Derived Invariance of Operations in Hochschild Theory

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$$- \overset{\mathbf{L}}{\otimes}_A X : D(A) \xrightarrow{\sim} D(B).$$
- We put $X^\vee = \mathbf{R}Hom_B(X, B)$.
- There are natural isomorphisms
$$u : A \xrightarrow{\sim} X \overset{\mathbf{L}}{\otimes}_B X^\vee \quad \text{and} \quad v : X^\vee \overset{\mathbf{L}}{\otimes}_A X \xrightarrow{\sim} B$$
in $D(A^e)$ and $D(B^e)$, respectively.

Notation

- The functor

$$F = - \overset{\mathbf{L}}{\otimes}_{A^e} (X \overset{\mathbf{L}}{\otimes}_k X^\vee) = X^\vee \overset{\mathbf{L}}{\otimes}_A - \overset{\mathbf{L}}{\otimes}_A X : D^b(A^e) \xrightarrow{\sim} D(B^e)$$

has the following quasi-inverse

$$G = - \overset{\mathbf{L}}{\otimes}_{B^e} (X^\vee \overset{\mathbf{L}}{\otimes}_k X) = X \overset{\mathbf{L}}{\otimes}_B - \overset{\mathbf{L}}{\otimes}_B X^\vee : D(B^e) \xrightarrow{\sim} D^b(A^e).$$

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- We denote $(-)^* = Hom(-, k)$ the k-dual functor.

The cup product

- Let N and M be A-bimodules. The cup product in the Hochschild complexes is

$$\cup : Hom(A^{\otimes n}, N) \otimes Hom(A^{\otimes m}, M) \to Hom(A^{\otimes(n+m)}, N \otimes_A M)$$

which is given by

$$\alpha \cup \beta(a_1 \otimes \cdots \otimes a_{n+m}) := \alpha(a_1 \otimes \cdots \otimes a_n) \otimes_A \beta(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

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- This operation passes to cohomology

$$\cup : H^n(A, N) \otimes H^m(A, M) \to H^{n+m}(A, N \otimes_A M)$$

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- It yields an interpretation (Rickard) of the cup product in the derived category

$$Hom_{D(A^e)}(A, A[n]) \otimes Hom_{D(A^e)}(A, A[m]) \to Hom_{D(A^e)}(A, A[n+m])$$

given by $f \cup g := g[n] \circ f$.

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- Let N be an A-bimodule, there is an isomorphism

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- We extend Rickard's interpretation of the cup product to

$$\widetilde{\cup} : Hom_{D(A^e)}(A, N[n]) \otimes Hom_{D(A^e)}(A, M[m])$$

$$\to Hom_{D(A^e)}(A, N \otimes_A M[n+m])$$

given by $f \widetilde{\cup} g := (1_N \otimes_A g[n]) \circ f$.

Result

Theorem

Let A and B be derived equivalent algebras and let N and M be A-bimodules such that FN and FM are concentrated in degree zero. The diagrams

$$\begin{tikzcd} HH^n(A) \otimes H^m(A,M) \arrow[r, "\cup_A"] \arrow[d, "\cong"'] & H^{n+m}(A,M) \arrow[d, "\cong"'] \\ HH^n(B) \otimes H^m(B,FM) \arrow[r, "\cup_B"'] & H^{n+m}(B,FM) \end{tikzcd}$$

and

$$\begin{tikzcd} H^n(A,N) \otimes HH^m(A) \arrow[r, "\cup_A"] \arrow[d, "\cong"'] & H^{n+m}(A,N) \arrow[d, "\cong"'] \\ H^n(B,FN) \otimes HH^m(B) \arrow[r, "\cup_B"'] & H^{n+m}(B,FN) \end{tikzcd}$$

are commutative.

Hochschild homology

- There is a canonical monomorphism

$$\varphi : HH_n(A,N) \to HH_n(A,N)^{**} \xrightarrow{\sim} H^n(A,N^*)^*$$

which is an isomorphism if $HH_n(A,N)$ is finite dimensional.

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- For example, if A is a finite dimensional algebra and N is a finite dimensional A^e-module.
- We also have that $F(A^*)$ is concentrated in degree zero and is isomorphic to B^* (Zimmermann).

The cap product

- The cap product in the Hochschild complexes

$$\cap : \left(N \otimes A^{\otimes n}\right) \otimes Hom(A^{\otimes m}, M) \to N \otimes_A M \otimes A^{\otimes (n-m)}$$

is defined as

$$z \cap \beta := (-1)^{nm} x \otimes_A \beta(a_1 \otimes \cdots \otimes a_m) \otimes a_{m+1} \otimes \cdots \otimes a_n$$

for $\beta \in Hom(A^{\otimes m}, M)$ and every $z = x \otimes a_1 \otimes \cdots \otimes a_n \in N \otimes A^{\otimes n}$.

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$z = x \otimes a_1 \otimes \cdots \otimes a_n \in N \otimes A^{\otimes n}$.

- The cap product also provides a well-defined cap product in (co)homology

$$\cap : H_n(A, N) \otimes H^m(A, M) \to H_{n-m}(A, N \otimes_A M).$$

The cap product

- We get an interpretation of the cap product in termins of Hochschild cohomology

$$H^n(A, N^*)^* \otimes H^m(A, M) \longrightarrow H^{n-m}(A, (N \otimes_A M)^*)^*.$$

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- In case $N = M = A$, this interpretation of the cap product uses the interpretation of the cup product with coefficients in A^*, which we already proved that is a derived invariant.

Result

Theorem

Let A and B be derived equivalent algebras over a field k. Assume that A has finite dimensional Hochschild homology $HH_n(A)$ for each $n \geq 0$. For every pair of integers n and m, the following diagram is commutative

$$\begin{tikzcd}
HH_n(A) \otimes HH^m(A) \arrow[r, "\cap_A"] \arrow[d, "\cong"'] & HH_{n-m}(A) \arrow[d, "\cong"'] \\
HH_n(B) \otimes HH^m(B) \arrow[r, "\cap_B"'] & HH_{n-m}(B).
\end{tikzcd}$$

Result (j.w. Keller '17)

Lemma

Let A be an algebra projective over a commutative ring k and let $[f] \in HH^m(A)$. There is a commutative diagram

$$\begin{tikzcd} H_n(A,M) \arrow[rr, "- \cap [f]"] \arrow[d, "\cong"'] & & HH_{n-m}(A,M) \arrow[d, "\cong"'] \\ H_0\left(M \overset{\mathbf{L}}{\otimes}_{A^e} A[-n]\right) \arrow[rr, "H_0(1 \otimes [f][-n])"'] & & H_0\left(M \overset{\mathbf{L}}{\otimes}_{A^e} A[m-n]\right). \end{tikzcd}$$

Derived invariance (j.w. Keller '17)

Theorem

Let A and B be derived equivalent algebras projective over a commutative ring k. Let M be an A-bimodule such that $N := FM$ is concentrated in degree zero. There are canonical isomorphisms

$$H_\bullet(A, M) \xrightarrow{\sim} H_\bullet(B, N) \quad \textit{and} \quad HH^\bullet(A) \xrightarrow{\sim} HH^\bullet(B)$$

such that the following diagram is commutative

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H_n(A, M) \otimes HH^m(A) \arrow[r, "\cap_A"] \arrow[d, "\cong"'] & H_{n-m}(A, M) \arrow[d, "\cong"'] \\
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\end{tikzcd}$$

for all $n, m \geq 0$.

Connes differential

- Connes differential is the map $B_A : HH_n(A) \to HH_{n+1}(A)$ given by

$$B_A([a_0 \otimes \cdots \otimes a_n]) = \sum_{i=0}^{n} (-1)^{in}([1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}]),$$

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- Connes periodicity long exact sequence (ISB-sequence)

$$\cdots HH_n(A) \xrightarrow{I_A} HC_n(A) \xrightarrow{S_A} HC_{n-2}(A) \xrightarrow{B'_A} HH_{n-1}(A) \cdots .$$

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- These maps are related by $B_A = B'_A I_A$.
- Keller proved derived invariance of $HC_\bullet(A)$.

Tamarkin-Tsygan calculus

Let A be a k-algebra. The datum

$$(HH_\bullet(A), HH^\bullet(A), \cup_A, [-,-]_A, \cap_A, B_A)$$

is a Tamarkin-Tsygan (differential) calculus, i.e.

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- The map

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  provides $HH_\bullet(A)$ with the structure of a graded $(HH^\bullet(A), \cup_A, [-,-]_A)$-module.
- For each $j \geq 0$ define $i_\alpha : HH_j(A) \to HH_{n-j}(A)$ by

$$i_\alpha(z) := (-1)^{jn} z \cap \alpha,$$

  for $\alpha \in HH^n(A)$. Let $\alpha \in HH^n(A)$ and $\beta \in HH^m(A)$, the map $B_A : HH_\bullet(A) \to HH_{\bullet+1}(A)$ is such that $B_A^2 = 0$ and

$$[[B_A, i_\alpha]_{gr}, i_\beta]_{gr} = i_{[\alpha,\beta]},$$

  where $[-,-]_{gr}$ is the commutator bracket.

Mixed complexes

- The triple (M, b, B) is called a *mixed complex* if M is a $\mathbb{Z}$-graded k-module, b and B are graded endomorphisms of M of degrees 1 and -1, respectively, that satisfy the equations $b^2 = 0$ and $B^2 = 0$ as well as $bB + Bb = 0$.

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- Let Λ be the DG-algebra (differential graded) $k[\varepsilon]/(\varepsilon^2)$ where the degree of ε is -1 and the differential vanishes.
- We identify the category of mixed complexes with the category of DG-Λ modules ($B \leftrightarrow \varepsilon$).

Cyclic homology

Cyclic homology is by definition the homology of the bicomplex $C(A)$ given by

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & \cdots \arrow[l] \\
A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & \cdots \arrow[l] \\
A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & A \arrow[l, "1-t"'] & \cdots . \arrow[l]
\end{tikzcd}
$$

In which the multiplicative cyclic group $< t >= C_n$ acts on $A^{\otimes n}$ by cyclic permutation of the tensor factors and $N = 1 + t + ... + t^{n-1}$.

The cyclic functor

- Let $\mathbf{Alg}_k$ be the category whose objects are the associative DG k-algebras A such that the functor $Hom(A, -)$ sends quasi-isomorphisms to isomorphisms, and whose morphisms are morphisms of DG k-algebras which do not necessarily preserve the unit.

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- Define $\mathbf{ALG}_k$ to be the category whose objects are those of $\mathbf{Alg}_k$ and morphisms from A to B are the isomorphism classes of objects of $rep(A, B)$. The composition of morphisms in $\mathbf{ALG}_k$ is given by the total derived tensor product.

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- Define $\mathbf{ALG}_k$ to be the category whose objects are those of $\mathbf{Alg}_k$ and morphisms from A to B are the isomorphism classes of objects of $rep(A,B)$. The composition of morphisms in $\mathbf{ALG}_k$ is given by the total derived tensor product.
- The derived mix category **DMix** is the derived category of the DG-algebra of dual numbers $\Lambda = k[\varepsilon]/(\varepsilon^2)$.

The cyclic functor

The *cyclic functor*

$$\mathscr{C} : \mathbf{Alg}_k \to \mathbf{DMix}$$

is defined as follows. Let A be an object of $\mathbf{Alg}_k$, then $\mathscr{C}(A)$ is the mixed complex whith underlying graded k-vector space the mapping cone over $1-t$ viewed as a morphism of complexes

$$1-t : (A^{\otimes \bullet+1}, b') \to (A^{\otimes \bullet+1}, b).$$

The first and second differentials of the mixed complex $\mathscr{C}(A)$ are

$$\begin{bmatrix} b & 1-t \\ 0 & -b' \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix},$$

respectively.

The cyclic functor

- Consider the diagram of DG-algebras

$$A \xrightarrow{\alpha_X} End_B(B \oplus X) \xleftarrow{\beta_X} B$$

and define a morphism in **DMix** by

$$\mathscr{C}(X) := \mathscr{C}(\beta_X)^{-1} \circ \mathscr{C}(\alpha_X) : \mathscr{C}(A) \to \mathscr{C}(B).$$

Theorem

(Keller) The functor $\mathscr{C}$: $\mathbf{Alg}_k \to \mathbf{DMix}$ extends uniquely to a functor $\mathscr{C}$: $\mathbf{ALG}_k \to \mathbf{DMix}$.

Results (j.w. Keller '19)

Theorem

Let A and B be derived equivalent algebras over a field k. There is an isomorphism of exact sequences induced by the cyclic functor

$$
\begin{tikzcd}
\cdots \arrow[r] & HC_{n-1}(A) \arrow[r, "B'_{n-1}"] \arrow[d, "\cong"'] & HH_n(A) \arrow[r, "I_n"] \arrow[d, "\cong"'] & HC_n(A) \arrow[r, "S_n"] \arrow[d, "\cong"'] & HC_{n-2}(A)\cdots \arrow[d, "\cong"'] \\
\cdots \arrow[r] & HC_{n-1}(B) \arrow[r, "B'_{n-1}"] & HH_n(B) \arrow[r, "I_n"] & HC_n(B) \arrow[r, "S_n"] & HC_{n-2}(B)\cdots
\end{tikzcd}
$$

Corollary

There is a commutative diagram

$$
\begin{tikzcd}
HH_n(A) \arrow[r, "B_A"] \arrow[d, "\cong"'] & HH_{n+1}(A) \arrow[d, "\cong"'] \\
HH_n(B) \arrow[r, "B_B"] & HH_{n+1}(B).
\end{tikzcd}
$$

Observations

- A priori, the isomorphism induced between Hochschild homologies in the last theorem is not the same than the one we used for derived invariance of the cap product.

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- But ...
- They are the same!

Result (j.w. Keller '19)

Theorem

Let A and B be derived equivalent algebras over a field k. The isomorphisms

$$HH_\bullet(X) : HH_\bullet(A) \to HH_\bullet(B)$$

and

$$HH^\bullet(X) : HH^\bullet(A) \to HH^\bullet(B)$$

define an isomorphism between the Tamarkin-Tsygan calculi of A and B.

Thank you!

References

- [1] Armenta, Marco and Keller, Bernhard, *Derived invariance of the cap product in Hochschild theory*. Comptes Rendus Mathematique 355 (2017), pp 1205-1207.
- [2] Armenta, Marco and Keller, Bernhard, *Derived invariance of the Tamarkin-Tsygan calculus of an algebra*, Comptes Rendus Mathematique 357 (2019), pp 236-240.