My trajectory in Argentina: A mathematical journey and other short stories

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Outline

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Spoiler Alert: An exposition without to reveal all tricks....

Introduction

In this Talk we are interested in presenting recent advancements in Nonlinear PDEs and their applications in free boundary problems and asymptotic analysis of PDEs.

Our focus will be explain some ideas of so-named Geometric Regularity Theory.

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Our focus will be explain some ideas of so-named Geometric Regularity Theory.

By way of illustration, we will treat the quasi-linear problem with strong absorption[a]:

$$-\Delta_p u(x) + \lambda_0(x) u^q \chi_{\{u>0\}}(x) = 0 \quad \text{in} \quad \Omega, \quad (1 < p < \infty) \tag{1.1}$$

- ✓ $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ the $p-$**Laplacian operator**;
- ✓ λ_0 is a bounded function away from zero and infinity (Thiele modulus);
- ✓ $q(p) \in [0, p-1)$ reaction factor;
- ✓ $\Omega \subset \mathbb{R}^N$ is a bounded and regular domain.

[a] One of our main motivations for studying such problems comes from chemical catalysis in certain isothermal catalytic reaction process, where the existence of Dead-cores, i.e., regions where the density of certain substance (reactant or gas) vanishes identically plays an important role in the formulation of such phenomena.

First insights into the theory

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$$u_r(x) := \frac{u(x_0 + rx)}{r^{\frac{p}{p-1-q}}} \quad (\text{Natural scaling of the problem})$$

satisfies the same PDE of original solution (with a possible changed Thiele modulus).

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satisfies the same PDE of original solution (with a possible changed Thiele modulus).

Now, if we fix $q \in (0, p-1)$ and $R > r > 0$, the Radial profile $u : B_R \to \mathbb{R}_+$

$$v(x) := \mathfrak{c}.(|x| - r)_+^{\frac{p}{p-1-q}}$$

(for an appropriate constant $\mathfrak{c} = \mathfrak{c}(N, p, q) > 0$) is a weak solution to

$$-\Delta_p v(x) + v^q(x) = 0 \quad \text{in} \quad B_R.$$

First insights into the theory

Remind that solutions to (1.1) are $C^{1,\gamma}_{\text{loc}}$ for some $\gamma \in (0,1) \quad \Leftrightarrow \quad |\nabla u| \in C^{0,\gamma}_{\text{loc}}$.

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However, fixed $1 < p < \infty$, the previous profile fulfills $v \in C^{\lfloor \alpha \rfloor, \beta}_{\text{loc}}$, where

$$\alpha(p,q) := \frac{p}{p-1-q} \quad \text{and} \quad \beta(p,q) := \alpha - \lfloor \alpha \rfloor.$$

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$$\alpha(p,q) := \frac{p}{p-1-q} \quad \text{and} \quad \beta(p,q) := \alpha - \lfloor \alpha \rfloor.$$

On the other hand, when $0 < q < p-1$,

$$\alpha(p,q) = \frac{p}{p-1-q} > \frac{p}{p-1} \quad \Rightarrow \quad \text{Improved estimates along FBP.}$$

Finally, remember that $\alpha_0 = \frac{p}{p-1}$ is the better regularity expected for sol. with L^∞-RHS.
Therefore, our solution presents an unexpected gain of smoothness.

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Mathematical goal

Our Impetus

Therefore, we will focus our attention in presenting sharp and improved growth (geometric) estimates for solutions to (1.1) along free boundary points.

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Therefore, we will focus our attention in presenting sharp and improved growth (geometric) estimates for solutions to (1.1) along free boundary points.

Such estimates play a fundamental role in proving (some applications):

1. Blow-up and Liouville type results;
2. Weak geometric properties of free boundary (e.g. Hausdorff measure estimates);
3. Asymptotic results as $p \to \infty$ (resp. $q \to \infty$).

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Non-degeneracy of solutions

Firstly we give the precise growth rate at which non-negative solutions leave their FBs.

Theorem (da S., Rossi & Salort, Calc. Var. PDEs'18 and J. London Math. Soc.'19)

[a] *Let u be a nonnegative, bounded solution to (1.1) in B_1 and let $x_0 \in \overline{\{u > 0\}} \cap B_{\frac{1}{2}}$ be a generic point in the closure of the non-coincidence set. Then for any $0 < r < \frac{1}{2}$, there holds*

$$\sup_{B_r(x_0)} u(x) \geq \left[\inf_{\Omega} \lambda_0(x) . \frac{(p-1-q)^p}{p^{p-1}(pq + N(p-1-q))}\right]^{\frac{1}{p-1-q}} . r^{\frac{p}{p-1-q}}.$$

a

J.V. da Silva, J. Rossi & A. Salort, *Regularity properties for $p-$dead core problems and their asymptotic limit as $p \to \infty$*. **J. London Math. Soc.** (2) 99 (2019) 69-96.

J.V. da Silva & A. Salort, *Sharp regularity estimates for quasi-linear elliptic dead core problems and applications*. **Calc. Var. Partial Differential Equations** 57 (2018), no. 3, 57: 83.

Growth rate for dead core solutions

The next result provides an improved estimate for solutions along FB points.

Theorem (da S., Rossi & Salort, J. London Math. Soc.'19)

Let u be a nonnegative, bounded solution to (1.1) in B_1 and let $x_0 \in \partial\{u > 0\} \cap B_{\frac{1}{2}}$ be a FB point. Then for any $0 < r < \frac{1}{2}$, there holds

$$\sup_{B_r(x_0)} u(x) \leq 2.2^{\frac{p}{p-1-q}} \left[\inf_{\Omega} \lambda_0(x). \frac{(p-1-q)^p}{p^{p-1}(pq + N(p-1-q))}\right]^{\frac{1}{p-1-q}} .r^{\frac{p}{p-1-q}}.$$

Limit as $p, q \to \infty$?

What should we expect from $p-$dead core solutions as $p, q \to \infty$?

1. Do these (limit) profiles exist? Would they be unique?
2. Do such (limit) profiles verify some (limit) PDE? In what sense?
3. Are these (limit) solutions regular ?
4. How do behave such (limit) solutions near free boundary points?

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Going to infinity....

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A final application: Asymptotic analysis as $p, q \to \infty$

Theorem (da S., Rossi & Salort, J. London Math. Soc. 19)

Let $(u_p)_{p\geq 2}$ be the family of uniformly bounded solutions to

$$\begin{cases} -\Delta_p u_p(x) + \lambda_0(x)(u_p)_+^{q(p)}(x) = 0 & \text{in } \Omega \\ u_p(x) = g(x) & \text{on } \partial\Omega, \end{cases}$$

with $g \in Lip(\partial\Omega)$. If $\ell := \lim_{p\to\infty} \frac{q(p)}{p} \in [0,1)$ exists. Then, $u_p \to u_\infty$ unif. in $\overline{\Omega}$. Furthermore, $u_\infty \in Lip(\overline{\Omega})$ is a viscosity solution to

$$\begin{cases} \max\left\{-\Delta_\infty u_\infty, -|\nabla u_\infty| + u_\infty^\ell\right\} = 0 & \text{in } \{u_\infty > 0\} \cap \Omega, \\ u_\infty = g & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

A final application: Asymptotic analysis as $p, q \to \infty$

Existence and Uniqueness assertions:

- ✓ Theorem 3 provides a device in order to obtain existence of viscosity solutions to (1.2).
- ✓ We highlight that whether $\ell = 0$ then (1.2) admits a unique Lipschitz visc. sol. [a]
- ✓ The case $\ell \in (0,1)$ remains as an open issue yet.

[a]

P. Blanc, J.V. da Silva and J.D. Rossi, *A limiting free boundary problem with gradient constraint and Tug-of-War games*. To appear in **Ann. Mat. Pura Appl.**

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A final application: Asymptotic analysis as $p, q \to \infty$

Theorem (**Sharp growth for limit solutions**, *J. London Math. Soc.'19*)

Let u_∞ be a uniform limit to solutions u_p of (1.2) Then, for any $x_0 \in \partial\{u_\infty > 0\} \cap \Omega'$ and $0 < r \ll 1$ the following estimate holds:

$$\sup_{B_r(x_0)} u_\infty(x) \leq 2 \cdot 2^{\frac{1}{1-\ell}} (1-\ell)^{\frac{1}{1-\ell}} r^{\frac{1}{1-\ell}}.$$

Theorem (**Strong non-degeneracy for limit solutions**, *J. London Math. Soc.'19*)

Let u_∞ be a uniform limit to solutions of (1.2). Then, for any $x_0 \in \{u_\infty > 0\} \cap \Omega'$ and any $0 < r \ll 1$, the following estimate holds:

$$\sup_{B_r(x_0)} u_\infty(x) \geq (1-\ell)^{\frac{1}{1-\ell}} r^{\frac{1}{1-\ell}}.$$

Comments about evolution problems with strong absorption

We are also able to analyse the parabolic counterpart[a], namely

$$\mathfrak{L}(x,t,Du,D^2u) - \frac{\partial u}{\partial t} = \lambda_0 u^q \chi_{\{u>0\}}$$

a

J.V. da Silva & P. Ochoa, *Fully noninear parabolic dead core problems*. To appear in **Pacific J. Math.** 2018.

J.V. da Silva, P. Ochoa & A. Silva, *Regularity for degenerate evolution equations with strong absorption*. **J. Differential Equations** 264 (2018), no. 12, 7270-7293.

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Final comments about evolution problems with strong absorption

Notwithstanding,

There ain't no such thing as a free lunch!

Let us present some difficulties for this scenery which we need overcome:

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There ain't no such thing as a free lunch!

Let us present some difficulties for this scenery which we need overcome:

1. Distinct homogeneities in space and time variables (We must make use of Intrinsic Scaling Techniques);
2. Absent of SMP for the homogeneous problem (A non-decreasing in time assumption is necessary combined with DiBenedetto's Hölder estimates);
3. In the "parabolic world" standard barriers function are not the radial profile (distance up to a fixed FB point) : _(. Therefore, building a barrier becomes a non-trivial task!

Presenting the problem

In this Lecture we are interested in studying quantitative features for evolution models of $p-$Laplacian type as follows:

$$\mathcal{Q}u := \frac{\partial u}{\partial t} - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x,t) \quad \text{in} \quad \Omega_T, \quad p > 2 \tag{2.1}$$

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where

- ✓ $\Omega_T := \Omega \times (0,T)$ with $\Omega \subset \mathbb{R}^n$ a bounded and regular domain;
- ✓ $f \in L^{q,r}(\Omega_T)$ (a **Lebesgue space with mixed norms**) endowed with the norm

$$\|f\|_{L^{q,r}(\Omega_T)} := \left(\int_0^T \left(\int_\Omega |f(x,t)|^q dx\right)^{\frac{r}{q}} dt\right)^{\frac{1}{r}}$$

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Some intuitions

A fundamental issue in linear/nonlinear PDEs consists of inferring which is the **expected regularity** to weak solutions.

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By way of motivation, let us visit the linear theory: Let u be a weak solution to:

$$\mathcal{H}u := \frac{\partial u}{\partial t}(x,t) - \mathrm{div}(a(x,t)\nabla u) = f(x,t) \text{ in } \quad Q_1^- := B_1 \times (-1,0]. \tag{2.2}$$

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There are two important aspects which we must take into account:

A priori estimate to Hom. problem with "frozen" coef. **Vs** Integrability of the source term

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As a matter of fact, $v(x,t) := \frac{u(\rho x, \rho^2 t)}{\rho^\kappa}, \quad \kappa \in (0,2]$ verifies in the weak sense:

$$\frac{\partial v}{\partial t} - \operatorname{div}(a(\rho x, \rho^2 t)\nabla v) = \rho^{2-\kappa} f(\rho x, \rho^2 t) := f_\rho(x,t) \;\Rightarrow\; \|f_\rho\|_{L^{q,r}(Q_1^-)} \le \rho^{2-\kappa-\left(\frac{n}{q}+\frac{2}{r}\right)} \|f\|_{L^{q,r}(Q_1^-)}.$$

Sharp regularity estimates: Linear scenario

More integrability of f and "regularity" of a(., .) $\Rightarrow$ More (local) regularity of u.

Theorem (**da S. and Teixeira, Math. Ann. 17**)

a *Let* u *be a bounded weak solution to* (2.2) *then*

$\mathbf{f \in L^{q,r}(Q_1^-)}$	**Sharp Regularity**

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$\frac{n}{q} + \frac{2}{r} = 1$	$C^{0,Log\text{-}Lip}_{loc}(Q_1^-)$

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$1 < \frac{n}{q} + \frac{2}{r} < 2$	$C_{loc}^{\zeta, \frac{\zeta}{2}}(Q_1^-)$
$\frac{n}{q} + \frac{2}{r} = 1$	$C_{loc}^{0,Log\text{-}Lip}(Q_1^-)$
$0 < \frac{n}{q} + \frac{2}{r} < 1$	$C_{loc}^{1+\zeta, \frac{1+\zeta}{2}}(Q_1^-)$

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$BMO \supset L^{\infty,\infty} \simeq L^{\infty}$	$C_{loc}^{1, Log\text{-}Lip}(Q_1^-)$

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a

J.V. da Silva & E.V. Teixeira, *Sharp regularity estimates for second order fully nonlinear parabolic equations*. **Math. Ann.** 369 (2017), no. 3-4, 1623-1648.

From linear to nonlinear theory

What we should expect from **Nonlinear Scenario** $(p \neq 2)$*?*

Recently, under the conditions $\frac{1}{r} + \frac{n}{pq} < 1 < \frac{2}{r} + \frac{n}{q}$ for $p > 2$ and by combining **geometric tangential methods** and **intrinsic scaling techniques** (cf. [2]), the sharp (geometric) $C^{\alpha, \frac{\alpha}{\theta}}_{\text{loc}}$ regularity estimate to (2.1) was established by Teixeira-Urbano[a], where

$$\alpha = \frac{p\left[1 - \left(\frac{1}{r} + \frac{n}{pq}\right)\right]}{p\left[1 - \left(\frac{1}{r} + \frac{n}{pq}\right)\right] + \left(\frac{2}{r} + \frac{n}{q}\right) - 1} \quad \text{and} \quad \theta := 2\alpha + (1 - \alpha)p.$$

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a

E.V. Teixeira & J.M. Urbano, *A geometric tangential approach to sharp regularity for degenerate evolution equations.* **Anal. PDE** 7 (2014), no. 3, 733-744.

Who wants to be a Millionaire?

Open scenarios: "One Million" Dollar Questions

Notwithstanding, Teixeira-Urbano's work leaves as open issues some scenarios:

$f \in L^{q,r}(Q_1^-)$	Regularity	$f \in L^{q,r}(Q_1^-)$	Regularity
$\frac{n}{q} + \frac{2}{r} = 1$	$C_{loc}^{0,\text{Log-Lip}}(Q_1^-)$	$\frac{n}{q} + \frac{2}{r} = 1 > \frac{1}{r} + \frac{n}{pq}$	Open Problem
$0 < \frac{n}{q} + \frac{2}{r} < 1$	$C_{loc}^{1+\zeta, \frac{1+\zeta}{2}}(Q_1^-)$	$0 < \frac{1}{r} + \frac{n}{pq} < 1,\ 0 < \frac{2}{r} + \frac{n}{q} < 1$	Open Problem
BMO $\supset L^{\infty,\infty}$	$C_{loc}^{1,\text{Log-Lip}}(Q_1^-)$	BMO $\supset L^{\infty,\infty}$	Open Problem
Linear Theory		**Nonlinear Theory**	

Remember that the regularity theory to models like (2.1) is available in:[a]

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E. DiBenedetto, *Degenerate parabolic equations. Universitext*. Springer-Verlag, New York, 1993. xvi+387 pp. ISBN: 0-387-94020-0.

J.M. Urbano, *The method of intrinsic scaling*. A systematic approach to regularity for degenerate and singular PDEs. Lecture Notes in Mathematics, 1930. Springer-Verlag, Berlin, 2008. x+150 pp. ISBN: 978-3-540-75931-7.

Our contributions

We give an **affirmative answer** to last two scenarios:

$\mathbf{f \in L^{q,r}(Q_1^-)}$	**Sharp Regularity**
(CC) $0 < \frac{1}{r} + \frac{n}{pq} < 1$ and $0 < \frac{2}{r} + \frac{n}{q} < 1$	$C_{\text{loc}}^{1+\min\left\{\frac{1-\left(\frac{n}{q}+\frac{2}{r}\right)}{p\left[1-\left(\frac{n}{pq}+\frac{1}{r}\right)\right]-\left[1-\left(\frac{n}{q}+\frac{2}{r}\right)\right]},\, \alpha_{\text{Hom}}^{-}\right\}}$
$\text{BMO} \supset L^{\infty,\infty} \simeq L^{\infty}$	$C_{\text{loc}}^{1+\min\left\{\frac{1}{p-1},\, \alpha_{\text{Hom}}^{-}\right\}}$

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Another pivotal question:

Are there significant changes between the Teixeira-Urbano's case and the above ones?[a]

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Another pivotal question:

Are there significant changes between the Teixeira-Urbano's case and the above ones?[a]

1. Degenerate character of operator is an obstacle (the Bad Guy is $\{|\nabla u| \ll 1\}$);
2. An adjusted intrinsic scaling of the equation will be necessary;
3. $C^{1+\alpha}$ regime via affine approximations scheme does not work in such scenarios.

[a] Cambia, Todo cambia...Mercedes Sosa. Todo cambia, Live in Europe, 1989.

Some possible implications

It is worth highlight that such estimates play a fundamental role in establishing[a]:

1. **Blow-up** and **Liouville type results**;
2. **Weak geometric properties** and **Hausdorff measure estimates** (in certain FBPs);

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J.V. da Silva, J. Rossi & A. Salort, *Regularity properties for $p-$dead core problems and their asymptotic limit as $p \to \infty$*. **J. London Math. Soc.** (2) 99 (2019) 69-96.

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J.V. da Silva, P. Ochoa & A. Silva, *Regularity for degenerate evolution equations with strong absorption*. **J. Differential Equations** 264 (2018), no. 12, 7270-7293.

Our Main Theorem

Theorem (**Israel J. Math. 19 and Nonlinear Anal. 19**)

[a] *Let* $K \subset\subset Q_1^-$, u *be a bounded weak solution of* (2.1) *in* Q_1^- *and suppose that (CC) are in force. Then* u *is* $C^{1+\alpha}$ *(in the parabolic sense), i.e., there exists a (universal) constant* $M > 0$ *such that*

$$[u]^*_{C^{1+\alpha}(K)} \le M.\left[\|u\|_{L^\infty(Q_1^-)} + \|f\|_{L^{q,r}(Q_1^-)}\right],$$

where

$$[u]^*_{C^{1+\alpha}(K)} := \sup_{0<\rho\le\rho_0}\left(\inf_{(x_0,t_0)\in \mathcal{C}^\alpha_\rho(Q_1^-)} \frac{\|u - \mathfrak{l}_{(x_0,t_0)}(u)\|_{L^\infty(\hat{Q}^-_\rho(x_0,t_0)\cap K)}}{\rho^{1+\alpha}}\right).$$

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[a]

M.D. Amaral, J.V. da Silva, G.C. Ricarte & R. Teymurazyan, *Sharp regularity estimates for quasilinear evolution equations*. To appear in **Israel J. Math.** (2019).

J.V. da Silva, *Geometric $C^{1+\alpha}$ regularity estimates for nonlinear evolution model.* **Nonlinear Anal.** 184 (2019), 95-115.

Optimal design problems

In Mathematics an Optimal Design Problem under constrained volume consists of following: For a $\Omega \subset \mathbb{R}^N$ (bounded domain), fix $0 < \alpha < \mathscr{L}^N(\Omega)$

$$\min \left\{\mathfrak{J}[u](\Xi) : \; u : \Omega \to \mathbb{R}, \; \Xi \subset \Omega \text{ and } 0 < \mathscr{L}^N(\Xi) \leq \alpha \right\}.$$

In a number of situations the functional $\mathfrak{J}[u]$ has an integral representation, whose involved functions are linked to the competing configuration Ξ via a prescribed PDE.

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State of the Art

1. 86's Aguilera, Alt and Caffarelli;
   $\mathfrak{J}_\alpha[v_\Xi] = \int_\Omega |\nabla v_\Xi|^2 dx$ with prescribed volume of the set $\Xi = \{u = 0\}$;
2. 87's Aguilera, Caffarelli and Spruck;
   Shape optimization problem in heat conduction: $\mathfrak{J}_\alpha[v_\Xi] = \int_{\partial\Xi} v_\nu d\mathcal{H}^{N-1}$;
3. 96's Lederman ;
   $\mathfrak{J}_g[v] = \int_\Omega (|\nabla v|^2 - gv) dx$ with prescribed volume of $\Xi = \{u > 0\}$;
4. 05,07,10's Teixeira;
   Nonlinear counterpart of ACS's manuscript: $\mathfrak{J}_\alpha[v_\Xi] = \int_{\partial\Xi} \Gamma(x, v_\nu) d\sigma$;
5. 06's Bonder, Martínez and Wolanski and Oliveira and Teixeira;
   $\mathfrak{J}_\alpha[v_\Xi] = \int_\Omega |\nabla v_\Xi|^p dx$ with prescribed volume of $\Xi = \{u > 0\}$;
6. 08's Martínez (Orlicz-Sobolev setting).

Theory of non-local operators

Problems driven by fractional diffusion appear naturally in models coming from jumping and Levy processes. A prototype is given by

$$(-\Delta)_p^s u(x) := C_{N,s,p}.\text{P.V.} \int_{\mathbb{R}^N} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{N+ps}} dy.$$

In order to deal with such operators we need to define appropriated Sobolev spaces!

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A minimization problem in fractional diffusion

Taking into account the previous motivation. We would like to solve the following problem:

$$\mathfrak{L}_p^s[\alpha] = \min_{\mathcal{K}_g^{s,p}} \mathcal{J}_p(v), \qquad (\mathfrak{P}_p^s)$$

where

$$\mathcal{J}_p(v) := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^p}{|y - x|^{N+sp}} dxdy \right)^{\frac{1}{p}}$$

and

$$\mathcal{K}_g^{s,p} := \left\{ v \in W^{s,p}(\mathbb{R}^N),\, v = g \geq 0 \text{ in } \Omega^c \text{ and } \mathscr{L}^N(\{v > 0\} \cap \Omega) \leq \alpha \right\}.$$

A minimization problem in fractional diffusion

Important issues

1: Are there any minimizers to $(\mathfrak{P}_p^s)$?
2: How regular are the minimizers to $(\mathfrak{P}_p^s)$?
3: What should be the expect limiting problem as $p \to \infty$ to $(\mathfrak{P}_p^s)$?

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A limiting configuration

Motivated by formal considerations, we will consider the following limit configuration:

$$\mathfrak{L}^s_\infty[\alpha] = \min_{\mathcal{K}^{s,\infty}_g} [v]_{C^{0,s}(\mathbb{R}^N)}, \qquad (\mathfrak{P}^s_\infty)$$

where

$$[v]_{C^{0,s}(\mathbb{R}^N)} := \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|v(y)-v(x)|}{|x-y|^s}$$

and

$$\mathcal{K}^{s,\infty}_g := \left\{ v \in W^{s,\infty}(\mathbb{R}^N),\ v = g \geq 0 \text{ in } \Omega^c \text{ and } \mathscr{L}^N(\{v>0\}\cap\Omega) \leq \alpha \right\}$$

A limiting configuration

Theorem (da S. & Rossi, Trans. Amer. Math. Soc.'19)

[a] Let u_p be a minimizer to $(\mathfrak{P}_p^s)$. Then, up to a subsequence, $u_p \to u_\infty$ as $p \to \infty$, uniformly in Ω and weakly in $W^{s,q}(\Omega)$ for all $1 < q < \infty$, where u_∞ minimizes $(\mathfrak{P}_\infty^s)$. Moreover, the limit u_∞ fulfils

$$-\mathscr{L}_\infty^s[u_\infty] = 0 \quad \text{in} \quad \{u_\infty > 0\} \cap \Omega,$$

in the viscosity sense, where $\mathscr{L}_\infty^s[u_\infty]$ is the Hölder $\infty-$Laplacian operator. Finally,

$$[u_\infty]_{C^{0,s}(\mathbb{R}^N)} \leq [g]_{C^{0,s}(\mathbb{R}^N \setminus \Omega)}.$$

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J.V. da Silva & J.D. Rossi, *The limit as $p \to \infty$ in free boundary problems with fractional $p-$Laplacians*. **Trans. Amer. Math. Soc.** 371 (2019), no. 4, 2739-2769.

A limiting configuration

Furthermore, uniqueness holds under the geometric compatibility condition:

$$\alpha \leq \mathscr{L}^N \left(\bigcup_{y \in \mathbb{R}^N \setminus \Omega} B_{\left(\frac{g(y)}{[g]_{C^{0,s}(\mathbb{R}^N \setminus \Omega)}} \right)^{\frac{1}{s}}}(y) \cap \Omega \right). \qquad \textbf{(Comp. Assump.)}$$

Theorem (da S. & Rossi, Trans. Amer. Math. Soc.'19)

Let v_∞ be given by $v_\infty(x) = \sup\limits_{\mathbb{R}^N \setminus \Omega} \left(g(y) - \mathfrak{H}^\sharp |x-y|^s \right)_+$. Assume that **(Comp. Assump.)** *holds. Let $\mathfrak{H}^\sharp$ be the unique positive number such that*

$$\Omega^\sharp := \bigcup_{y \in \mathbb{R}^N \setminus \Omega} B_{\left(\frac{g(y)}{\mathfrak{H}^\sharp} \right)^{\frac{1}{s}}}(y) \cap \Omega \quad \textit{fulfils} \quad \mathscr{L}^N \left(\Omega^\sharp \right) = \alpha.$$

Then v_∞ is the unique minimizer for $(\mathfrak{P}^s_\infty)$.

Other contributions in the PDE's research group

- J.V. da Silva, G.C. Ricarte & R. Teymurazyan, *Cavity type problems ruled by infinity Laplacian operator*. **J. Differential Equations** 262, no. 3 (2017), 2135-2157.
- J.V. da Silva, J.D. Rossi & A. Salort, *The ∞-Fučík spectrum*. **Ann. Acad. Sci. Fenn. Math.** 43 (1), (2018), 293-310.
- J.V. da Silva & J.D. Rossi, *A limit case in non-isotropic two-phase minimization problems driven by $p-$Laplacians*. **Interfaces Free Bound**. 20 (2018), no. 3, 379-406.
- J.V. da Silva & D. dos Prazeres, *Schauder type estimates for "flat" viscosity solutions to non-convex fully nonlinear parabolic equations and applications*. **Potential Anal.** 50 (2019), no. 2, 149-170.
- J.V. da Silva, J.D. Rossi & A. Salort, *Maximal solutions for the $\infty-$eigenvalue problem*. **Adv. Calc. Var.** 12 (2019), no. 2, 181-191.
- J.V. da Silva & G.C. Ricarte, *An asymptotic treatment for non-convex fully non-linear elliptic equations: Global Sobolev and BMO type estimates*. To appear in **Comm. Contemp. Math.**.

Some current directions

Let us present some current directions:

- J.V. da Silva, L. Del Pezzo & J.D Rossi, *An optimization problem with volume constraint with applications to optimal mass transport*.
- J.V. da Silva, R.A. Leitão & G.C. Ricarte, *Geometric regularity estimates for fully nonlinear elliptic equations with free boundaries*.
- J.V. da Silva, P. Ochoa & A. Silva, *Fractional elliptic problems with nonlinear gradient sources and measures*.
- J.V. da Silva & A. Salort, *A limiting obstacle type problem for the inhomogeneous $p-$fractional Laplacian*.
- J.V. da Silva, A. Salort, A. Silva & J. Spedaletti, *A constrained shape optimization problem in Orlicz-Sobolev spaces*.

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Some future directions

Finally, let us present some future directions for the next months:

1. An optimization problem for the $1-$Laplacian and its connections with Cheeger problem (with L. Del Pezzo and J.D. Rossi).
2. Existence and non-existence results of dead-core for fully nonlinear elliptic problems (with D.S. dos Prazeres and H.R. Quoirin).
3. Sharp regularity estimates for fully nonlinear elliptic equations with super-linear gradient growth (with G.S. Nornberg).
4. Asymptotic analysis in sandpile type models (with J.D. Rossi).
5. A limit problem for local-nonlocal $p-$Laplacians with concave-convex non-linearities (with A. Salort).
6. The obstacle problem for a degenerate fully nonlinear equation (with H. Vivas).
7. Sharp regularity for degenerate obstacle type problems (with H. Vivas).

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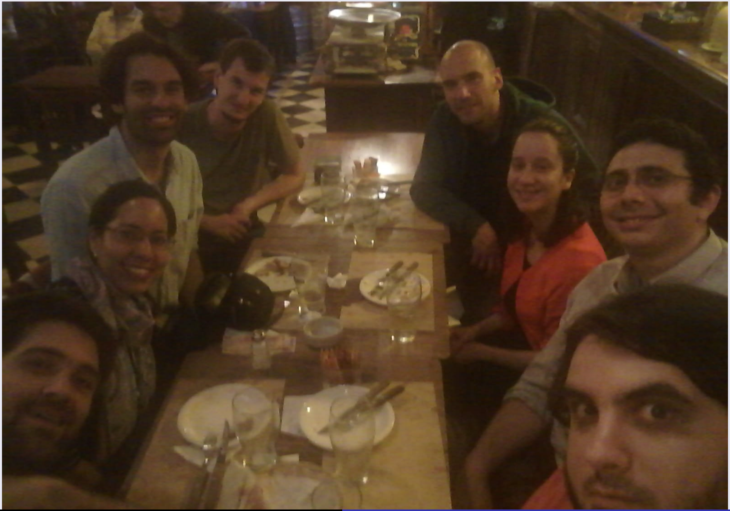
Good memories.... Reunión de la UMA 2018

Good memories.... Cumpleaños 2018

Good memories.... Otro Cumpleaños



Good memories.... ICM Rio de Janeiro 2018

I'll be eternally grateful to everybody here in Argentina $: -)$!

Thank you for everything, I'll never forget!

This chapter finishes here, but surely other ones will be written!

Aires