Hydrodynamics of the N-BBM process

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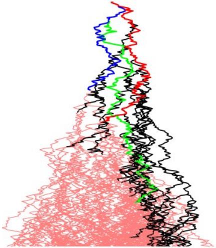

Illustration by Eric Brunet

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Brunet and Derrida 1997

N branching particles in $\mathbb{R}$ with selection:

discrete time.

Particle at x dies and creates random offsprings around x.

Select the rightmost N particles.

iterate

Maillard 2016 N-BBM.

N particles move as independent Brownian motions in $\mathbb{R}$,

Each particle, at rate 1, creates a new particle at its current position.

At each branching time, the left-most particle is removed.

The number N of particles is conserved.

Brunet Derrida 1997 Shift in the velocity of a front due to a cutoff PRE

Brunet, Derrida, Mueller, Munier (2006). Noisy traveling waves: effect of selection on genealogies. EPL + (06) + (07)

Bérard, Gouéré 2010 Brunet-Derrida behavior of branching-selection particle systems CMP.

Bérard, Maillard 2014 Limiting NBRW with polynomial tails EJP.

Durrett, Remenik 2011 Brunet-Derrida, FBP and Wiener-Hopf AOP.

Derrida, Shi 2017 Large deviations for BBM with selection or coalescence.

Berestycki, Brunet, Derrida 2017 Solution and asymptotics of F-KPP front

Berestycki, Brunet, Penington 2018 Global existence for FBP.

Hydrodynamics

Density ρ with $\int_{L_0}^{\infty} \rho(x)dx = 1$ and $\rho(x) = 0$ for $x \leq L_0 \in \mathbb{R}$.

Time zero: iid variables with density ρ.

$X_t :=$ set of positions of N-BBM particles at time t.

Theorem 1. *[DFPS 2017 Existence]*

For every $t \geq 0$, there is a density function $\psi(\cdot, t) : \mathbb{R} \to \mathbb{R}^+$ such that,

$$\lim_{N\to\infty} \frac{\big|X_t \cap [a, \infty)\big|}{N} = \int_a^\infty \psi(r, t)dr, \quad \textit{a.s. and in } L^1.$$

for any $a \in \mathbb{R}$.

Free boundary problem. FBP.

Density ρ with $\int_{L_0}^{\infty} \rho(x)dx = 1$ and $\rho(x) = 0$ for $x \leq L_0 \in \mathbb{R}$.

FBP1: Find $(u, L) = \big((u(r,t))_{r\in\mathbb{R}}, L_t\big)_{t\geq 0}$ such that

$$\begin{aligned}
\partial_t u &= \tfrac{1}{2}\partial_r^2 u + u, \qquad \text{in} \quad (L_t, +\infty);\\
u(r,0) &= \rho(r);\\
u(L_t,t) &= 0,\\
\int_{L_t}^{\infty} u(r,t)dr &= 1.
\end{aligned}$$

Theorem 2 (DFPS 2017). *If*

$$(u, L) = ((u(r,t), L_t) : t \in [0,T])$$

is a solution of the free boundary problem, with $L \in C^1$, then the hydrodynamic limit ψ coincides with u:

$$\psi(\cdot, t) = u(\cdot, t), \quad t \in [0,T].$$

Remark. Our proof of existence of ψ does not use existence of solution.

FBP has solution!

Lee (2017) proved that if $\rho \in C_c^2([L_0, \infty))$ and $\rho'_{L_0} = 2$ then there exist $T > 0$ and a solution (u, L) of the free boundary problem with the following properties:

- $\{L_t : t \in [0, T]\}$ is in $C^1[0, T]$, $L_{t=0} = L_0$
- $u \in C(D_{L,T}) \cap C^{2,1}(D_{L,T})$,
  where $D_{L,T} = \{(r, t) : L_t < r, 0 < t < T\}$.

Theorem 3 (Berestycki, Brunet, Penington (2018)). *Non-increasing* $v \in [0,1]$ *with* $\lim_{x\to-\infty} v(x) = 1$ *and* $\lim_{x\to\infty} v(x) = 0$.

$L_0 = \inf\{x : v(x) = 1\} \in \mathbb{R} \cup \{-\infty\}$

There exist a solution for

FBP2: find $(U, L) = \big((U(r,t))_{r\in\mathbb{R}}, L_t\big)_{t\geq 0}$ *such that*

$$\begin{aligned} \partial_t U &= \tfrac{1}{2}\partial_r^2 U + U, \qquad x \geq L_t,\ t > 0 \\ U(r,0) &= v(r); \\ \partial_r U(L_t, t) &= 0, \\ U(r,t)dr &= 1, \qquad \text{for } r \leq L_t \end{aligned}$$

with $L_t \in \mathbb{R}$, $U(\cdot, t) \in C^1$ *for all* $t > 0$ *and* $U \in C^1(\mathbb{R}, \mathbb{R}_+)$.

$L_t \in \mathbb{R}$ *for all* $t > 0$ *and continuous.* $U \in C^{2,1}(\{(r,t) : r > L_t, t > 0\})$

And $\partial_r U = u$ *satisfies the FBP 1.*

Berestycki, Brunet, Penington (2018)

Idea of proof: Define U_n as the solution of F-KPP

$$\partial_t U_n = \tfrac{1}{2}\partial_r^2 U_n + U_n - U_n^n, \qquad x \geq L_t,\ t > 0$$
$$U_n(r,0) = \int_r^\infty \rho(r')dr';$$

Known: U_n exists and it is unique. $U_n(r,t) \in (0,1)$.

(not a free boundary problem)

Define

$$U(r,t) = \lim_n U_n(r,t) \tag{1}$$

Then, U satisfies the FBP 2 (and $u = \partial_x U$ satisfies FBP 1).

General strategy to prove Theorem 1

We use a Trotter-Kato approximation as upper and lower bounds.

Durrett and Remenik 2011 upperbound for the Brunet-Derrida model. Leftmost particle motion is *increasing*: natural lower bounds.

Upper and lower bounds method was used in several papers:

- De Masi, F and Presutti 2015 Symmetric simple exclusion process with free boundaries. PTRF

- Carinci, De Masi, Giardinà, and Presutti 2016 Free boundary problems in PDEs and particle systems. SpringerBriefs in Mathematical Physics.

We introduce labelled versions of the processes and a coupling of trajectories to prove the lowerbound.

Ranked BBM, a tool

Let $(Z_0^1, \dots, Z_0^N)$ BBM initial positions.

$B_0^{i,1} = Z_0^i$, iid with density ρ.

N_t^i: is the size of the ith BBM family.

$B_t^{i,j}$: is the j-th member of the i-th family at time t, $1 \le j \le N_t^i$.

birth-time order.

$$\text{BBM:} \quad Z_t = \{B_t^{i,j} : 1 \le j \le N_t^i, \, 1 \le i \le N\}$$

$B_{[0,t]}^{i,j}$ trajectory coincides with ancestors before birth.

(i,j) is the rank of the jth particle of i-family

N-BBM as subset of BBM

Let $X_0 = Z_0$, $\tau_0 = 0$

τ_n branching times of BBM.

$$X_t := \{B_t^{i,j} \ : B_{\tau_n}^{i,j} \geq L_{\tau_n}, \text{ for all } \tau_n \leq t\}$$

$$L_{\tau_n} := \text{ defined iteratively such that } |X_t| = N \text{ for all } t$$

X_t has the law of N-BBM.

Stochastic barriers.

Fix $\delta > 0$

$X_0^{\delta,\pm} = Z_0$.

The upper barrier. Post-selection at time $k\delta$.

$$X_{k\delta}^{\delta,+} := N \text{ rightmost } \{B_{k\delta}^{i,j} : B_{(k-1)\delta}^{i,j} \in X_{(k-1)\delta}^{\delta,+}\}$$

$$L_{k\delta}^{N,\delta,+} := \min X_{k\delta}^{\delta,+}$$

The number of particles in $X_{k\delta}^{\delta,+}$ is exactly N for all k.

The lower barrier.

Pre selection at time $(k-1)\delta$.

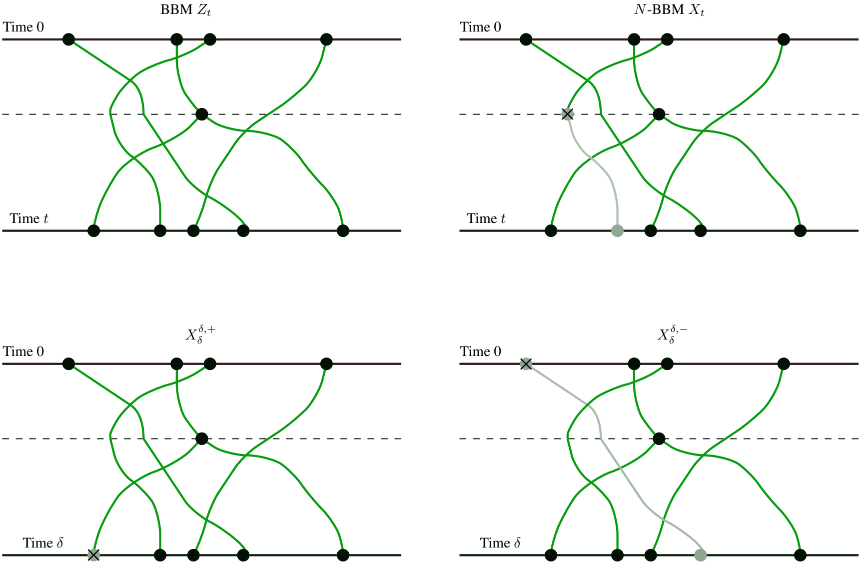
Select maximal number of rightmost particles at time $(k-1)\delta$ keeping no more than N particles at time $k\delta$.

$$L^{N,\delta,-}_{(k-1)\delta} := \text{cutting point at time } (k-1)\delta$$

$$X^{\delta,-}_{k\delta} := \{B^{i,j}_{k\delta} : B^{i,j}_{(k-1)\delta} \in X^{\delta,-}_{(k-1)\delta} \cap [L^{N,\delta,-}_{(k-1)\delta}, \infty)\}$$

Only entire families of particles at time $(k-1)\delta$ are kept at time $k\delta$.

The number of particles in $X^{\delta,-}_{k\delta}$ is $N - O(1)$.


BBM Z_t
Time 0
Time t
N-BBM X_t
Time 0
Time t
$X_\delta^{\delta,+}$
Time 0
Time δ
$X_\delta^{\delta,-}$
Time 0
Time δ

Mass transport partial order

$$X \preccurlyeq Y \quad \text{if and only if} \quad |X \cap [a,\infty)| \leq |Y \cap [a,\infty)| \quad \forall a \in \mathbb{R}.$$

Proposition 4. *Coupling* $\big((\hat{X}^{\delta,-}_{k\delta}, \hat{X}_{k\delta}, \hat{X}^{\delta,+}_{k\delta}) : k \geq 0\big)$ *such that*

$$\hat{X}^{\delta,-}_{k\delta} \preccurlyeq \hat{X}_{k\delta} \preccurlyeq \hat{X}^{\delta,+}_{k\delta}, \quad k \geq 0.$$

$\hat{X}^{\delta,-}_t$ is a subset of $\hat{Z}_t$, a BBM with the same law as Z_t.

Deterministic barriers. $u \in L^1(\mathbb{R}, \mathbb{R}_+)$.

Gaussian kernel: $G_t u(a) := \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(a-r)^2/2t} u(r)\, dr.$

$$e^t G_t \rho \text{ solves } u_t = \tfrac{1}{2} u_{rr} + u \text{ with initial } \rho.$$

Cut operator: $C_m u(a) := u(a) \mathbf{1}\left\{ \int_a^{\infty} u(r) dr \le m \right\},$

so that

$$\|C_m u\|_1 = m \wedge \|u\|_1.$$

For $\delta > 0$ and $k \in \mathbb{N}$, define *upper and lower barriers*:

$$S_0^{\delta,\pm}\rho := \rho \quad \text{Initial condition}$$

$$S_{k\delta}^{\delta,+}\rho := \left(C_1\left(e^{\delta}G_\delta\right)\right)^k \rho \quad \text{(diffuse \& grow) + cut;}$$

$$S_{k\delta}^{\delta,-}\rho := \left(\left(e^{\delta}G_\delta\right)C_{e^{-\delta}}\right)^k \rho \quad \text{cut + (diffuse \& grow)}$$

We have $\left\|S_{k\delta}^{\delta,\pm}\rho\right\|_1 = \|\rho\|_1 = 1$ for all k.

Hydrodynamics of δ-barriers

We prove that for fixed δ

the stochastic barriers converge to the macroscopic barriers:

Theorem 5. *Conditions of Theorem 1 and fixed δ:*

$$\lim_{N\to\infty} \frac{\left|X_{k\delta}^{\delta,\pm} \cap [r,\infty)\right|}{N} = \int_a^\infty S_{k\delta}^{\delta,\pm}\rho, \quad \textit{a.s. and in } L^1.$$

The same is true for the coupling marginals $\hat{X}_{k\delta}^{\delta,\pm}$.

Convergence of macroscopic barriers

Partial order: Take $u, v : \mathbb{R} \to \mathbb{R}^+$ and denote

$$u \preccurlyeq v \quad \text{iff} \quad \int_a^\infty u \leq \int_a^\infty v \quad \forall a \in \mathbb{R}.$$

Fix t and take diadic $\delta = t2^{-n}$. We prove

- $S_t^{\delta,-}\rho$ is increasing and $S_t^{\delta,+}\rho$ decreasing in n (diadics).
- $\left\|S_t^{\delta,+}\rho - S_t^{\delta,-}\rho\right\|_1 \leq c\delta$.
- There exists a continuous function ψ such that for any $t > 0$,

$$\lim_{n\to\infty} \|S_t^{\delta,\pm}\rho - \psi(\cdot,t)\|_1 = 0.$$

Sketch of proof of Theorem 1

By coupling $\hat{X}_t^{\delta,-} \preccurlyeq \hat{X}_t \preccurlyeq \hat{X}_t^{\delta,+}$.

Convergences in the sense of the Theorem 1:

$N \to \infty$:
The stochastic barriers $\hat{X}_t^{\delta,\pm}$ converge to the macroscopic barriers $S_t^{\delta,\pm}$.

$\delta \to 0$:
The macroscopic barriers converge to a function ψ, along diadics $\delta \to 0$.

Corollary:
N-BBM $\hat{X}_t$ converge to ψ as $N \to \infty$.

This is Theorem 1.

Sketch of proof of Theorem 2

We show that for continuous L_t, the solution u of the free boundary problem is in between the barriers:

$$S^{\delta,-}_{k\delta}\rho \preccurlyeq u(\cdot, k\delta) \preccurlyeq S^{\delta,+}_{k\delta}\rho.$$

We use the Brownian representation of solution with initial condition ρ:

$$\int_a^\infty u(r,t)dr = \frac{\int \rho(r)P_r(B_t \geq a\,,\, \tau_L > t)dr}{\int \rho(r)P_r(\tau_L > t)dr}$$

$$\tau_L := \inf\{t > 0 : B_t \leq L_t\}.$$

Hydrodynamic limit for the barriers

Macroscopic left boundaries

For $\delta > 0$ and $\ell \le k$ denote

$$L_{\ell\delta}^{\delta,+} := \sup_r \Big\{ \int_{-\infty}^{r} S_{\ell\delta}^{\delta,+} \rho(r') dr' = 0 \Big\};$$
$$L_{\ell\delta}^{\delta,-} := \sup_r \Big\{ \int_{-\infty}^{r} S_{\ell\delta}^{\delta,-} \rho(r') dr' < 1 - e^{-\delta} \Big\}. \tag{2}$$

Brownian representation of macroscopic barriers:

$B_{[0,t]} = (B_s : s \in [0,t])$ Brownian motion with

B_0, *random* variable with density ρ.

Lemma 6. *For test function $\varphi \in L^\infty(\mathbb{R})$ and $t > 0$,*

$$\int \varphi S_{k\delta}^{\delta,+} \rho \,=\, e^{k\delta} E\big[\varphi(B_{k\delta}) \mathbf{1}\{B_{\ell\delta} > L_{\ell\delta}^{\delta,+} : 1 \le \ell \le k\}\big].$$

$$\int \varphi S_{k\delta}^{\delta,-} \rho \,=\, e^{k\delta} E\big[\varphi(B_{k\delta}) \mathbf{1}\{B_{\ell\delta} > L_{\ell\delta}^{\delta,-} : 0 \le \ell \le k-1\}\big].$$

Generic LLN over trajectories of BBM

Let $B_0^{i,1}$ iid with density ρ.

$N_t^i :=$ size at time t of the i-th BBM family. $EN_t^i = e^t$.

Proposition 7. *Let g be bounded. Then*

$$\mu_t^N g := \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N_t^i} g(B_{[0,t]}^{i,j}) \underset{N\to\infty}{\longrightarrow} e^t E g(B_{[0,t]}), \quad \text{a.s. and in } L^1. \tag{1}$$

a.s. and in L^1.

Proof. By the many-to-one Lemma we have

$$E\mu_t^N g = EN_t \, Eg(B_{[0,t]}) = e^t Eg(B_{[0,t]}), \tag{2}$$

(The variance of $\mu_t^N g$ is order $1/N$, by family independence.) $\square$

Corollary 8 (Hydrodynamics of the BBM)**.**

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N_t^i}\varphi(B_t^{i,j}) &= e^t E\varphi(B_t) \qquad \textit{a.s. and in } L^1. \\ &= e^t\int\varphi(r)G_t\rho(r)dr, \qquad (3)\end{aligned}$$

Proof of Hydrodynamics for barriers

Proof of Theorem 5 BBM representation of stochastic barriers:

$$\pi_{k\delta}^{N,\delta,+}\varphi = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N_{k\delta}^{i}}\varphi(B_{k\delta}^{i,j})\mathbf{1}\{B_{\ell\delta}^{i,j} \geq L_{\ell\delta}^{N,\delta,+} : 1 \leq \ell \leq k\}$$

$$\pi_{k\delta}^{N,\delta,-}\varphi = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N_{k\delta}^{i}}\varphi(B_{k\delta}^{i,j})\mathbf{1}\{B_{\ell\delta}^{i,j} \geq L_{\ell\delta}^{N,\delta,-} : 0 \leq \ell \leq k-1\}.$$

As $N \to \infty$ $L_{\ell\delta}^{N,\delta,\pm}$ can be replaced by $L_{\ell\delta}^{\delta,\pm}$, and use the generic LLN.

For the replacement use that the random left boundaries are exact quantiles of 1. $\square$

Proof of Theorem 2 *The limit function ψ is the solution of the free boundary problem.*

The local solution of the free boundary problem is in between the barriers:

Theorem 9. *Let $t \in (0, T]$, $\delta \in \{2^{-n}t, n \in \mathbb{N}\}$. Then*

$$S_t^{\delta,-}\rho \preccurlyeq u(\cdot, t) \preccurlyeq S_t^{\delta,+}\rho, \qquad t = k\delta$$

The upperbound is immediate. The lower bound reduces to show the following stochastic order between conditioned probability measures:

$$P_{u_0}(B_t \geq r | \tau^L \leq \delta) \leq P_{u_1}(B_t \geq r | \tau^L > \delta) \tag{4}$$

where $u_1 = C_{e^{-\delta}}u$, $u_0 = u - u_1$,

$$P_{u_i}(B_t \in A) := \frac{1}{\|u_i\|_1} \int u_i(x) P_x(B_t \in A) dx. \tag{5}$$

and τ^L is the hitting time of the boundary.

Stationary N-BBM X_t is N-BBM. Process as seen from front:

$$X'_t := \{x - \min X_t : x \in X_t\}$$

Durrett and Remenik for Brunet-Derrida:

Theorem 10. *X'_t is Harris recurrent.*

ν_N unique invariant measure.

Speed: $\alpha_N = (N-1)\,\nu_N\big[\min(X \setminus \{0\})\big]$.

Law of X'_t starting with any initial condition converges to ν_N and

$$\lim_{t\to\infty} \frac{\min X_t}{t} = \alpha_N.$$

α_N converges to asymtotic speed of the first particle in BBM:

$$\lim_{N\to\infty} \alpha_N = \sqrt{2}.$$ *Berard and Gouéré*

Travelling wave solutions $u(r,t) = w(r - \alpha t)$, where w satisfies

$$\frac{1}{2}w'' + \alpha w' + w = 0, \quad w(0) = 0, \quad \int_0^\infty w(r)dr = 1. \tag{5}$$

Groisman and Jonckheere (2013): for each $\alpha \geq \sqrt{2}$, w_α given by

$$w_\alpha(r) = \begin{cases} M_\alpha\, r e^{-\alpha r} & \text{if } \alpha = \sqrt{2} \\ M_\alpha\, e^{-\alpha r} \sinh\left(r\sqrt{\alpha^2 - 2}\right) & \text{if } \alpha > \sqrt{2} \end{cases}$$

is solution to 5, where M_α is a normalization constant.

w_α is the unique qsd for Brownian motion with drift $-\alpha$ and absorption rate 1 ($w'(0) = 1$); see Martínez and San Martín (1994).

Popular open problems.

(1) Let X_t be N-BBM process with $X_0 \sim$ stationary measure ν^N.

Show that the empirical distribution of X_t converges to $w_{\sqrt{2}}(t\sqrt{2}+\cdot)$, as $N \to \infty$. (*strong selection principle*)

X_0 iid with density $w_{\sqrt{2}}$, and $w_{\sqrt{2}}(t\sqrt{2}+\cdot)$ strong solution of FBP imply convergence, as before.

Control particle-particle correlations in $X_0 \sim \nu_N$ and LLN as before?

(2) “Yaglom limit”? Does $u(\cdot - L_t, t) \to_t w_\alpha$ for some $\alpha \geq \sqrt{2}$?

(3) “Domain of attraction”? Fix α, which conditions must satisfy ρ to converge to w_α?

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