_g$.
When q is close to 1, its behavior is similar to that of the eigenvalue problem.
When q is very large however, linear theory is no longer useful.
The exponent in the Yamabe equation is the critical value below which the equation can be solved by classical methods and above which it may be unsolvable.

Yamabe constant

Yamabe equation is the Euler-Lagrange equation for the Hilbert-Einstein functional restricted to $[g]$:

$$Q(\tilde{g}) = \frac{\int_M S_{\tilde{g}} dV_{\tilde{g}}}{(\int_M dV_{\tilde{g}})^{2/p}}.$$

The Yamabe constant of $(M, [g])$:

$$Y(M, [g]) = \inf \left\{ Q(\tilde{g}) : \tilde{g} \in [g] \right\}$$

is always achieved H. Yamabe- N. Trudinger- T. Aubin- R. Schoen. There is always at least one (volume 1) solution of the Yamabe equation.

1. Solution is unique if $Y(M,[g]) \leq 0$.
2. In general multiple solutions when $Y(M,[g]) > 0$

Examples of nonuniqueness:

- $(\mathbb{S}^n,[g_0])$.
- Riemannian products with constant scalar curvature $(M \times N, [g + \delta h])$, with $\delta > 0$ small cannot be a minimizer.

Multiplicity for products

Let (M^n, g) be any closed manifold and (N^m, h) a manifold of constant positive scalar curvature s_h. We will be interested in positive solutions of the Yamabe equation for the product manifold $(M \times N, g + \epsilon^2 h)$:

$$-a(\Delta_g + \Delta_{\epsilon^2 h})u + (s_g + \epsilon^{-2} s_h)u = u^{p-1}, \tag{2}$$

with $a = a_{m+n} = \frac{4(m+n-1)}{m+n-2}$, $p = p_{m+n} = \frac{2(m+n)}{m+n-2}$, s_g the scalar curvature of (M^n, g), and ϵ small enough so that the scalar curvature $s_g + \epsilon^{-2} s_h$ is positive. The conformal metric $u^{p-2}(g + \epsilon^2 h)$ then has constant scalar curvature.

We restrict our study to functions that depend only on the first factor, $u : M \to \mathbb{R}$. We normalize h so that $s_h = a_n$. Then u solves the Yamabe equation if and only if (after renormalizing)

$$-\epsilon^2 \Delta_g u + \left(\lambda s_g \epsilon^2 + 1\right) u = u^{p-1} \tag{3}$$

with $\lambda = a_n^{-1}$. Note that $p = p_{m+n} < p_n$. So the problem becomes a subcritical problem on M.

Positive solutions of this equation are the critical points of the functional $J_\epsilon : H^{1,2}(M) \to \mathbb{R}$, given by

$$J_\epsilon(u) = \epsilon^{-n} \int_M \left(\frac{1}{2} \epsilon^2 |\nabla u|^2 + \frac{1}{2} \left(\epsilon^2 \lambda s_g + 1 \right) u^2 - \frac{1}{p} (u^+)^p \right) dV_g,$$

where $u^+(x) = \max\{u(x), 0\}$.

We will build solutions of (3) by using the Lyapunov-Schmidt reduction procedure which was applied by several authors. For example:

- E. N. Dancer, A. M. Michelett,i A. Pistoia, *Multipeak solutions for some singularly perturbed nonlinear elliptic problems on Riemannian manifolds*, Manuscripta Math. 128 (2009), 163-193.
- A. M. Micheletti, A. Pistoia, *The role of the scalar curvature in a nonlinear elliptic problem on Riemannian manifolds*, Calc. Var. 34 (2009), 233-265.

In these articles the procedure is used to build solutions of a similar elliptic equation under certain conditions on the scalar curvature.

One first considers what will be called the limit equation in $\mathbb{R}^n$. Recall that for $2 < q < \frac{2n}{n-2}$, $n > 2$, the equation

$$-\Delta U + U = U^{q-1} \text{ in } \mathbb{R}^n \tag{4}$$

has a unique (up to translations) positive solution $U \in H^1(\mathbb{R}^n)$ that vanishes at infinity. It is known that

- U is radial and exponentially decreasing at infinity,
- U' is exponentially decreasing at infinity.
- And the function $U_\epsilon(x) = U(\frac{x}{\epsilon})$, is a solution of
$$-\epsilon^2 \Delta U_\epsilon + U_\epsilon = U_\epsilon^{q-1}.$$

Approximate solutions

Positive solutions of (4) are the critical points of the functional $E : H^1(\mathbb{R}^n) \to \mathbb{R}$,

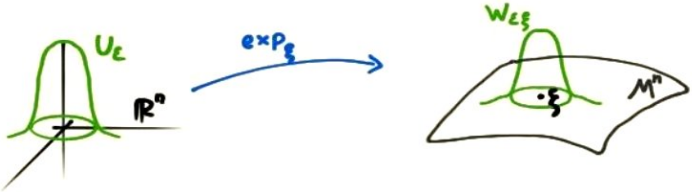
$$E(f) = \int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla f|^2 + \frac{1}{2} f^2 - \frac{1}{p} (f^+)^p \right) dx.$$

Let $S_0 = \nabla E : H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$.
Eq. (4) is equivalent to $S_0(U) = 0$ and the solution U is non-degenerate in the sense that Kernel$(S_0'(U))$ is spanned by

$$\psi^i(x) := \frac{\partial U}{\partial x_i}(x)$$

with $i = 1, \ldots, n$.

For any $x \in M$ consider the exponential map $\exp_x : T_x M \to M$.
Since M is closed we can fix $r_0 > 0$ such that
$\exp_x \big|_{B(0,r_0)} : B(0, r_0) \to B_g(x, r_0)$ is a diffeomorphism.
Here $B(0, r)$ is the ball in $\mathbb{R}^n$ centered at 0 with radius r and
$B_g(x, r)$ is the geodesic ball in M centered at x with radius r.
Let χ_r be a smooth radial cut-off function.



Let us define on M the functions

$$Z^i_{\epsilon,\xi}(x) := \begin{cases} \psi^i_\epsilon(\exp_\xi^{-1}(x))\chi_r(\exp_\xi^{-1}(x)) & \text{if} \quad x \in B_g(\xi, r) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$W_{\epsilon,\xi}(x) = \begin{cases} U_\epsilon(\exp_\xi^{-1}(x))\chi_r(\exp_\xi^{-1}(x)) & \textit{if} \quad x \in B_g(\xi, r) \\ 0 & \text{otherwise.} \end{cases}$$

One considers $W_{\epsilon,\xi}$ as an approximate solution to equation (3) which concentrates around ξ.

Let $k_0 \geq 0$ be a fixed integer and denote $\xi = (\xi_1, \dots, \xi_{k_0}) \in M^{k_0}$. Then

$$V_{\epsilon,\tilde{\xi}}(x) := \sum_{i=1}^{k_0} W_{\epsilon,\xi_i^\epsilon}$$

is our approximate (k_0-peaks) solution.
We will find exact solutions by perturbing these approximate solutions.
Let β be the constant:

$$\beta := \lambda \int_{\mathbb{R}^n} U^2(z)\, dz - \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\nabla U(z)|^2 |z|^2\, dz. \qquad (5)$$

Theorem

Assume that $\beta_\lambda < 0$ then let ξ_0 be an isolated local maximum point of the scalar curvature S_g. For each positive integer k_0, there exists $\epsilon_0 = \epsilon_0(k_0) > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ there exist points $\xi_1^\epsilon, \ldots, \xi_{k_0}^\epsilon \in M$ such that

$$\frac{d_g(\xi_i^\epsilon, \xi_j^\epsilon)}{\epsilon} \to +\infty \quad \text{and} \quad d_g(\xi_0, \xi_j^\epsilon) \to 0. \tag{6}$$

and a solution u_ϵ of problem (3) such that

$$\|u_\epsilon - \sum_{i=1}^{k_0} W_{\epsilon,\xi_i^\epsilon}\|_\epsilon \to 0,$$

- We have not been able to find an analytical proof that $\beta \neq 0$ but we give the numerical computation of β for low values of m and n. In all cases $\beta < 0$.
- It has been proved by A. M. Micheletti and A. Pistoia [**?**] that for a generic Riemannian metric the critical points of the scalar curvature are non-degenerate and in particular isolated.
- Then, by the Theorem we show the multiplicity of metrics of constant scalar curvature in the product manifold $(M \times N, g + \epsilon^2 h)$.

The reduction of the equation

For $\epsilon > 0$ and $\overline{\xi} = (\xi_1, ..., \xi_{k_0}) \in M^{k_0}$ let

$$K_{\epsilon,\overline{\xi}} := \text{ span } \{Z^i_{\epsilon,\xi_j} : i = 1, \dots, n, j = 1, \dots, k_0\}$$

and

$$K^{\perp}_{\epsilon,\overline{\xi}} := \{\phi \in H_\epsilon : \langle \phi, Z^i_{\epsilon,\xi_j} \rangle_\epsilon = 0, i = 1, \dots, n, j = 1 \dots, k_0\}.$$

Let $\Pi_{\epsilon,\overline{\xi}} : H_\epsilon \to K_{\epsilon,\overline{\xi}}$ and $\Pi^{\perp}_{\epsilon\overline{\xi}}: H_\epsilon \to K^{\perp}_{\epsilon,\overline{\xi}}$ be the orthogonal projections. In order to solve equation (3) we call

$$S_\epsilon = \nabla J_\epsilon : H_\epsilon \to H_\epsilon.$$

Equation (3) is then $S_\epsilon(u) = 0$.
The idea is that the kernel of $S'_\epsilon(V_{\epsilon,\bar{\xi}})$ should be close to $K_{\epsilon,\bar{\xi}}$ and then the linear map $\phi \mapsto \Pi^{\perp}_{\epsilon\bar{\xi}} S'_\epsilon(V_{\epsilon,\bar{\xi}})(\phi) : K^{\perp} \to K^{\perp}$ should be invertible.
Then the inverse function theorem would imply that there is a unique small $\phi = \phi_{\epsilon,\bar{\xi}} \in K^{\perp}_{\epsilon,\bar{\xi}}$ such that

$$\Pi^{\perp}_{\epsilon,\bar{\xi}}\{S_\epsilon(V_{\epsilon,\bar{\xi}} + \phi)\} = 0. \tag{7}$$

And then we have to solve the finite dimensional problem

$$\Pi_{\epsilon,\bar{\xi}}\{S_\epsilon(V_{\epsilon,\bar{\xi}} + \phi)\} = 0. \tag{8}$$

The finite dimensional equation

Let $\overline{J_\epsilon} : M^{k_0} \to \mathbb{R}$ be defined by

$$\overline{J_\epsilon}(\overline{\xi}) := J_\epsilon(V_{\epsilon,\overline{\xi}} + \phi_{\epsilon,\overline{\xi}}) \, .$$

(8) is equivalent to finding critical points of $\overline{J_\epsilon}$, ie, if $\overline{\xi}_\epsilon$ is a critical point of $\overline{J_\epsilon}$, then the function $V_{\epsilon,\overline{\xi}_\epsilon} + \phi_{\epsilon,\overline{\xi}_\epsilon}$ is a solution to problem (3).

Let $\xi_0 \in M$ be an isolated local minimum point of the scalar curvature. We find critical points of $\overline{J}$ in the open set

$$D^{k_0}_{\epsilon,\rho} := \Big\{ \overline{\xi} \in M^{k_0} \; / \; d_g(\xi_0, \xi_i) < \rho, i = 1, \dots, k_0, \\ \sum_{i \neq j}^{k_0} U_\epsilon \left(\exp^{-1}_{\xi_i} \xi_j \right) < \epsilon^2 \Big\}. \tag{9}$$

The infinite dimensional reduction

We rewrite eq. (7) as

$$-R_{\epsilon,\bar{\xi}} + L_{\epsilon,\bar{\xi}}(\phi) - N_{\epsilon,\bar{\xi}}(\phi) = 0$$

with the first term being independent of ϕ, the second term, the linear operator:

$$L_{\epsilon,\bar{\xi}}(\phi) = \Pi^{\perp}_{\epsilon,\bar{\xi}}\{S'_{\epsilon}(V_{\epsilon,\bar{\xi}})\ \phi\}$$

and the last term a remainder.
eq. (7) can be written as

$$L_{\epsilon,\bar{\xi}}(\phi) = N_{\epsilon,\bar{\xi}}(\phi) + R_{\epsilon,\bar{\xi}}.$$

If L is invertible, we turn eq. (7) into a fixed point problem, for the operator

$$T_{\epsilon,\bar{\xi}}(\phi) := L^{-1}_{\epsilon,\bar{\xi}}(N_{\epsilon,\bar{\xi}}(\phi) + R_{\epsilon,\bar{\xi}})$$

.

We proved that $T_{\epsilon,\bar{\xi}}$ has a fixed point in a small enough ball in $K^{\perp}_{\epsilon,\bar{\xi}}$, centered at 0. And for such fixed point, we have

$$||\phi_{\epsilon,\bar{\xi}}||_{\epsilon} = \|T_{\epsilon,\bar{\xi}}(\phi)\|_{\epsilon} \leq c\Big(\epsilon^2 + \sum_{i\neq j} e^{-\frac{1+\sigma}{2}\frac{d_g(\xi_i,\xi_j)}{\epsilon}}\Big).$$

Finally we have

$$\|u_{\epsilon} - \sum_{i=1}^{k_0} W_{\epsilon,\xi_i^{\epsilon}}\|_{\epsilon} \to 0,$$

Gracias!