Control Óptimo para Ecuaciones No Lineales de tipo Schrödinger

IMAS-CONICET y DM, FCEyN, UBA

22 de mayo de 2018

Constanza S. F. de la Vega

The equations

- Bilinear control

$$\begin{cases} i\partial_t u = -\Delta u + V_0(x)u + \lambda|u|^{2\sigma}u + Wu, & t \in [0, T], x \in \Omega \subset \mathbb{R}^n \\ u(0,x) = u_0(x) \text{ for all } x \in \Omega \end{cases}$$

- Distributed control

$$\begin{cases} i\partial_t u = -\Delta u + V_0(x)u + \lambda|u|^{2\sigma}u + h, & t \in [0, T], x \in \Omega \subset \mathbb{R}^n \\ u(0,x) = u_0(x) \text{ for all } x \in \Omega \end{cases}$$

Well posedness

T. Cazenave; "Semilinear Schrödinger Equation"; Courant Lecture Notes 10, AMS, 2003.

For $\Omega = \mathbb{R}^n$ we can solve the nonlinear problem

$$\begin{cases} i\partial_t u = -\Delta u + h(u), t \in [0,T], \, x \in \mathbb{R}^n \\ u(0,x) = u_0(x) \text{ for all } x \in \mathbb{R}^n \end{cases}$$

For the linear problem, $h(u) = 0$

- For $u_0 \in L^2$, there exists solution in $C(\mathbb{R}, L^2) \cap C^1(\mathbb{R}, H^{-2})$.
- Smoothing effect: For $u_0 \in L^2$, then $u(t) \in H^{1/2}_{loc}(\mathbb{R}^n)$ aa.
- The same results holds for $-\Delta + V_0$, with potentials $V_0 \in C^\infty(\mathbb{R}^n)$ nonnegative and subquadratic.

For $h(u) = Vu$, $h(u) = \lambda |u|^{2\sigma} u$ and h of Hartree type, we have local well posedness in H^1.

With time dependent potentials

R. Carles; "Nonlinear Schrödinger Equation with time dependent potential"; Communications Math. Sci. 2011.

Given the nonlinear equation

$$\begin{cases} i\partial_t u = -\Delta u + V(t,x)u + \lambda |u|^{2\sigma} u, t \in [0, T], \, x \in \mathbb{R}^n \\ u(0,x) = u_0(x) \end{cases}$$

$V(t) \in C^\infty(\mathbb{R}^n)$ locally bounded in time and subquadratic in space.

It is proved the global existence of solution in the energy space

$$\Sigma = \{u \in H^1(\mathbb{R}^n) : xu \in L^2(\mathbb{R}^n)\},$$

- For $\lambda \in \mathbb{R}$, $0 < \sigma < 2/n$.
- For $\lambda > 0$, $2/n \leq \sigma < 2/(n-2)$ and more regularity on V.

Optimal control problems

We will study the problem of proving the existence of a solution and first order necessary conditions for

$$\text{mín}\, \mathcal{J}(u, h)$$

subject to the condition that the state u is the solution of a type Schrödinger equation for a given control h.

Optimal control of quantum systems

A. Pierce and M. Dahleh; " Optimal control of quantum-mechanical systems: Existence, numerical approximation, and applications"; Physical Review A, 1988.

$$\text{mín}\, \|u(T) - \hat{u}\|^2_{L^2(\Omega)} + \alpha \|v\|^2_{L^2([0,T],\Omega\times\Omega)}$$

subject to

$$i\partial_t u = -\Delta u + (V_0 + W)u, \quad t \in [0, T], \, x \in \Omega$$

$$u(0) = u_0$$

- $V_0(x)$ is a potential for which $-\Delta + V_0$ generates a C_0 semigroup on $L^2(\Omega)$.
- W is a linear Hilbert Schmidt operator given by

$$W\, u(t,x) := \int_\Omega v(t,x,x')u(t,x')dx', \; v \in L^2([0, T], \Omega \times \Omega).$$

Laser control of chemical reactions

E. Cances, C. Le Bris and M. Pilot; "Control Optimal Bilineare d'une equation de Schrödinger"; C. R. Acad. Sci. Paris, 2000.

$$\text{mín}\, \|u(T) - \hat{u}\|^2_{L^2(\mathbb{R}^3)} + \alpha \|E\|^2_{L^2([0,T],\mathbb{R})}$$

subject to

$$i\partial_t u = -\Delta u - \frac{1}{|x|} u + \left(|u|^2 * \frac{1}{|x|} \right) u + (E(t)x)u, \quad t \in [0, T],\ x \in \mathbb{R}^3$$

$$u(0) = u_0$$

- Well posedness in $\Sigma = \{f \in H^2 : \sqrt{1+|x|^2} f \in L^2\}$.

Linear modelling of a hydrogen atom

L. Baudouin, O. Kavian and J.P. Puel; " Regularity for a Schrödinger equation with a singular potentials and application to bilinear optimal control"; J. Differential Equations, 2005.

$$\text{mín} \, \|u(T) - \hat{u}\|^2_{L^2(\mathbb{R}^3)} + \alpha \|V_1\|^2_{H^1(0,T;W)}$$

$$\text{s. t.} \quad i\partial_t u = -\Delta u + \frac{1}{|x - a(t)|} u + V_1(t,x)u, \quad t \in [0, T], \, x \in \mathbb{R}^3$$

$$u(0) = u_0$$

- $a \in W^{1,1}(0, T)$.
- V_1 y $\frac{\partial V_1}{\partial t}$ are subquadratic in space.
- Well posedness in $\Sigma = \{f \in H^2 : |x|^2 f \in L^2\}$.

Abstract linear Schrödinger equation

K. Ito and K. Kunisch; "Optimal Bilinear Control of an Abstract Schrödinger Equation"; SIAM Journal on Control and Optimization, 2007.

$$\begin{aligned} &\text{máx}\langle u(T), Au(T)\rangle - \alpha\|\mu\|^2_{L^2(0,T;\mathcal{L}(H))} \\ \text{s. t.}\quad & i\partial_t u = H_0 u - \mu(t)u, \quad t\in[0,T],\, x\in\Omega \\ & u(0) = u_0 \end{aligned}$$

- H_0 is densely defined self adjoint positive semidefinite operator in H real Hilbert.
- A is the observable operator (self adjoint positive definite) that encodes the goal.

BEC for dilute gases

M. Hintermuller, D. Marahrens, P. Markowich and C. Sparber; "Optimal Bilinear Control of Gross-Pitaevskii Equations"; SIAM Journal on Control and Optimization, 2013.

$$\text{mín}\langle u(T), Au(T)\rangle_{L^2(\mathbb{R}^n)} + \alpha_1 \int_0^T (\dot{E}(t))^2 dt + \alpha_2 \int_0^T (\dot{\alpha}(t))^2 dt$$

$$\text{s.t. } i\partial_t u = -\Delta u + U(x)u + \lambda |u|^{2\sigma} u + \alpha(t) V(x) u, \; t \in [0, T], \; x \in \mathbb{R}^n$$

$$u(0) = u_0 \qquad \text{for } n = 1, 2, 3$$

- $\lambda \geq 0$, $0 < \sigma < 2/(n-2)$, $\alpha_1 \geq 0$, $\alpha_2 > 0$.
- $U \in C^\infty(\mathbb{R}^n)$ subquadratic potential and $V \in W^{1,\infty}(\mathbb{R}^n)$.
- The energy space $\Sigma = \left\{u \in H^1(\mathbb{R}^n) : xu \in L^2(\mathbb{R}^n)\right\} \hookrightarrow L^2(\mathbb{R}^n)$.
- $E(t) = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla u(t)|^2 + \frac{\lambda}{\sigma+1}|u(t)|^{2\sigma+2} + (U(x) + \alpha(t)V(x))|u(t)|^2 dx$.
- $\dot{E}(t) = \dot{\alpha}(t) \int_{\mathbb{R}^n} V(x)|u(t,x)|^2 dx$.

Quantum control via external potentials

B. Feng, D. Zhao and P. Chen; "Optimal Bilinear Control of nonlinear Schrödinger equations with Singular Potentials"; SIAM Journal on Control and Optimization, 2013.

$$\text{mín}\langle u(T), Au(T)\rangle_{L^2(\mathbb{R}^n)} + \alpha_1 \int_0^T (\dot{E}(t))^2 dt + \alpha_2 \int_0^T (\dot{\alpha}(t))^2 dt$$

subject to

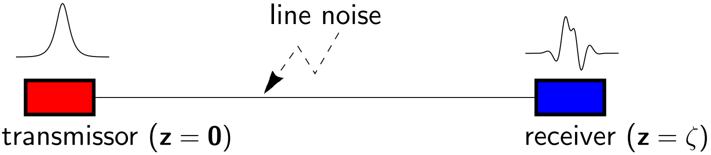
$$i\partial_t u = -\Delta u + \lambda |u|^{2\sigma} u + \alpha(t) V(x) u, t \in [0, T], \; x \in \mathbb{R}^n$$

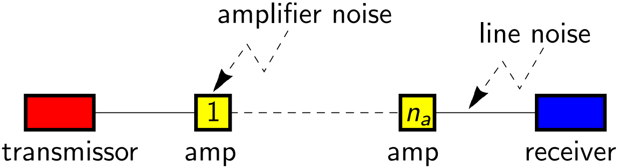
$$u(0) = u_0$$

- For $\lambda < 0$, $0 \leq \sigma < 2/(n-2)$ and for $\lambda > 0$, $0 \leq \sigma < 2/n$.
- $V \in L^p(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$.
- The energy space $H^1(\mathbb{R}^n)$.
- $E(t) = \int_{\mathbb{R}^n} \frac{1}{2} |\nabla u(t)|^2 + \frac{\lambda}{\sigma+1} |u(t)|^{2\sigma+2} + \alpha(t) V(x) |u(t)|^2 dx.$

Transmission line

R.O. Moore, G. Biondini, W.L. Kath, "A Method to Compute Statistics of Large, Noise-Induced Perturbations of Nonlinear Schrödinger Solitons", SIAM Review, 2008.





Transmission with noise

$$\begin{cases} \partial_z u = i\partial_t^2 u + i|u|^2 u + g, & z \in [0,\zeta],\ t \in \mathbb{R} \\ u(0,t) = u_0(t) \end{cases}$$

- $g \in L^2([0,\zeta], L^2(\mathbb{R}))$.
- $u_0 \in L^2(\mathbb{R})$, $|u_0|^2$ initial signal.
- The L^2 norm is not conserved:

$$\|u(z)\|_{L^2(\mathbb{R})}^2 = \|u_0\|_{L^2(\mathbb{R})}^2 + \int_0^z 2\mathrm{Re}\left(\langle u(z'), g(z')\rangle_{L^2(\mathbb{R})}\right) dz'$$

Signal error

- σ: temporal window, $\sigma(t) = \alpha e^{-\beta(t-T)^2}$, $T = \zeta/c$.
- Given u_0, v_0 initial data and u_ζ, v_ζ the solution without noise repectively we choose η such that

$$\int_{\mathbb{R}} \sigma^2(t) \left| u_\zeta(t) - v_\zeta(t) \right|^2 dt > 2\eta.$$

- Assume that the signal with initial data u_0 with noise is recognized if

$$\int_{\mathbb{R}} \sigma^2(t) \left| u[u_0, g](\zeta, t) - u_\zeta(t) \right|^2 dt \leq \eta.$$

- We say that an error occurs when

$$\int_{\mathbb{R}} \sigma^2(t) \left| u[u_0, g](\zeta, t) - v_\zeta(t) \right|^2 dt \leq \eta.$$

The optimal control problem

$$\text{mín}\, \kappa \|\sigma(u(\zeta) - v_\zeta)\|^2_{L^2(\mathbb{R})} + \|g\|^2_{L^2([0,\zeta],L^2(\mathbb{R}))}$$

subject to

- $g \in L^2([0,\zeta], L^2(\mathbb{R}))$.
- $u = u[g] \in C([0,\zeta], L^2(\mathbb{R}))$ is the solution of the nonlinear Schrödinger equation

$$\begin{cases} \partial_z u = i\partial_t^2 u + i|u|^2 u + g \\ u(0,t) = u_0(t). \end{cases}$$

- $\|\sigma(u[g](\zeta) - v_\zeta)\|^2_{L^2(\mathbb{R})} \leq \eta$.

The results

D. Rial, C. Sánchez de la Vega, "Optimal distributed control problem for the cubic nonlinear Schrödinger equation", sent for publication, 2017.

It is proved:

- Well posedness.
- Regularizing effect of the solution.
- Existence of minimizer.
- Fréchet differentiability of the solution with respect to the control.
- First order necessary conditions.

Integral equation

Given the equation

$$\begin{cases} \partial_z u = i\partial_t^2 u + i|u|^2 u + g \\ u(0,t) = u_0(t). \end{cases}$$

- Let $S(z)$ be the unitary group generated by $i\partial_t^2$.
- A mild solution for the state equation with noise is

$$u(z) = S(z)u_0 + \int_0^z S\left(z - z'\right)\left(i|u(z')|^2 u(z') + g(z')\right)dz'.$$

Local existence

Space of solutions: $\mathcal{X}_z = C([0,z], L^2(\mathbb{R})) \cap L^6([0,z], L^6(\mathbb{R}))$.

Theorem

Given $u_0 \in L^2(\mathbb{R})$*, let* $r = \text{máx}\left\{\|u_0\|_{L^2}, \|g\|_{L^1(0,\zeta,L^2)}\right\}$*. Then, there exist* $z = z(r) \in (0,\zeta]$ *and* $u \in \mathcal{X}_z$ *solution of the integral equation*

$$u(z) = S(z)u_0 + \int_0^z S(z-z')\left(\mathrm{i}|u(z')|^2 u(z') + g(z')\right) dz'.$$

The solution u *depends continuously on* u_0*,* g *and*

$$\|u\|_{C(0,z,L^2)} \leq \|u_0\|_{L^2} + 2\,\|g\|_{L^1(0,\zeta,L^2)}.$$

Global existence

Theorem

Given $u_0 \in L^2(\mathbb{R})$ *and* $g \in L^1([0,\zeta], L^2(\mathbb{R}))$*, there exists a unique* $u \in \mathcal{X}_\zeta$ *solution of the state equation which verifies*

$$\|u\|_{\mathcal{X}_\zeta} \leq C\left(\zeta, \|u_0\|_{L^2}, \|g\|_{L^1(0,\zeta,L^2)}\right).$$

Furthermore, $u \in W^{1,1}([0,\zeta], H^{-2}(\mathbb{R}))$*,*

$$\|u\|_{W^{1,1}(0,\zeta,H^{-2})} \leq C\left(\zeta, \|u_0\|_{L^2}, \|g\|_{L^1(0,\zeta,L^2)}\right)$$

and the state equation is posed in H^{-2} *for a.e.* $z \in [0,\zeta]$.

Compactness

Theorem

Let $u \in \mathcal{X}_\zeta$ be the solution of the initial value problem with $u_0 \in L^2$, then for any $\omega \in \mathcal{S}(\mathbb{R})$ it is verified that $\omega u \in L^2([0,\zeta], H^{1/2})$ and

$$\|\omega u\|_{L^2([0,\zeta],H^{1/2})} \leq C(\omega, \zeta, \|g\|_{L^1([0,\zeta],L^2)})$$

Corollary

Let g_k be a sequence of controls bounded in $L^1([0,\zeta], L^2(\mathbb{R}))$, then there exist a subsequence u_k and u^* such that $u_k \to u^*$ in $L^2([0,\zeta], L^2_{loc}(\mathbb{R}))$.

Minimizing sequence

$$\kappa\|\sigma(u_k(\zeta) - v_\zeta\|^2_{L^2} + \|g_k\|^2_{L^2(L^2)} \to \inf \kappa\|\sigma(u(\zeta) - v_\zeta\|^2_{L^2} + \|g\|^2_{L^2(L^2)}$$

- Then $g_k \rightharpoonup g^*$ en $L^2([0,\zeta], L^2(\mathbb{R}))$.
- From the estimates for solution u_k associated to g_k,

$$\|u_k\|_{\mathcal{X}_\zeta} \leq C.$$

- There exists $u^* \in \mathcal{X}_\zeta$

$$u_k \to u^* \text{ in } L^2([0,\zeta], L^2_{loc}(\mathbb{R})).$$

- Then

$$|u_k|^2 u_k \rightharpoonup |u^*|^2 u^* \text{ in } L^2([0,\zeta], L^2(\mathbb{R})).$$

- u^* the associated solution to the control g^*.
- g^* is admisible and is optimal.

$u[g]$ es Fréchet differentiable

Recall $u[g]$ is the solution of the integral equation

$$u(z) = S(z)u_0 + \int_0^z S(z-z')\left(\mathrm{i}|u(z')|^2 u(z') + g(z')\right) dz'.$$

Theorem

Let $u_0 \in L^2(\mathbb{R})$ and $g \in L^1\left([0,\zeta], L^2(\mathbb{R})\right)$, then $u[g]$ is Fréchet differentiable and $D_g u[g](\delta g) \in \mathcal{X}_\zeta$ is the solution of the linear integral equation

$$y(z) = \int_0^z S(z-z')\left(2\mathrm{i}\mathrm{Re}\left(\overline{u[g]}\, y\right) u[g] + \mathrm{i}\left|u[g]\right|^2 y + \delta g\right)(z')dz'.$$

Abstract theorem

Theorem (Casas 1993)

Given G and Z Banach spaces and $\mathcal{U} \subset Z$ a convex subspace with nonempty interior. Let g_* be a solution of the problem

$$\begin{cases} \min \mathcal{J}(g) \\ g \in G, \Lambda(g) \in \mathcal{U} \end{cases}$$

where $\mathcal{J} : G \to (-\infty, +\infty]$ and $\Lambda : G \to Z$ are Gateaux differentiable. Then, there exist $\lambda \geq 0$ and $\mu_* \in Z'$ such that

- $\lambda + \|\nu\|_{Z'} > 0$
- $\langle \nu, z - \Lambda(g_*) \rangle \leq 0$ *for all* $z \in \mathcal{U}$
- $\lambda \mathcal{J}'(g_*) + [D\Lambda(g_*)]^\star \nu = 0.$

Optimal control problem

Applied to our problem we have

$$\begin{cases} \text{mín}\, \kappa \|\sigma(u(\zeta) - v_\zeta)\|^2_{L^2(\mathbb{R})} + \|g\|^2_{L^2([0,\zeta],L^2(\mathbb{R}))} \\ g \in L^2([0,\zeta], L^2(\mathbb{R})), \Lambda(g) = \sigma(u[g](\zeta) - v_\zeta) \in \bar{B}_{L^2(\mathbb{R})}(0, \sqrt{\eta}) \end{cases}$$

Then, there exist $\lambda \geq 0$ and $\nu \in L^2(\mathbb{R})$ such that

- $\lambda + \|\nu\|_{L^2(\mathbb{R})} > 0$
- $\langle \nu, z - \sigma(u[g](\zeta) - v_\zeta)\rangle \leq 0$ for all $z \in \bar{B}_{L^2(\mathbb{R})}(0, \sqrt{\eta})$
- $\lambda \mathcal{J}'(g_*) + (D\Lambda(g_*))^\star \nu = 0.$

Dual problem

We compute $(D\Lambda(g))^\star : L^2(\mathbb{R}) \to L^2([0,\zeta], L^2(\mathbb{R}))$:

- Given $g \in L^2\left([0,\zeta], L^2\left(\mathbb{R}\right)\right)$, $u[g] \in \mathcal{X}_\zeta$ the associated state, and $\nu \in L^2(\mathbb{R})$, let $\mu \in \mathcal{X}_\zeta$ be the solution of the dual equation

$$\partial_z \mu = \mathrm{i}\partial_t^2 \mu + 2\mathrm{i}|u|^2\mu - \mathrm{i}u^2\overline{\mu},$$
$$\mu\left(\zeta\right) = \sigma\nu.$$

- From

$$\langle \nu, D\Lambda(g)(\delta g)\rangle_{L^2} = \langle \nu, \sigma D_g u[g](\delta g)(\zeta)\rangle_{L^2} = \int_0^\zeta \langle \mu, \delta g\rangle_{L^2}.$$

- Then $(D\Lambda(g))^\star \nu = \mu$.

Necessary conditions

Let g be an optimal control and $u = u[g]$ its associated state

$$\partial_z u = \mathrm{i}\partial_t^2 u + \mathrm{i}|u|^2 u + g,$$
$$\partial_z g = \mathrm{i}\partial_t^2 g + 2\mathrm{i}|u|^2 g - \mathrm{i}(u)^2 \overline{g},$$
$$u(0) = u_0,$$
$$g(\zeta) = \beta\sigma^2(u(\zeta) - v_\zeta) \text{ with } \beta < 0$$
$$\|\sigma(u(\zeta) - v_\zeta)\|_{L^2}^2 = \eta$$

The physical models

$$iu_t = -\Delta u + V(h(t), x) + \lambda |u|^2 u$$
$$u(0, x) = u_0$$

- S. van Frank et al, "Interferometry with non classical motional states of the Bose Einstein condensate": Nature Communications, 2014.
  - $n = 1$.
  - $h : [0, T] \to \mathbb{R}$ is the displacement inflected on the BEC $\Rightarrow$ $V(h(t), x) = \alpha(h(t) - x)^2 + \ldots.$
- J.F. Mennemann et al, 'Optimal control of Bose Einstein condensates in three dimensions": New Journal of Physics, 2015.
  - $n = 3$.
  - $h : [0, T] \to \mathbb{R}^2$ such that $V(h(t), x, y, z) = m\left((w_x(h_1(t)))^2 x^2 + (w_y(h_2(t)))^2 y^2 + (w_z)^2 z^2\right).$

Ongoing work in colaboration with D. Rial

$$\text{mín}\langle u(T), Au(T)\rangle_{L^2(\mathbb{R}^n)} + \alpha_1 \int_0^T (\dot{E}(t))^2 dt + \alpha_2 \int_0^T (\dot{\alpha}(t))^2 dt$$

subject to

$$i\partial_t u = -\Delta u + V(h(t), x)u + \lambda |u|^{2\sigma} u, t \in [0, T], x \in \mathbb{R}^n$$

$$u(0) = u_0 \qquad \text{for } n = 1, 2, 3$$

$h \in H^1(0, T)$ and $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is bounded in time and subquadratic in space.

It is proved the existence of a minimizer for $\alpha_1 = 0$.