Long-time solvability for the 2D dispersive SQG equation with borderline regularity

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The dispersive inviscid surface quasi-geostrophic equation (DSQG) in $\mathbb{R}^2$

$$(DSQG): \ \partial_t\theta + u\cdot\nabla\theta + A\mathcal{R}_1\theta = 0 \quad \text{in } \mathbb{R}^2\times(0,\infty), \quad \text{and } \ \theta(0,x)=\theta_0(x)$$

- θ is the surface temperature, u is the velocity field, $div\, u = 0$;
- $\psi := \Lambda^{-1}\theta =$ stream function, $\Lambda = (-\Delta)^{1/2} = \int_{\mathbb{R}^2} \frac{f(x)-f(y)}{|x-y|^{2+\frac{1}{2}}}\,dy$, and

$$u = \nabla^{\perp}\psi = \nabla^{\perp}\Lambda^{-1}\theta = (-\partial_{x_2}\psi, \partial_{x_1}\psi) = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta),$$

with $\mathcal{R}_j =$ Riesz transform on the j-th component, so that $\widehat{\mathcal{R}_j\theta}(\xi) = i\frac{\xi_j}{|\xi|}\widehat{\theta}(\xi)$.

- $A^{-1} > 0$ plays an analogous role to the Rossby number from the Coriolis force,

  in fact: A^{-1} **represents the meridional variation of this force.**

The SQG system ("between 2D and 3D dynamics") is derived from the classical 3D QG
[Pierrehumbert, et al.]; high rotation, high stratification limit
to model large-scale motions in the ocean and atmosphere.

We want to study this system in the limit $A\to\infty$ as this has a further stabilization effect in $\mathbb{R}^2$.

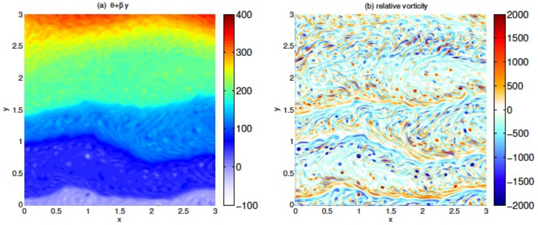
Geophysics applications: Zonal jets and Rossby wave propagation

- **Relation with "classical" QG model for potential vorticity (PV)**:
  The QG model comes from 3D Navier-Stokes-Coriolis under conservation of PV.
  In turn, SQG comes from 3DQG: Surface (potential) buoyancy ($\theta = \partial_z \psi|_{z=0}$),
  i.e. **temperature θ plays an analogous role to PV in the interior ("PV sheet").**

- **Thermocline ($u \cdot \nabla\theta$): dynamics in the upper layers of the ocean, small scales are driven by surface temperature**; frontogenesis, interaction of ocean eddies, phytoplankton...
  (SQG, Guillaume Lapeyre '17).

- The buoyancy gradient generates dispersive waves, which provides a **model for the interaction between waves and (nonlocal) turbulent motions in 2D**.
  (Smith and Sukhatme)

- When A **is larger than the nonlinear advective terms, the solutions are the equivalent to Rossby waves** of barotropic turbulence. **When they are both comparable**, the Rossby waves will interact with the nonlinearities of the flow, and **zonal jets will develop**.



Introduction: Related results

- **Inviscid SQG ($\partial_t \theta + u \cdot \nabla \theta = 0$) are the 2D analogue of 3D Euler equations, and Critical SQG eqns. ($\partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0$) are the correct 2D analogue of the 3D Navier-Stokes.**
  $\nabla \theta$ in 2D plays an analogous role to the vorticity ω in 3D.
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- **Global regularity problem:** Formation of sharp fronts suggested a possible singularity.
  - **Critical SQG** (Hard: nonlinearity $\equiv$ dissipation at all scales): but **Solved: globally well-posed**, Caffarelli-Vasseur, Kiselev-Nazarov-Volberg, Constantin-Vicol.
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- **Dispersive Critical SQG ($\partial_t\theta + u\cdot\nabla\theta + \Lambda\theta + A\mathcal{R}_1\theta = 0$) are 2D analogue of 3D Navier-Stokes-Coriolis:**
  $$\partial_t + u\cdot\nabla u - \nu\Delta u + \nabla p + \Omega^{-1}(u\times e_3) = 0.$$
  In the limit $|\Omega| \to 0$ the **fast rotating term produces a stabilization effect and ensures the global well-posedness** of strong solutions with large initial data. This effect drives the system **toward 2D dynamics** (Taylor columns), basically 2DNS with damping (due to friction at the boundary, i.e. Ekman pumping).
  Refs: Babin-Mahalov-Nicolaenko, Chemin-Desjardins-Gallagher-Grenier, Dutrifoy.

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- **Regularization via large dispersion for *DSQG* in $\mathbb{R}^2$:**
  $\to$ Kiselev-Nazarov '10: Global regularity for dispersive critical SQG;
  $\to$ Cannone-Miao-Xue '13: Global regularity for dissipative super-critical SQG via large dispersive forcing;
  Asymptotic behavior toward linear part of the system: Rossby waves.
  $\to$ Elgindi-Widmayer '15, Stabilization of inviscid SQG with dispersion;
  $\to$ Wan-Chen '17: Global regularity for inviscid SQG via large dispersive forcing in H^s, $s > 2 + \delta$, not critical.

Motivation: Global stabilization in critical Besov spaces

- DSQG is 2D analogue of 3D Euler-Coriolis.

- **Global results in Sobolev spaces**: *DSQG* (Wan-Chen), 3*DEC* (Koh-Lee-Takada) .
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- Reminiscent of a *folkloric* (now solved) problem:
  For $d = 2, 3$ if we perturb any given smooth initial data in H^{s_c} norm, $s_c = d/2 + 1$, then the corresponding solution can have infinite H^{s_c} norm instantaneously at $t > 0$.
  (J. Bourgain and D. Li, Strong ill-posedness of the incompressible Euler equation in borderline Sobolev spaces, Inventiones Mathematicae 201 (1) (2015), 97-157.)

- On the other hand, **2D Euler has global solvability in critical Besov spaces $B^{2/p+1}_{p,1}$,**
  $1 < p < \infty$. (3DE only local solvability). (M. Vishik. Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type. Ann. Sci. Ecole Norm. Sup. (4) 32 (1999), no. 6, 769-812.)

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- We want this type of result **BUT** **smoothing property of $e^{iAt\mathcal{R}_1}$ (i.e. Strichartz estimates) is not enough to control the nonlinearity** via a contraction argument.

Plan

- **Main idea: Parabolic Regularization** (Kato, Lions), add auxiliary viscous term, $\varepsilon\Delta$.
  $\rightarrow$ Local well-posedness via energy method, using commutator estimates.

- Show the local time of existence is independent of ε and A to obtain a solution as $\varepsilon \rightarrow 0^+$.

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- Cannot use dispersive effects because the dispersive term has zero energy (see below), direct analogy to the skew-symmetry of the Coriolis term.

- **Dispersive effects enter (only) in the global regularity via a BKM blow-up criterion**, control the nonlinearity by bounding $\|\nabla\theta\|_{L^\infty}$ with appropriate (Besov) norms.

Main theorem

Following analogous results for 3D Euler-Coriolis by Angulo-Castillo and Ferreira, we

"Exploit" the embedding $B^s_{2,q} \hookrightarrow L^\infty$, which allows us to have $\nabla u \in L^\infty$ for $s_0 = 2/2 + 1 = 2$ and **control** $\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau$ **globally in time for large** A when $s_1 = s_0 + 1 = 3$, to obtain

Main theorem

Let s and q be real numbers such that $s > 2$ with $1 \leq q \leq \infty$ or $s = 2$ with $q = 1$.

(i) (Local solvability) Let $\theta_0 \in B^s_{2,q}(\mathbb{R}^2)$. There exists $T = T(\|\theta_0\|_{B^s_{2,q}}) > 0$ such that (**??**) has a unique solution $\theta \in C([0,T]; B^s_{2,q}(\mathbb{R}^2)) \cap C^1([0,T]; B^{s-1}_{2,q}(\mathbb{R}^2))$, for all $A \in \mathbb{R}$.

(ii) (Long-time solvability) Let $T \in (0,\infty)$ and $\theta_0 \in B^{s+1}_{2,q}(\mathbb{R}^2)$. There exists $A_0 = A_0(T, \|\theta_0\|_{B^{s+1}_{2,q}}) > 0$ such that if $|A| \geq A_0$ then (**??**) has a unique solution $\theta \in C([0,T]; B^{s+1}_{2,q}(\mathbb{R}^2)) \cap C^1([0,T]; B^s_{2,q}(\mathbb{R}^2))$.

Littlewood-Paley decomposition and Besov spaces

Let $\phi_0 \geq 0$ be a radial function in $\mathcal{S}(\mathbb{R}^2)$ such that supp $\widehat{\phi}_0 \subset \{\xi \in \mathbb{R}^2 : \frac{1}{2} \leq |\xi| \leq 2\}$, $0 \leq \widehat{\phi}_0(\xi) \leq 1$ for all $\xi \in \mathbb{R}^2$, and

$$\sum_{j \in \mathbb{Z}} \widehat{\phi}_j(\xi) = 1 \quad \text{for all} \quad \xi \in \mathbb{R}^2 \backslash \{0\}, \tag{1}$$

where $\phi_j(x) := 2^{2j}\phi_0(2^j x)$. For $k \in \mathbb{Z}$, let $S_k \in \mathcal{S}$ be defined via Fourier transform by

$$\widehat{S}_k(\xi) = 1 - \sum_{j \geq k+1} \widehat{\phi}_j(\xi),$$

For each $j \in \mathbb{Z}$, define the Fourier localization operator $\Delta_j : \mathcal{S}'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^2)$ by $\widehat{\Delta_j f} = \widehat{\phi}_j \widehat{f}$.

The Littlewood-Paley decomposition of f is defined by,

$$f := \sum_{j \in \mathbb{Z}} \Delta_j f \quad \text{in } \mathcal{S}'(\mathbb{R}^2)/\mathcal{P}, \qquad \mathcal{P} = \{\text{polynomials in } \mathbb{R}^2\}$$

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For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the **homogeneous and inhomogeneous Besov spaces are**

$$\dot{B}^s_{p,q}(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2)/\mathcal{P} : \|f\|_{\dot{B}^s_{p,q}} := \|\{2^{sj}\|\Delta_j f\|_{L^p}\}_{j\in\mathbb{Z}}\|_{l^q(\mathbb{Z})} < \infty \right\}$$

and

$$B^s_{p,q}(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2) : \|f\|_{B^s_{p,q}} := \|\{2^{sj}\|\Delta_j f\|_{L^p}\}_{j\in\mathbb{N}}\|_{l^q(\mathbb{N})} + \|S_0 * f\|_{L^p} < \infty \right\}.$$

The spaces $\dot{B}^s_{p,q}$ and $B^s_{p,q}$ endowed with $\|\cdot\|_{\dot{B}^s_{p,q}}$ and $\|\cdot\|_{B^s_{p,q}}$ are Banach spaces.
For $s > 0$, these norms are equivalent:

$$\|f\|_{B^s_{p,q}} \sim \|f\|_{\dot{B}^s_{p,q}} + \|f\|_{L^p}. \tag{2}$$

Bernstein's inequality and Embeddings to control $\|\nabla\theta\|_{L^\infty}$

Bernstein's ineq: Let $f \in L^p$, $1 \le p \le \infty$, be such that supp $\widehat{f} \subset \{\xi \in \mathbb{R}^2 : 2^{j-2} \le |\xi| < 2^j\}$. Then,

$$C^{-1} 2^{jk} \|f\|_{L^p} \le \|D^k f\|_{L^p} \le C 2^{jk} \|f\|_{L^p}, \qquad \text{where } C = C(k) > 0.$$

Using Bernstein's inequality one can prove the equivalence: $\|f\|_{\dot{B}^{s+k}_{p,q}} \sim \|D^k f\|_{\dot{B}^s_{p,q}}$.

Moreover, for $s > n/p$ with $1 \le p, q \le \infty$, or $s = n/p$ with $1 \le p \le \infty$ and $q = 1$, we then have

$$\|f\|_{L^\infty} \le C\|f\|_{B^s_{p,q}} \Rightarrow \|\nabla f\|_{L^\infty} \le C\|\nabla f\|_{B^{s-1}_{p,q}} \le C\|f\|_{B^s_{p,q}}.$$

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Commutator estimates

Let $1 < p < \infty$ and $1 \le q \le \infty$. There exists positive universal constants C_1 and C_2 such that

(*i*) with $s > 0$, $v_1 \in \dot{B}^s_{p,q}(\mathbb{R}^n)$, $v_2 \in \dot{B}^s_{p,q}(\mathbb{R}^n)$, $\nabla v_1 \in L^\infty(\mathbb{R}^n)$, $\nabla \cdot v_1 = 0$, $\nabla v_2 \in L^\infty(\mathbb{R}^n)$

$$\left(\sum_{j\in\mathbb{Z}} 2^{sjq}\|[v_1\cdot\nabla, \Delta_j]v_2\|^q_{L^p}\right)^{1/q} \le C_1 \left(\|\nabla v_1\|_{L^\infty}\|v_2\|_{\dot{B}^s_{p,q}} + \|\nabla v_2\|_{L^\infty}\|v_1\|_{\dot{B}^s_{p,q}}\right).$$

(*ii*) with $s > -1$, $v_1 \in \dot{B}^{s+1}_{p,q}(\mathbb{R}^n)$, $v_2 \in \dot{B}^s_{p,q}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\nabla v_1 \in L^\infty(\mathbb{R}^n)$ and $\nabla \cdot v_1 = 0$

$$\left(\sum_{j\in\mathbb{Z}} 2^{sjq}\|[v_1\cdot\nabla, \Delta_j]v_2\|^q_{L^p}\right)^{1/q} \le C_2 \left(\|\nabla v_1\|_{L^\infty}\|v_2\|_{\dot{B}^s_{p,q}} + \|v_2\|_{L^\infty}\|v_1\|_{\dot{B}^{s+1}_{p,q}}\right).$$

No dispersive effect on energy, analogy with the Coriolis term

Since $|\hat{\theta}(\xi)|^2 = \hat{\theta}(\xi)\hat{\theta}(-\xi)$, by Plancherel's formula we have

$$\int_{\mathbb{R}^2} \Lambda^s \mathcal{R}_1 \theta(x) \Lambda^s \theta(x) \, dx = \langle \Lambda^s \mathcal{R}_1 \theta, \Lambda^s \theta \rangle_{\mathbb{R}^2} = \int_{\mathbb{R}^2} i \, \xi_1 |\xi|^{2s-1} |\hat{\theta}(\xi)|^2 \, d\xi = 0.$$

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Strichartz estimates from Elgindi-Widmayer (generalized by Wan-Chen)

Let $4 \le \gamma \le \infty$ and $2 \le r \le \infty$ be such that $\frac{1}{\gamma} + \frac{1}{2r} \le \frac{1}{4}$. Then,

$$\|e^{\mathcal{R}_1 t} f\|_{L^\gamma(\mathbb{R}^+, L^r(\mathbb{R}^2))} \le C \|f\|_{\dot{B}^{1-\frac{2}{r}}_{2,1}(\mathbb{R}^2)}, \qquad \text{for all } f \in \dot{B}^{1-\frac{2}{r}}_{2,1}(\mathbb{R}^2).$$

By the change of variable $t \to At$, we get $\|e^{\mathcal{R}_1 At} f\|_{L^\gamma(\mathbb{R}^+, L^r(\mathbb{R}^2))} \le C|A|^{-\frac{1}{\gamma}} \|f\|_{\dot{B}^{1-\frac{2}{r}}_{2,1}(\mathbb{R}^2)}$.

Furthermore, let $s, t, A \in \mathbb{R}$, then: $\left\|e^{\pm tA\mathcal{R}_1} f\right\|_{L^\gamma(0,\infty; \dot{B}^s_{r,q})} \le C|A|^{-\frac{1}{\gamma}} \|f\|_{\dot{B}^{s+1-\frac{2}{r}}_{2,q}}$.

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Parabolic regularization for Kato-Fujita mild formulation

$$\textit{Regularized DSQG}: \begin{cases} \partial_t \theta^\varepsilon + (u^\varepsilon \cdot \nabla)\theta^\varepsilon + A u_2^\varepsilon - \varepsilon \Delta \theta^\varepsilon = 0, \\ u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) = (-\mathcal{R}_2 \theta^\varepsilon, \mathcal{R}_1 \theta^\varepsilon), \\ \theta^\varepsilon(0, x) = \theta_0(x),\ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \qquad \theta_0 \in B^s_{2,q}(\mathbb{R}^2) \end{cases} \tag{3}$$

$$\theta^\varepsilon(t) = e^{\varepsilon t \Delta} \theta_0 - \int_0^t e^{\varepsilon(t-\tau)\Delta} A u_2^\varepsilon(\tau)\, d\tau - \int_0^t e^{\varepsilon(t-\tau)\Delta} (u^\varepsilon(\tau) \cdot \nabla) \theta^\varepsilon(\tau)\, d\tau. \tag{4}$$

Smoothing estimates

Let $1 \le p, q \le \infty$ and $s_0 \le s_1$. Then, for all $f \in B^{s_0}_{p,q}(\mathbb{R}^n)$, there is a universal constant $C > 0$ s.t.

$$\|e^{t\Delta} f\|_{B^{s_1}_{p,q}} \le C(1 + t^{-\frac{1}{2}(s_1 - s_0)})\|f\|_{B^{s_0}_{p,q}},$$

Lemma: Smoothing the dispersive term

Let $0 < \varepsilon < 1$, $0 < T \le \infty$, $1 \le p, q \le \infty$, $A \in \mathbb{R}$ and $s \in \mathbb{R}$. $\exists\, C_1, C_2 > 0$ such that

$$(i) \quad \sup_{0 \le t \le T} \left\| \int_0^t e^{\varepsilon(t-\tau)\Delta} A v(\tau)\ d\tau \right\|_{B^s_{p,q}} \le C_1\, |A|\, T \sup_{0 \le t \le T} \|v(t)\|_{B^s_{p,q}}, \quad \forall v \in C([0, T]; B^s_{p,q}(\mathbb{R}^2)).$$

$$(ii) \quad \left\| \int_0^t e^{\varepsilon(t-\tau)\Delta} A v(\tau)\ d\tau \right\|_{L^1(0,T;B^{s+1}_{p,q})} \le C_2\, |A|\, T\, \|v(t)\|_{L^1(0,T;B^{s+1}_{p,q})}, \quad \forall v \in L^1(0, T; B^{s+1}_{p,q}(\mathbb{R}^2)).$$

Lemma: Smoothing the Nonlinear term

Let $0 < \varepsilon < 1$, $0 < T \le \infty$ and $1 \le p \le \infty$. Assume $s > \frac{2}{p}$ with $1 \le q \le \infty$ or $s = \frac{2}{p}$ with $q = 1$.

$$(i) \quad \sup_{0 \le t \le T} \left\| \int_0^t e^{\varepsilon(t-\tau)\Delta} (u(\tau) \cdot \nabla)\, \theta(\tau)\ d\tau \right\|_{B^s_{p,q}} \le C \sup_{0 \le t \le T} \|u(t)\|_{B^s_{p,q}}\, \|\theta\|_{L^1(0,T;B^{s+1}_{p,q})}$$

$$(ii) \quad \left\| \int_0^t e^{\varepsilon(t-\tau)\Delta} (u(\tau) \cdot \nabla)\, \theta(\tau)\ d\tau \right\|_{L^1(0,T;B^{s+1}_{p,q})} \le C(T + \varepsilon^{-\frac{1}{2}} T^{\frac{1}{2}}) \sup_{0 \le t \le T} \|u(t)\|_{B^s_{p,q}}\, \|\theta\|_{L^1(0,T;B^{s+1}_{p,q})}$$

for all $u \in C([0, T]; B^s_{p,q}(\mathbb{R}^2))$ and $\theta \in L^1(0, T; B^{s+1}_{p,q}(\mathbb{R}^2))$.

Proposition (Local solution via regularization)

Let $\varepsilon \in (0,1)$ and $A \in \mathbb{R}$. Assume $s > 2$ with $1 \le q \le \infty$ or $s = 2$ with $q = 1$, and $\theta_0 \in B^s_{2,q}(\mathbb{R}^2)$. Then, $\exists\, T = T(\|\theta_0\|_{B^s_{2,q}}) > 0$ such that (Regularized *DSQG*) has a unique strong solution θ^ε

$$\theta^\varepsilon \in C([0,T]; B^s_{2,q}(\mathbb{R}^2)) \cap AC([0,T]; B^{s-1}_{2,q}(\mathbb{R}^2)) \cap L^1(0,T; B^{s+1}_{2,q}(\mathbb{R}^2)).$$

Furthermore, $\{\theta^\varepsilon\}_{\varepsilon \in (0,1)}$ is bounded in $C([0,T]; B^s_{2,q}(\mathbb{R}^2))$, uniformly w.r.t. ε.

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Proof Sketch: (1) Show the map $\Gamma(\theta^\varepsilon)(t)$ is a contraction on the complete metric space W_T

$$\Gamma(\theta^\varepsilon)(t) = e^{\varepsilon t\Delta}\theta_0 - \int_0^t e^{\varepsilon(t-\tau)\Delta} A u_2^\varepsilon(\tau)\, d\tau - \int_0^t e^{\varepsilon(t-\tau)\Delta}(u^\varepsilon(\tau)\cdot\nabla)\theta^\varepsilon(\tau)\, d\tau$$

$$W_T = \left\{\theta \in C([0,T]; B^s_{p,q}) \cap L^1(0,T; B^{s+1}_{p,q}) : \|\theta\|_{W_T} \leq 2C_0\|\theta_0\|_{B^s_{p,q}}\right\},$$

$$\text{where } \|\theta\|_{W_T} := \sup_{0\leq t\leq T}\|\theta\|_{B^s_{p,q}} + (T + \varepsilon^{-\frac{1}{2}} T^{\frac{1}{2}})\|\theta\|_{L^1(0,T;B^{s+1}_{2,q})}.$$

This gives a unique solution $\theta^\varepsilon \in W_{T_{\varepsilon,A}}$ for Regularized $DSQG$.

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This gives a unique solution $\theta^\varepsilon \in W_{T_{\varepsilon,A}}$ for Regularized *DSQG*.

(2) Show $\theta^\varepsilon \in W_{T_{\varepsilon,A}}$ is strong in $C([0,T_{\varepsilon,A}]; B^s_{2,q}) \cap AC([0,T_{\varepsilon,A}]; B^{s-1}_{2,q}) \cap L^1(0,T_{\varepsilon,A}; B^{s+1}_{2,q})$.
For this, we see that for the function

$$\omega^\varepsilon(t) := -\int_0^t e^{\varepsilon(t-\tau)\Delta}\left\{A u_2^\varepsilon(\tau) + (u^\varepsilon(\tau)\cdot\nabla)\theta^\varepsilon(\tau)\right\}\, d\tau$$

we have $\partial_t\omega^\varepsilon \in L^1(0,T_{\varepsilon,A}; B^{s-1}_{2,q}) \Rightarrow \omega^\varepsilon \in AC([0,T_{\varepsilon,A}]; B^{s-1}_{2,q})$.

Since $e^{\varepsilon t\Delta}\theta_0 \in AC([0,T_{\varepsilon,A}]; B^{s-1}_{2,q})$ this implies $\theta^\varepsilon \in AC([0,T_{\varepsilon,A}]; B^{s-1}_{2,q})$.

(3): Show the boundedness of $\theta^\varepsilon \in C([0,T]; B^s_{2,q}(\mathbb{R}^2))$. We have

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta^\varepsilon(t)\|^2_{L^2} + \varepsilon\langle -\Delta\Delta_j\theta^\varepsilon(t), \Delta_j\theta^\varepsilon(t)\rangle_{L^2} = -\langle \Delta_j(u^\varepsilon(t)\cdot\nabla)\theta^\varepsilon(t), \Delta_j\theta^\varepsilon(t)\rangle_{L^2}, \tag{5}$$

where we used $\langle \Delta_j \mathcal{R}_1\theta^\varepsilon, \Delta_j\theta^\varepsilon\rangle_{L^2} = 0$ (by Plancherel's Theorem.)

Using $\nabla\cdot u^\varepsilon = 0$ and the definition of the commutator $[u^\varepsilon(t)\cdot\nabla, \Delta_j]$, we get

$$\frac{d}{dt}\|\Delta_j\theta^\varepsilon(t)\|_{L^2} \le \|[u^\varepsilon(t)\cdot\nabla, \Delta_j]\Delta_j\theta^\varepsilon(t)\|_{L^2}. \tag{6}$$

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$$\begin{aligned}\Rightarrow \frac{d}{dt}\|\theta^\varepsilon(t)\|_{\dot{B}^s_{2,q}} &\le \left(\sum_{j\in\mathbb{Z}} 2^{sjq}\left(\|[u^\varepsilon(t)\cdot\nabla, \Delta_j]\Delta_j\theta^\varepsilon(t)\|_{L^2}\right)^q\right)^{\frac{1}{q}}\\ &\le C\left(\|\nabla u^\varepsilon(t)\|_{L^\infty}\|\theta^\varepsilon(t)\|_{\dot{B}^s_{2,q}} + \|\nabla\theta^\varepsilon(t)\|_{L^\infty}\|u^\varepsilon(t)\|_{\dot{B}^s_{2,q}}\right)\\ &\le C\left(\|u^\varepsilon(t)\|_{B^s_{2,q}}\|\theta^\varepsilon(t)\|_{\dot{B}^s_{2,q}} + \|\theta^\varepsilon(t)\|_{B^s_{2,q}}\|u^\varepsilon(t)\|_{\dot{B}^s_{2,q}}\right) \le C\|\theta^\varepsilon(t)\|^2_{B^s_{2,q}}.\end{aligned} \tag{7}$$

This yields

$$\|\theta^\varepsilon(t)\|_{B^s_{2,q}} \le \frac{\|\theta_0\|_{B^s_{2,q}}}{1 - C\|\theta_0\|_{B^s_{2,q}}t} \text{ for all } 0 \le t < \frac{1}{C\|\theta_0\|_{B^s_{2,q}}}.$$

Taking $T = T\left(\|\theta_0\|_{B^s_{2,q}}\right) = \frac{1}{2C\|\theta_0\|_{B^s_{2,q}}}$ we get: $\|\theta^\varepsilon(t)\|_{B^s_{2,q}} \le 2\|\theta_0\|_{B^s_{2,q}}$ for all $0 \le t \le T$.

(3): Show the boundedness of $\theta^\varepsilon \in C([0,T]; B^s_{2,q}(\mathbb{R}^2))$. We have

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\Rightarrow \frac{d}{dt}\|\theta^\varepsilon(t)\|_{\dot{B}^s_{2,q}} &\le \left(\sum_{j\in\mathbb{Z}} 2^{sjq}\left(\|[u^\varepsilon(t)\cdot\nabla, \Delta_j]\Delta_j\theta^\varepsilon(t)\|_{L^2}\right)^q\right)^{\frac{1}{q}} \\
&\le C\left(\|\nabla u^\varepsilon(t)\|_{L^\infty}\|\theta^\varepsilon(t)\|_{\dot{B}^s_{2,q}} + \|\nabla\theta^\varepsilon(t)\|_{L^\infty}\|u^\varepsilon(t)\|_{\dot{B}^s_{2,q}}\right) \\
&\le C\left(\|u^\varepsilon(t)\|_{B^s_{2,q}}\|\theta^\varepsilon(t)\|_{\dot{B}^s_{2,q}} + \|\theta^\varepsilon(t)\|_{B^s_{2,q}}\|u^\varepsilon(t)\|_{\dot{B}^s_{2,q}}\right) \le C\|\theta^\varepsilon(t)\|^2_{B^s_{2,q}}.
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If $T_{\varepsilon,A} < T$, we can take $T'_{\varepsilon,A} = T'_{\varepsilon,A}(\|u_0\|_{B^s_{2,q}}) > 0$ sufficiently small and obtain a solution with initial data $\theta^\varepsilon(T_{\varepsilon,A}) \in B^s_{2,q}(\mathbb{R}^2)$ on the interval $[T_{\varepsilon,A}, T_{\varepsilon,A} + T'_{\varepsilon,A}]$.

Thus, the solution θ^ε can be extended to $[0, T_{\varepsilon,A} + T'_{\varepsilon,A}]$ and the same argument can be repeated **to obtain a solution θ^ε on $[0,T]$ verifying (??).**

Proof of local solvability, item (i) of Main Theorem: we want to show

$$\exists\, \theta \in C([0,T]; B^s_{2,q}(\mathbb{R}^2)) \text{ such that } \theta^\varepsilon(t) \to \theta(t) \text{ in } B^s_{2,q} \text{ uniformly for } t \in [0,T]. \tag{8}$$

Consider an auxiliary system for $\theta^{\varepsilon_1} - \theta^{\varepsilon_2}$ with $0 < \varepsilon_1 < \varepsilon_2 < 1$ given by

$$\begin{cases} \partial_t(\theta^{\varepsilon_1} - \theta^{\varepsilon_2}) - \varepsilon_1 \Delta(\theta^{\varepsilon_1} - \theta^{\varepsilon_2}) + (\varepsilon_2 - \varepsilon_1)\Delta\theta^{\varepsilon_2} = -A(u_2^{\varepsilon_1} - u_2^{\varepsilon_2}) - ((u^{\varepsilon_1} - u^{\varepsilon_2}) \cdot \nabla)\theta^{\varepsilon_1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - (u^{\varepsilon_2} \cdot \nabla)(\theta^{\varepsilon_1} - \theta^{\varepsilon_2}), \\ u^{\varepsilon_1} = (-\mathcal{R}_2\theta^{\varepsilon_1}, \mathcal{R}_1\theta^{\varepsilon_1}), \quad u^{\varepsilon_2} = (-\mathcal{R}_2\theta^{\varepsilon_2}, \mathcal{R}_1\theta^{\varepsilon_2}), \quad (\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(0,x) = 0. \end{cases} \tag{9}$$

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Energy method:, using $\nabla \cdot u^{\varepsilon_1} = \nabla \cdot u^{\varepsilon_2} = 0$, Cauchy-Schwartz ineq, continuity of $\mathcal{R}_j$ in L^2, $B^s_{2,q} \hookrightarrow W^{1,\infty}$, and Gronwall's lemma yield

$$\sup_{0 \le t \le T} \|(\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t)\|_{L^2} \le C\varepsilon_2 T \|\theta_0\|_{B^s_{2,q}} \exp\left(C\|\theta_0\|_{B^s_{2,q}} T\right) \to 0, \qquad \text{as } \varepsilon_2 \to 0^+ \tag{10}$$

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Let $0 < \gamma < 1$ and $s_1, s_2, s_3 \ge 0$ be such that $s_3 = (1-\gamma)s_1 + \gamma s_2$. By interpolation, we estimate

$$\|(\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t)\|_{B^{s_3}_{2,q}} \le C\|(\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t)\|^{1-\gamma}_{B^{s_1}_{2,2}} \|(\theta^{\varepsilon_1} - \theta^{\varepsilon_2})(t)\|^{\gamma}_{B^{s_2}_{2,q}}. \tag{11}$$

Taking $s_1 = 0$, $s_2 = s$ and $s_3 = \theta\, s_2$ in **(??)**, $\|\theta^\varepsilon(t)\|_{B^s_{2,q}} \le 2\|\theta_0\|_{B^s_{2,q}}$, $B^0_{2,2} = L^2$ and **(??)**,

$$\|\theta^{\varepsilon_1} - \theta^{\varepsilon_2}\|_{L^\infty(0,T;B^{s_3}_{2,q})} \le C\, \|\theta_0\|^{\gamma}_{B^s_{2,q}}\, \|\theta^{\varepsilon_1} - \theta^{\varepsilon_2}\|^{1-\gamma}_{L^\infty(0,T;L^2)} \to 0, \text{ as } \varepsilon_2 \to 0^+,$$

It follows that $\theta^\varepsilon \to \theta$ in $L^\infty(0,T; B^{\tilde{s}}_{2,q})$ for all $0 < \tilde{s} < s$.
We deduce the convergence **(??)** by considering $\tilde{s} = s - 1$ and using that $\theta^\varepsilon \in C([0,T]; B^{s-1}_{2,q}(\mathbb{R}^2))$.

Moreover, since $\{\theta^\varepsilon\}_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(0,T;B^s_{2,q}(\mathbb{R}^2))$ we also obtain,

$$\theta \in L^\infty(0,T;B^s_{2,q}(\mathbb{R}^2)) \cap C([0,T];B^{s-1}_{2,q}(\mathbb{R}^2)), \text{ and } \|\theta\|_{L^\infty(0,T;B^s_{2,q})} \le 2\|\theta_0\|_{B^s_{2,q}}. \tag{12}$$

One can then show **θ satisfies *DSQG* locally because as $\varepsilon \to 0^+$**

$$\begin{aligned}
&\int_0^t (u^\varepsilon(\tau)\cdot\nabla)\theta^\varepsilon(\tau)\, d\tau \to \int_0^t (u(\tau)\cdot\nabla)\theta(\tau)\, d\tau \ \text{ in } \ L^\infty(0,T;B^{s-2}_{2,q}),\\
&\varepsilon\int_0^t \|-\Delta\theta^\varepsilon(\tau)\|_{B^{s-2}_{2,q}}\, d\tau \le \varepsilon\int_0^t \|\theta^\varepsilon(\tau)\|_{B^s_{2,q}}\, d\tau \le C\varepsilon T\|\theta_0\|_{B^s_{2,q}} \to 0\\
&\int_0^t \|A(u_2^\varepsilon - u_2)(\tau)\|_{B^{s-2}_{2,q}}\, d\tau \le CT|A| \sup_{0\le t\le T} \|(\theta^\varepsilon-\theta)(t)\|_{B^{s-1}_{2,q}} \to 0,
\end{aligned} \tag{13}$$

We then have that θ is a solution in $AC([0,T];B^{s-1}_{2,q}(\mathbb{R}^2)) \cap L^\infty(0,T;B^s_{2,q}(\mathbb{R}^2))$.

Moreover, since $\{\theta^\varepsilon\}_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(0,T;B^s_{2,q}(\mathbb{R}^2))$ we also obtain,

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One can then show θ **satisfies** ***DSQG*** **locally because as** $\varepsilon \to 0^+$

$$\begin{aligned} &\int_0^t (u^\varepsilon(\tau)\cdot\nabla)\theta^\varepsilon(\tau)\, d\tau \to \int_0^t (u(\tau)\cdot\nabla)\theta(\tau)\, d\tau \text{ in } L^\infty(0,T;B^{s-2}_{2,q}),\\ &\varepsilon\int_0^t \|-\Delta\theta^\varepsilon(\tau)\|_{B^{s-2}_{2,q}}\, d\tau \leq \varepsilon\int_0^t \|\theta^\varepsilon(\tau)\|_{B^s_{2,q}}\, d\tau \leq C\varepsilon T\|\theta_0\|_{B^s_{2,q}} \to 0\\ &\int_0^t \|A(u_2^\varepsilon - u_2)(\tau)\|_{B^{s-2}_{2,q}}\, d\tau \leq CT|A| \sup_{0\leq t\leq T}\|(\theta^\varepsilon-\theta)(t)\|_{B^{s-1}_{2,q}} \to 0, \end{aligned} \tag{13}$$

We then have that θ is a solution in $AC([0,T];B^{s-1}_{2,q}(\mathbb{R}^2)) \cap L^\infty(0,T;B^s_{2,q}(\mathbb{R}^2))$.

To see that $\theta \in C([0,T];B^s_{2,q}(\mathbb{R}^2))$ for initial data $\theta_0 \in B^s_{2,q}(\mathbb{R}^2)$, we use the approximation $w_k := S_k\theta$ for each $k \in \mathbb{N}$, and show $\{w_k\}_{k\in\mathbb{N}}$ converges to θ in $L^\infty(0,T;B^{s-1}_{2,q}(\mathbb{R}^2))$.

Using the commutator estimates one can get $\partial_t\theta \in L^\infty(0,T;B^{s-1}_{2,q}(\mathbb{R}^2))$. Therefore,

$$\theta \in W^{1,\infty}([0,T];B^{s-1}_{2,q}(\mathbb{R}^2)) \subset C([0,T];B^{s-1}_{2,q}(\mathbb{R}^2)).$$

The uniqueness follows by a standard argument.

Blow-up criterion: Assume $\theta_0 \in B^s_{2,q}(\mathbb{R}^2)$ and θ is the corresponding solution of $(DSQG)$ in $C([0,T);B^s_{2,q}(\mathbb{R}^2)) \cap C^1([0,T);B^{s-1}_{2,q}(\mathbb{R}^2))$ satisfying $\int_0^T \|\nabla u(\tau)\|_{L^\infty}\,d\tau < \infty$. **Then,** $\exists\, T' > T$ **s.t.** θ **can be extended to** $[0,T')$ **with** $\theta \in C([0,T');B^s_{2,q}(\mathbb{R}^2)) \cap C^1([0,T');B^{s-1}_{2,q}(\mathbb{R}^2))$.

Proof. Applying Δ_j in $(DSQG)$, multiplying by $\Delta_j\theta$ and using $\langle((u(t)\cdot\nabla)\Delta_j\theta, \Delta_j\theta\rangle_{L^2} = 0$,

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta(t)\|^2_{L^2} = -\langle(\Delta_j(u(t)\cdot\nabla)\theta(t), \Delta_j\theta(t)\rangle_{L^2} = \langle[(u(t)\cdot\nabla),\Delta_j]\theta(t), \Delta_j\theta(t)\rangle_{L^2}.$$

Multiplying by 2^{sj}, taking the $l^q(\mathbb{Z})$-norm and Integrating over $(0,t)$, we have

$$\|\theta(t)\|_{\dot{B}^s_{2,q}} \leq \|\theta_0\|_{\dot{B}^s_{2,q}} + \int_0^t \left(\sum_{j\in\mathbb{Z}} 2^{sjq}\,\|[(u(t)\cdot\nabla),\Delta_j]\theta(\tau)\|^q_{L^2}\right)^{\frac{1}{q}} d\tau.$$

Blow-up criterion: Assume $\theta_0 \in B^s_{2,q}(\mathbb{R}^2)$ and θ is the corresponding solution of $(DSQG)$ in $C([0,T); B^s_{2,q}(\mathbb{R}^2)) \cap C^1([0,T); B^{s-1}_{2,q}(\mathbb{R}^2))$ satisfying $\int_0^T \|\nabla u(\tau)\|_{L^\infty}\, d\tau < \infty$. **Then, $\exists\, T' > T$ s.t. θ can be extended to $[0,T')$ with $\theta \in C([0,T'); B^s_{2,q}(\mathbb{R}^2)) \cap C^1([0,T'); B^{s-1}_{2,q}(\mathbb{R}^2))$.**

Proof. Applying Δ_j in $(DSQG)$, multiplying by $\Delta_j\theta$ and using $\langle ((u(t)\cdot\nabla)\Delta_j\theta, \Delta_j\theta\rangle_{L^2} = 0$,

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta(t)\|^2_{L^2} = -\langle (\Delta_j(u(t)\cdot\nabla)\theta(t), \Delta_j\theta(t)\rangle_{L^2} = \langle [(u(t)\cdot\nabla), \Delta_j]\theta(t), \Delta_j\theta(t)\rangle_{L^2}.$$

Multiplying by 2^{sj}, taking the $l^q(\mathbb{Z})$-norm and Integrating over $(0,t)$, we have

$$\|\theta(t)\|_{\dot{B}^s_{2,q}} \le \|\theta_0\|_{\dot{B}^s_{2,q}} + \int_0^t \left(\sum_{j\in\mathbb{Z}} 2^{sjq}\, \|[(u(t)\cdot\nabla), \Delta_j]\theta(\tau)\|^q_{L^2}\right)^{\frac{1}{q}} d\tau.$$

Since $\|\theta(t)\|^2_{L^2} = \|\theta_0\|^2_{L^2}$, using the **commutator estimates**, $\exists\, C > 0$ such that

$$\|\theta(t)\|_{B^s_{2,q}} \le \|\theta_0\|_{B^s_{2,q}} + C\int_0^t \|\nabla\theta(\tau)\|_{L^\infty}\|u(\tau)\|_{B^s_{2,q}} + \|\nabla u(\tau)\|_{L^\infty}\|\theta(\tau)\|_{B^s_{2,q}}\, d\tau.$$

By $B^s_{2,q} \hookrightarrow W^{1,\infty}$, $\|\theta\|_{L^\infty(0,T;B^s_{2,q})} \le 2\|\theta_0\|_{B^s_{2,q}}$ and the continuity of $\mathcal{R}_j$ in $B^s_{2,q}$, $\exists\, C_3, C_4 > 0$

$$\|\theta(t)\|_{B^s_{2,q}} \le \|\theta_0\|_{B^s_{2,q}} + C_3 t\|\theta_0\|^2_{B^s_{2,q}} + C_4\int_0^t \|\nabla u(\tau)\|_{L^\infty}\|\theta(\tau)\|_{B^s_{2,q}}\, d\tau.$$

By Gronwall's inequality: $\|\theta(t)\|_{B^s_{2,q}} \le \|\theta_0\|_{B^s_{2,q}}\left(1 + C_3 T\|\theta_0\|_{B^s_{2,q}}\right)\exp\left\{C_4\int_0^T \|\nabla u(\tau)\|_{L^\infty}\, d\tau\right\}$ for all $t \in [0,T)$. By standard arguments, using $\int_0^T \|\nabla u(\tau)\|_{L^\infty}\, d\tau < \infty$, we are done.

Proof of long-time solvability, item (ii) of Main Theorem: Let $\theta_0 \in B^{s+1}_{2,q}(\mathbb{R}^2)$, with the solution

$$\theta \in C([0, T^*); B^{s+1}_{2,q}(\mathbb{R}^2)) \cap C^1([0, T^*); B^{s}_{2,q}(\mathbb{R}^2)) \text{ with maximal existence time } T^* > 0.$$

If $T^* = \infty$, we are done. Assume that $T^* < \infty$. By Duhamel's principle, we have

$$\theta(t) = e^{A\mathcal{R}_1 t}\theta_0 - \int_0^t e^{A\mathcal{R}_1(\tau - t)}(u \cdot \nabla\theta)(\tau)\, d\tau.$$

For $0 \le t \le T^*$, define $\mathcal{M}(t) := \int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau$. Consider the case $s = 2$ with $q = 1$, we have

$$\mathcal{M}(t) \le C \max_{l=1,2} \int_0^t \|\mathcal{R}_l e^{A\mathcal{R}_1 \tau}\theta_0\|_{\dot{B}^1_{\infty,1}}\, d\tau + C \max_{l=1,2} \int_0^t \left\| \int_0^\tau \mathcal{R}_l e^{A\mathcal{R}_1(\tau' - \tau)}(u(\tau') \cdot \nabla)\theta(\tau')\, d\tau' \right\|_{\dot{B}^1_{\infty,1}} d\tau := K_1 + K_2.$$

Proof of long-time solvability, item (ii) of Main Theorem: Let $\theta_0 \in B^{s+1}_{2,q}(\mathbb{R}^2)$, with the solution

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Using the **Strichartz estimate with** $r=\infty$, Hölder's inequality and the continuity of $\mathcal{R}_l$

$$\begin{aligned} K_1 &= C\sum_{j\in\mathbb{Z}} 2^j \int_0^t \max_{l=1,2}\|\mathcal{R}_l e^{A\mathcal{R}_1\tau}\Delta_j\theta_0\|_{L^\infty}\, d\tau \le Ct^{1-\frac{1}{\gamma}} \sum_{j\in\mathbb{Z}} 2^j \max_{l=1,2}\|e^{A\mathcal{R}_1\tau}\Delta_j\mathcal{R}_l\theta_0\|_{L^\gamma(\mathbb{R};L^\infty)} \\ &\le C\, t^{1-\frac{1}{\gamma}} A^{-\frac{1}{\gamma}} \sum_{j\in\mathbb{Z}} 2^j \max_{l=1,2}\|\Delta_j\mathcal{R}_l\theta_0\|_{\dot{B}^1_{2,1}} \le C\, t^{1-\frac{1}{\gamma}} A^{-\frac{1}{\gamma}} \max_{l=1,2}\|\mathcal{R}_l\theta_0\|_{\dot{B}^2_{2,1}} \le C\, t^{1-\frac{1}{\gamma}} A^{-\frac{1}{\gamma}} \|\theta_0\|_{B^3_{2,1}}. \end{aligned}$$

$$\begin{aligned} K_2 &\le C\max_{l=1,2}\sum_{j\in\mathbb{Z}} 2^j \int_0^t\int_0^\tau \|\mathcal{R}_l e^{A\mathcal{R}_1(\tau'-\tau)}\Delta_j(u(\tau')\cdot\nabla)\theta(\tau')\|_{L^\infty}\, d\tau'\, d\tau \\ &\le Ct^{1-\frac{1}{\gamma}} \max_{l=1,2}\sum_{j\in\mathbb{Z}} 2^j \int_0^t \|e^{A\mathcal{R}_1(\tau'-\tau)}\mathcal{R}_l\Delta_j(u(\tau')\cdot\nabla)\theta(\tau')\|_{L^\gamma(\tau',t;L^\infty_x)}\, d\tau' \\ &\le C\, t^{1-\frac{1}{\gamma}} A^{-\frac{1}{\gamma}} \int_0^t \max_{l=1,2}\sum_{j\in\mathbb{Z}} 2^j \|\mathcal{R}_l\Delta_j(u(\tau')\cdot\nabla)\theta(\tau')\|_{\dot{B}^1_{2,1}}\, d\tau' \le C\, t^{1-\frac{1}{\gamma}} A^{-\frac{1}{\gamma}} \int_0^t \|\theta(\tau')\|^2_{B^3_{2,1}}\, d\tau'. \end{aligned}$$

The last inequality can be shown using Bony's paraproduct and the continuity of $\mathcal{R}_l$.

Thus, for each $0 < t < T^*$, we have

$$\mathcal{M}(t) \leq Ct^{1-\frac{1}{\gamma}} A^{-\frac{1}{\gamma}} \left(\|\theta_0\|_{B^{s+1}_{2,q}} + \int_0^t \|\theta(\tau')\|^2_{B^{s+1}_{2,q}}\, d\tau' \right)$$

$$\leq Ct^{1-\frac{1}{\gamma}} A^{-\frac{1}{\gamma}} \left(\|\theta_0\|_{B^{s+1}_{2,q}} + \|\theta_0\|^2_{B^{s+1}_{2,q}} \left(1 + C_3 T \|\theta_0\|_{B^{s+1}_{2,q}}\right)^2 \int_0^t e^{C_4 \mathcal{M}(\tau')} d\tau' \right)$$

$$\leq Ct^{1-\frac{1}{\gamma}} A^{-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}} \left(1 + \|\theta_0\|_{B^{s+1}_{2,q}} \left(1 + C_3 T \|\theta_0\|_{B^{s+1}_{2,q}}\right)^2 t e^{C_4 \mathcal{M}(t)} \right).$$

Thus, for each $0 < t < T^*$, we have

$$\begin{aligned}\mathcal{M}(t) &\leq Ct^{1-\frac{1}{\gamma}} A^{-\frac{1}{\gamma}} \left(\|\theta_0\|_{B^{s+1}_{2,q}} + \int_0^t \|\theta(\tau')\|^2_{B^{s+1}_{2,q}}\, d\tau' \right) \\ &\leq Ct^{1-\frac{1}{\gamma}} A^{-\frac{1}{\gamma}} \left(\|\theta_0\|_{B^{s+1}_{2,q}} + \|\theta_0\|^2_{B^{s+1}_{2,q}} \left(1 + C_3 T \|\theta_0\|_{B^{s+1}_{2,q}}\right)^2 \int_0^t e^{C_4 \mathcal{M}(\tau')} d\tau' \right) \\ &\leq Ct^{1-\frac{1}{\gamma}} A^{-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}} \left(1 + \|\theta_0\|_{B^{s+1}_{2,q}} \left(1 + C_3 T \|\theta_0\|_{B^{s+1}_{2,q}}\right)^2 t e^{C_4 \mathcal{M}(t)} \right).\end{aligned}$$

Next, for each $0 < T < \infty$ we define $\hat{T} = \sup D_T$, where

$$D_T = \{t \in [0, T] \cap [0, T^*) \mid \mathcal{M}(t) \leq C_5 T^{1-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}}\}.$$

We first show that $\hat{T} = \min\{T, T^*\}$. We proceed by contradiction.

Assume to the contrary that $\hat{T} < \min\{T, T^*\}$. Then $\exists\, T_1$ such that $\hat{T} < T_1 < \min\{T, T^*\}$. It follows that $\theta \in C([0, T_1]; B^{s+1}_{2,q}(\mathbb{R}^2))$, $\mathcal{M}(t)$ is uniformly continuous on $[0, T_1]$ and

$$\mathcal{M}(\hat{T}) \leq C_5\, T^{1-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}}. \tag{14}$$

We now take $A > 0$ large enough so that

$$A^{\frac{1}{\gamma}} \geq 2 \left(1 + \|\theta_0\|_{B^{s+1}_{2,q}} \left(1 + C_3 T \|\theta_0\|_{B^{s+1}_{2,q}}\right)^2 T \exp\left(C_4 C_5 T^{1-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}} \right) \right). \tag{15}$$

Using (13), (**??**) and (**??**), we can estimate

$$\begin{aligned}\mathcal{M}(\hat{T}) &\le C_5(\hat{T})^{1-\frac{1}{\gamma}} A^{-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}} \left(1 + \|\theta_0\|_{B^{s+1}_{2,q}} \left(1 + C_3 \hat{T} \|\theta_0\|_{B^{s+1}_{2,q}}\right)^2 \hat{T} \exp\left(C_4 \mathcal{M}(\hat{T})\right)\right) \\ &\le C_5 T^{1-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}} A^{-\frac{1}{\gamma}} \left(1 + \|\theta_0\|_{B^{s+1}_{2,q}} \left(1 + C_3 T \|\theta_0\|_{B^{s+1}_{2,q}}\right)^2 T \exp\left(C_4 C_5 T^{1-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}}\right)\right) \\ &\le \frac{1}{2} C_5 T^{1-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}}. \end{aligned} \tag{16}$$

Thus, we can choose T_2 such that $\hat{T} < T_2 < T_1$ with $\mathcal{M}(T_2) \le C_5 T^{1-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}}$.

This contradicts the definition of $\hat{T}$.

Using (13), (**??**) and (**??**), we can estimate

$$\begin{aligned}
\mathcal{M}(\hat{T}) &\leq C_5(\hat{T})^{1-\frac{1}{\gamma}} A^{-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}} \left(1 + \|\theta_0\|_{B^{s+1}_{2,q}} \left(1 + C_3 \hat{T} \|\theta_0\|_{B^{s+1}_{2,q}}\right)^2 \hat{T} \exp\left(C_4 \mathcal{M}(\hat{T})\right)\right) \\
&\leq C_5 T^{1-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}} A^{-\frac{1}{\gamma}} \left(1 + \|\theta_0\|_{B^{s+1}_{2,q}} \left(1 + C_3 T \|\theta_0\|_{B^{s+1}_{2,q}}\right)^2 T \exp\left(C_4 C_5 T^{1-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}}\right)\right) \\
&\leq \frac{1}{2} C_5 T^{1-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}}. \qquad (16)
\end{aligned}$$

Thus, we can choose T_2 such that $\hat{T} < T_2 < T_1$ with $\mathcal{M}(T_2) \leq C_5 T^{1-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}}$.

This contradicts the definition of $\hat{T}$.

It follows that $\hat{T} = \min\{T, T^*\}$ when A verifies (**??**).

If $T^* < T$, we have that $T^* = \hat{T} = \sup D_T$ and

$$\mathcal{M}(t) = \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \leq C_5 T^{1-\frac{1}{\gamma}} \|\theta_0\|_{B^{s+1}_{2,q}} < \infty, \text{ for all } 0 \leq t < T^*.$$

It follows that $\mathcal{M}(T^*) < \infty$ which contradicts the maximality of T^* because of the blow-up criterion. This concludes the proof.

GRACIAS!