Quantum Cluster Algebra Structures on Double Bruhat Cells

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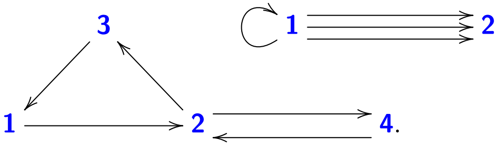
Cluster algebras

Introduced by Fomin and Zelevinsky in 2000. An axiomatic class of algebras with rich combinatorial structure, linked to problems in many diverse areas of mathematics, including:

- Representation Theory, Combinatorics,
- Algebraic and Poisson geometry,
- Topology and Mathematical Physics.

Input: A quiver (a directed graph) without loops and 2-cycles. Its vertices are indexed by $1, \ldots, m$.

Example. The following quivers are not allowed:



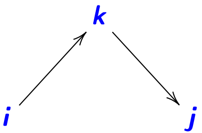
Quiver mutation

Given a quiver Q, for $k = 1, \ldots, m$, define its mutation $\mu_k(Q)$ at the vertex k:

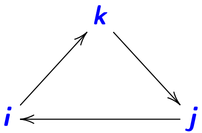
Step I: Reverse all arrows to and from the vertex k.

Step II: Complete

the 2-paths



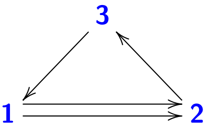
to triangles



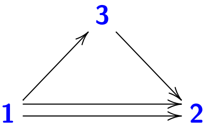
Step III: Cancel out pairs of opposite arrows.

An example: $\mu_3(Q)$

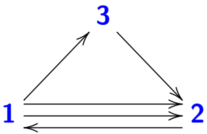
$Q :=$



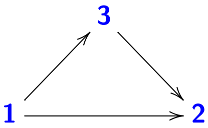
I (reverse arrows to/from 3):



II (2-path through 3):



III: $\mu_3(Q) =$



Definition of CA

Fix a quiver Q with m vertices and consider $\mathbb{K} := \mathbb{C}(y_1, \ldots, y_m)$; call

$$\Sigma := (y_1, \ldots, y_m; Q) \quad \text{the initial seed.}$$

Define the mutation of the seed at the vertex k by

$$\mu_k(\Sigma) = (y_1, \ldots y_{k-1}, y_k', y_{k+1}, \ldots, y_m; \mu_k(Q)), \quad y_k' := \frac{1}{y_k} \left(\prod_{j \to k} y_j + \prod_{i \leftarrow k} y_i \right).$$

Choose $n \leq m$, call $1, \ldots, n$ mutable vertices and $n+1, \ldots, m$ frozen vertices. Mutate the initial seed Σ in all mutable directions:

$$\mu_{k_1} \ldots \mu_{k_l}(\Sigma), \quad k_1, \ldots, k_l \in [1, n], \quad l = 1, 2, \ldots$$

Definition

The cluster algebra $\mathcal{A}(Q)$ is the subalgebra of $\mathbb{K}$ generated by the cluster variables in all seeds (infinitely many).

Quantum cluster algebras

Example. The space of 2×2 matrices. Its coordinate ring $\mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}]$ has cluster structure with only 2 clusters:

$$(x_{11}, x_{12}, x_{21}, \Delta), \quad \left(x_{22} = \frac{x_{12}x_{21} + \Delta}{x_{11}}, x_{12}, x_{21}, \Delta \right).$$

The variables x_{12}, x_{21}, Δ are frozen. The variables x_{11} and x_{22} are mutable.

Quantum cluster algebras $\mathcal{A}_q(Q)$

Introduced by Berenstein and Zelevinsky in 2004.

Idea: Replace all Laurent polynomial rings by quantum tori:

$$\mathcal{T} := \frac{\mathbb{C}\langle y_1^{\pm 1}, \ldots, y_m^{\pm 1} \rangle}{(y_j y_k - q_{jk} y_k y_j)}$$

for some $q_{jk} \in \mathbb{C}^*$.

Statement of the conjecture

Let G be an arbitrary complex simple Lie group and $B_{\pm}$ be a pair of opposite Borel subgroups. Denote the Weyl group of G by W. Define the double Bruhat cells

$$G^{u,w} = B_{+}uB_{+} \cap B_{-}wB_{-}, \quad u, w \in W.$$

Theorem [Berenstein–Fomin–Zelevinsky, 2003]

For all double Bruhat cells, $\mathbb{C}[G^{u,w}]$ is an upper cluster algebra.

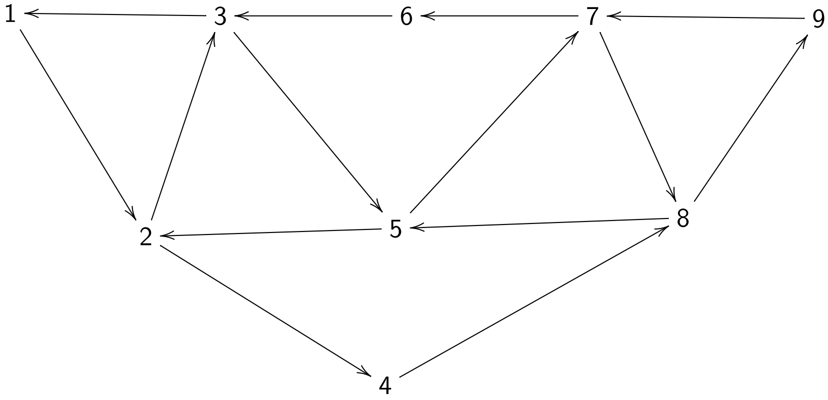
Conjecture [Berenstein–Zelevinsky, 2004]

For all double Bruhat cells $R_q[G^{u,w}]$ is an upper quantum cluster algebra.

Previous result: [Geiss–Leclerc–Schröer] Case $G = A, D, E$ and $w = 1$.

Example

Let $G = SL_4$ and $u = w = s_1 s_2 s_1 s_3 s_2 s_1 \in S_4$.
The Berenstein–Zelevinsky conjecture for the double Bruhat cell $R_q[SL_4^{w,w}]$ involves the quiver



Definitions

Lemma [Nagata] 1958

A noetherian integral domain R is a UFD if and only if every nonzero prime ideal contains a prime element.

Definition [Chatters] 1983

Let R be a noncommutative noetherian domain.

- A nonzero, nonunit element $p \in R$ is prime if $pR = Rp$ and R/pR is a domain.
- R is called a noetherian UFD if every nonzero prime ideal of R contains a prime element.

Quantum nilpotent algebras

For a nilpotent Lie algebra $\mathfrak{n}$, there exists a chain of ideals

$$\mathfrak{n} = \mathfrak{n}_m \rhd \mathfrak{n}_{m-1} \rhd \ldots \rhd \mathfrak{n}_1 \rhd \mathfrak{n}_0 = \{0\} \ \text{with} \ \dim(\mathfrak{n}_k/\mathfrak{n}_{k-1}) = 1, \ \text{and}$$

$$\mathcal{U}(\mathfrak{n}) \cong \mathbb{C}[x_1][x_2; \mathrm{id}, \delta_2] \ldots [x_m; \mathrm{id}, \delta_m]$$

for any $x_k \in \mathfrak{n}_k$, $x_k \notin \mathfrak{n}_{k-1}$; all derivations $\delta_k = \mathrm{ad}_{x_k}$ are locally nilpotent.

Definition [Cauchon–Goodearl–Letzter] late 90's

A quantum nilpotent algebra is a $\mathbb{C}$-algebra with an action of a torus H

$$R := \mathbb{C}[x_1][x_2; (h_2\cdot), \delta_2] \cdots [x_m; (h_m\cdot), \delta_m]$$

for some $h_k \in H$, satisfying the following conditions:

- all δ_k are locally nilpotent $(h_k\cdot)$-derivations,
- all x_k are H-eigenvectors, the eigenvals $h_k \cdot x_k = \lambda_k x_k$ are not roots of unity.

Examples

- Quantum Schubert cell algebras, (coideal subalgebras)

$$U_q(\mathfrak{n}_+ \cap w(\mathfrak{n}_-)) := U_q(\mathfrak{n}_+) \cap T_w(U_q(\mathfrak{n}_-)), \quad w \in W.$$

defined by Lusztig, De Concini–Kac–Procesi. Here $U_q(\mathfrak{n}_\pm) \subset U_q(\mathfrak{g})$ a quantized univ env alg, T_w denotes Lusztig's braid group action.

- Quantum Weyl algebras.
- Quantum double Bruhat cells (nontrivial presentation)

$$R_q[G^{u,w}] = (\mathcal{U}_q(\mathfrak{n}_- \cap u(\mathfrak{n}_+))^{\mathrm{op}} \bowtie \mathcal{U}_q(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))[E^{-1}].$$

Theorem [Launois–Lenagan–Rigal] 2005

All quantum nillpotent algebras are UFDs (technical point H-UFD).

Definitions

Definition I

A quantum nilpotent algebra is symmetric if for all $i < k$,

$$x_k x_i - \lambda_{ki} x_i x_k \in \mathbb{C}\langle x_{i+1}, \dots, x_{k-1} \rangle.$$

All mentioned examples are symmetric.

Definition II

Define the subset of the symmetric group S_m,

$$\Omega_m = \{\tau \in S_m \mid \tau([1,k]) \text{ is an interval for all } 1 \leq k \leq m\}.$$

Clusters on Quantum Nilpotent Algebras

Theorem [Goodearl-Y]

$R =$ an arbitrary symmetric quantum nilpotent algebra. Chain of subalgebras $R_1 \subset R_2 \subset \ldots \subset R_m$.

- Each R_k has a unique homogeneous (under H) prime element y_k that does not belong to R_{k-1}.
- Each such quantum nilpotent algebra R has a quantum cluster algebra structure with initial cluster $(y_1, \ldots, y_m)$.
- For $\tau \in \Omega_m$, adjoin the generators of R in the order $x_{\tau(1)}, \ldots, x_{\tau(m)}$. Chain of subalgebras $R_{\tau,1} \subset R_{\tau,2} \subset \ldots \subset R_{\tau,m}$. The sequence of primes $(y_{\tau,1}, \ldots, y_{\tau,m})$ is another cluster Σ_τ.
- The cluster algebra R is generated by the primes in the finitely many clusters Σ_τ for $\tau \in \Omega_m$.

An Application

Berenstein–Zelevinsky Conjecture [Goodearl-Y]

For all complex simple Lie groups G and Weyl groups elements w and u, the quantized coordinate ring of the double Bruhat cell $R_q[G^{u,w}]$ has a canonical cluster algebra structure.

Many other applications.