Some inequalities in the interface between harmonic analysis and differential equations

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Seminario de Ecuaciones Diferenciales y Análisis Numérico
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This presentation is based on the work

- *Extremizers for Fourier restriction on hyperboloids,* (with E. Carneiro, and D. Oliveira e Silva), to appear at Ann. Henri Poincaré, arXiv:1708.03826 (2017).

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But some events may be changed for dramatization effects.

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Given a subset $S \subset \mathbb{R}^d$, and a measure σ supported in S, we define formally the *Fourier extension operator* T, associated to the pair (S, σ), as

$$Tf(\xi) = \int_S f(x) e^{ix \cdot \xi} \mathrm{d}\sigma(x).$$

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Remark: When $S = \mathbb{R}^d$ and σ is the Lebesgue measure $Tf = \widehat{f}$, i.e, the more familiar Fourier transform of f. It will be useful today, but not the focus.

Why these operators in a PDE seminar?

Again, we wish to study operators of the form

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One of the oldest cases of interest is when $S = \mathcal{P} = \{(x, |x|^2);\ x \in \mathbb{R}^d\}$ and $\mathrm{d}\sigma(x,t) = \delta(t - |x|^2)\mathrm{d}x\mathrm{d}t$, i.e, the paraboloid with projection measure. In this case, by identifying $f : \mathcal{P} \to \mathcal{C}$ with a function in $\mathbb{R}^d$, one has

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When $f \in L^2(\mathbb{R}^d)$, one can see that $u(\xi,\tau) = Tf(\xi,\tau)$ is obviously the solution to the following Schrödinger equation

$$i\partial_t u = \Delta_x u, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R};$$
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(If this last part was not so obvious to you, look at the board)

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- The cubic surface $\mathcal{A} = \{(x, x^3) : x \in \mathbb{R}\}$ with projection measure generates solutions to Airy's equation (KdV for some).

PDE x Restriction estimate

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The same paradigm is true for the other equations: a $L^2(\sigma) \to L^p(\mathbb{R}^{d+1})$ is equivalent to a $H^s_x \to L^p_{x,t}$ estimate. The cone case for instance corresponds to this:

$$\|u\|_{L^p(\mathbb{R}^d \times \mathbb{R})} \lesssim \|u(\cdot, 0)\|_{\dot{H}^{1/2}(\mathbb{R}^d)} + \|\partial_t u(\cdot, 0)\|_{\dot{H}^{-1/2}(\mathbb{R}^d)}$$

where u is a solution to $\partial_{tt} u = \Delta_x u$ and $p = 2 + \frac{4}{d-1}$.

Back to restriction

Consider the hyperboloid $\mathbb{H}^d \subset \mathbb{R}^{d+1}$ defined by

$$\mathbb{H}^d = \left\{(y, y') \in \mathbb{R}^d \times \mathbb{R} : y' = \sqrt{1+|y|^2}\right\},$$

equipped with the measure

$$\mathrm{d}\sigma(y, y') = \boldsymbol{\delta}\left(y' - \sqrt{1+|y|^2}\right)\frac{\mathrm{d}y\,\mathrm{d}y'}{\sqrt{1+|y|^2}}.$$

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The *Fourier extension operator* on $\mathbb{H}^d$ is defined by

$$\begin{aligned} Tf(x, t) &= \int_{\mathbb{H}^d} f(y) e^{i(x,t)\cdot y} \mathrm{d}\sigma(y) \\ &= \int_{\mathbb{R}^d} e^{ix\cdot y} e^{it\sqrt{1+|y|^2}} f(y, \sqrt{1 + |y|^2}) \frac{\mathrm{d}y}{\sqrt{1 + |y|^2}}, \end{aligned}$$

where $(x, t) \in \mathbb{R}^d \times \mathbb{R}$, and $f : \mathbb{H}^d \to \mathcal{C}$ is in a appropriate dense class.

Hyperbolic setting

The classical work of Strichartz establishes that

$$\|Tf\|_{L^p(\mathbb{R}^{d+1})} \leq \mathbf{H}_{d,p} \|f\|_{L^2(\mathrm{d}\sigma)},$$

with a finite constant $\mathbf{H}_{d,p}$ (independent of f), in the following admissible range

$$\begin{cases} 6 \leq p < \infty, \text{ if } d = 1; \\ 2 + \frac{4}{d} \leq p \leq 2 + \frac{4}{d-1}, \text{ if } d \geq 2. \end{cases}$$

Hyperbolic setting

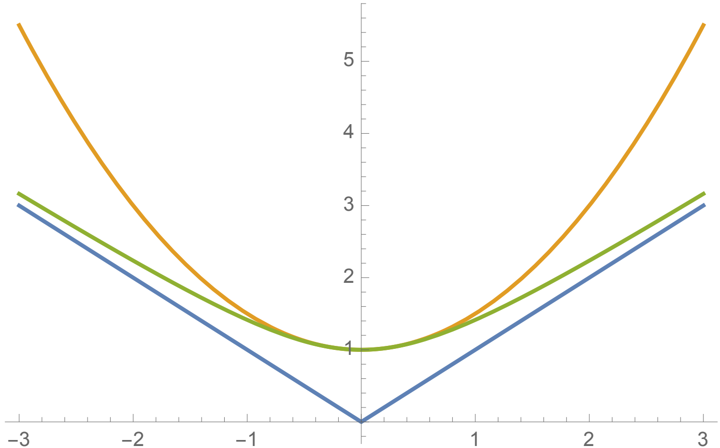

Figure: The paraboloid $y = 1 + \frac{|x|^2}{2}$ osculates the hyperboloid $y = \sqrt{1 + |x|^2}$ at its vertex. The cone $y = |x|$ approximates the same hyperboloid at infinity.

Klein–Gordon equation

The restriction operator T is intimately connected with the Klein-Gordon equation

$$\partial_t^2 u = \Delta_x u - u, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R};$$
$$u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x).$$

trough the following operator, the *Klein–Gordon propagator*,

$$e^{it\sqrt{1-\Delta}} g(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{g}(\xi)\, e^{ix\cdot\xi}\, e^{it\sqrt{1+|\xi|^2}} \mathrm{d}\xi.$$

One can see that solutions to the equation can be written as

$$u(\cdot,t) = \frac{1}{2}\left(e^{it\sqrt{1-\Delta}} u_0(\cdot) - i e^{it\sqrt{1-\Delta}} (\sqrt{1-\Delta})^{-1} u_1(\cdot)\right) + \frac{1}{2}\left(e^{-it\sqrt{1-\Delta}} u_0(\cdot) + i e^{-it\sqrt{1-\Delta}} (\sqrt{1-\Delta})^{-1} u_1(\cdot)\right),$$

and for $f(x) = \widehat{g}(x)\sqrt{1+|x|^2}$ (or $\widehat{f} = (1-\Delta)^{1/2} g$)

$$Tf(x,t) = (2\pi)^d\, e^{it\sqrt{1-\Delta}} g(x),$$

Klein-Gordon equation

This relation implies that the restriction estimate

$$\|Tf\|_{L^p(\mathbb{R}^{d+1})} \leq \mathbf{H}_{d,p}\,\|f\|_{L^2(\mathrm{d}\sigma)}$$

amounts to the following inequality

$$\|e^{it\sqrt{1-\Delta}}g\|_{L^p_{x,t}(\mathbb{R}^d\times\mathbb{R})} \leq (2\pi)^{-d}\,\mathbf{H}_{d,p}\,\|g\|_{H^{\frac{1}{2}}(\mathbb{R}^d)},$$

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and although these are equivalent, each inequality has its advantages.

Problems to come

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We are interested in sharp instances of the extension inequality on the hyperboloid. More precisely, given a pair (d, p) in the admissible range, we study *extremizers* and *extremizing sequences* for the restriction inequality , and are interested in the value of the *optimal constant*

$$\mathbf{H}_{d,p} := \sup_{0 \neq f \in L^2(\mathbb{H}^d)} \frac{\|Tf\|_{L^p(\mathbb{R}^{d+1})}}{\|f\|_{L^2(\mathrm{d}\sigma)}}.$$

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- Sphere: Constants are the only real extremizers for the 2-sphere endpoint of Tomas-Stein and there are extremizers for the circle. Every even exponent case in the mixed norm setting ($L^2 \to L^p_{rad}(L^2_{ang})$) has constants as unique extremizers and the same is true after some big exponent. (Foschi, Christ-Shao, Shao, Foschi-Oliveira e Silva, Carneiro-Oliveira e Silva-S.).

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To see a picture of the history of sharp Fourier restriction theory, one can check the recent survey *"Some recent progress on in sharp Fourier restriction theory"* by D. Foschi and D. Oliveira e Silva.

What about the hyperboloid?

Theorem (Quilodrán (2015))

Let $(d, p) \in \{(2,4), (2,6), (3,4)\}$. *Then*

$$\mathbf{H}_{2,4} = 2^{\frac{3}{4}}\pi, \ \mathbf{H}_{2,6} = (2\pi)^{\frac{5}{6}}, \ \textit{and } \mathbf{H}_{3,4} = (2\pi)^{\frac{5}{4}},$$

and there are no extremizers for the restriction inequality in these cases.

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Oberve that these cases are the only ones where $d \geq 2$ and p is an even integer. In his paper, Quilodrán left the question of what happens with the rest the even integer cases, which are many, since for $d = 1$ every integer bigger than 6 is an admissible exponent.

Our contribution

Theorem 1 (Carneiro, Oliveira e Silva, and S. (2017))

The value of the optimal constant in the case $(d, p) = (1, 6)$ is

$$\mathbf{H}_{1,6} = 3^{-\frac{1}{12}}(2\pi)^{\frac{1}{2}}.$$

Moreover, extremizers do not exist in this case.

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Theorem 2 (Carneiro, Oliveira e Silva, and S. (2017))

Extremizers for the restriction inequality do exist in the following cases:

(a) $d = 1$ *and* $6 < p < \infty$.

(b) $d = 2$ *and* $4 < p < 6$.

Strategy: the case of Theorem 1

From Plancherel's Theorem it follows that

$$\|Tf\|^3_{L^6(\mathbb{R}^2)} = \|(\widehat{f\sigma})^3\|_{L^2(\mathbb{R}^2)} = \|(f\sigma * f\sigma * f\sigma)\widehat{\ }\|_{L^2(\mathbb{R}^2)} = 2\pi\|f\sigma * f\sigma * f\sigma\|_{L^2(\mathbb{R}^2)},$$

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which in particular implies that

$$\mathbf{H}_{1,6}^3 = 2\pi \sup_{0\neq f\in L^2(\mathbb{H}^1)} \frac{\|f\sigma * f\sigma * f\sigma\|_{L^2(\mathbb{R}^2)}}{\|f\|_{L^2(\mathbb{H}^1)}^3}.$$

We are thus led to studying convolution measure $\sigma * \sigma * \sigma$ in order to obtain sharp bounds. In this case, we have access to the machinery of δ-calculus introduced by Foschi.

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- Step 3: Produce an extremizing sequence that converges to the candidate.

Strategy: the case of Theorem 2

To produce extremizers, we consider extremizing sequences and show that they converge after suitable modifications. Recent work of Fanelli-Vega-Visciglia suggest that in non endpoint cases concentration-compactness arguments can lead to the existence of extremizers.

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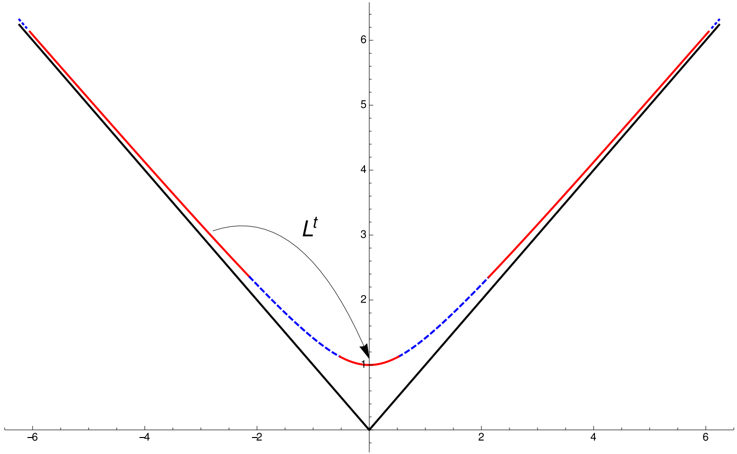
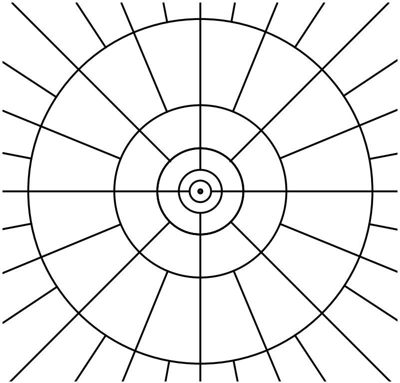
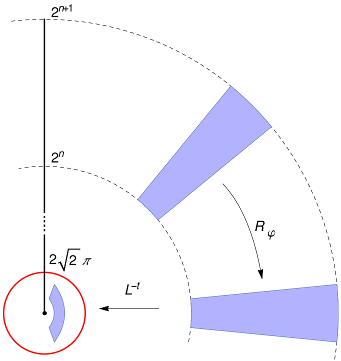
The heart of the matter lies in the construction of a *special cap*, i.e. a set that contains a positive universal proportion of the total mass in an extremizing sequence, possibly after applying the symmetries of the problem. This rules out the possibility of "mass concentration at infinity", and is the missing part in a observation of Quilodrán, that had originally outlined the proof of a dichotomy for extremizing sequences.

1D setting



2D setting





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- Step 1: Prove a bilinear estimate that show how caps interact.
- Step 2: Use the bilinear inequality to prove a improved linear inequality.
- Step 3: Use the improved inequality to prove that there is a special cap. Nonendpointness will produce a extremizer.

Thank you!

¡Gracias!