# Matrix representations for toric parametrizations

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# Motivation: Matrix representations for curves

- A planar rational curve $\mathscr{C}$ is given as the image of a map

$$\begin{array}{ccc} \mathbb{P}^1 & \overset{\phi}{\dashrightarrow} & \mathbb{P}^2 \\ (s,\bar{s}) & \mapsto & (f_1(s,\bar{s}) : f_2(s,\bar{s}) : f_3(s,\bar{s})) \end{array}$$

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  where $f_i \in \mathbb{K}[s,\bar{s}]$ are homogeneous polynomials of degree $d$ such that $\gcd(f_1, f_2, f_3) = 1$ and $\mathbb{K}$ is a field.
- A (linear) **syzygy** is a linear form $L = g_1T_1 + g_2T_2 + g_3T_3$ in the variables $T_1, T_2, T_3$ and with polynomial coefficients $g_i \in \mathbb{K}[s,\bar{s}]$ such that

$$\sum_{i=1,2,3} g_i f_i = 0$$

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- The set $\mathrm{Syz}(\phi)$ of all linear syzygies is a graded $\mathbb{K}[s, \bar{s}]$-module and for any integer $\nu$ the graded part $\mathrm{Syz}(\phi)_\nu$ is a finite-dimensional $\mathbb{K}$-vector space with a basis $(L_1, \ldots, L_k)$.

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- The matrix $M_\nu$ of coefficients with respect to a $\mathbb{K}$-basis of $\mathbb{K}[s,\bar{s}]_\nu$ is

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- For $\nu \geq d-1$, a point $P \in \mathbb{P}^2$ lies on $\mathscr{C}$ iff the rank of $M_\nu(P)$ drops.

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- Recent paper of Aruliah/Corless/Gonzalez-Vega/Shakoori: intersection problems are solved by using eigenvalue techniques
- Better suited for numerical methods

# Matrix representations of surfaces

- A rational surface $\mathscr{S}$ is given as the closed image of a map

$$\begin{array}{ccc} \mathcal{V} & \stackrel{\phi}{\dashrightarrow} & \mathbb{P}^3 \\ P & \mapsto & (f_1(P) : f_2(P) : f_3(P) : f_4(P)) \end{array}$$

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- Definition
  A **matrix representation** $M$ of $\mathscr{S}$ is a matrix with entries in $\mathbb{K}[T_1, T_2, T_3, T_4]$, generically of full rank, such that the rank of $M(P)$ drops iff the point $P \in \mathbb{P}^3$ lies on $\mathscr{S}$.

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- Some special classes of surfaces (e.g. ruled surfaces, canal surfaces): square matrix representations exist
- Two main approaches:
  - Use quadratic relations to construct square matrices
  - Only use linear syzygies and accept non-square matrices

# Linear and quadratic syzygies

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- Disadvantages:
  - require several additional geometric assumptions on the parametrization
  - require the computation of quadratic syzygies

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  - require only minimal assumptions on the parametrization
  - only linear syzygies have to be computed (efficient linear algebra methods)
- Disadvantages:
  - non-square matrix representations
  - previously only for $\mathcal{V} = \mathbb{P}^2$ (our goal: generalize the method for a larger class of varieties).

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- Surface parametrization of $\mathscr{S}$ given by

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  where $f_i \in \mathbb{K}[s,t]$ are polynomials such that $\gcd(f_1, \ldots, f_4) = 1$ and $\mathbb{K}$ is a field.
- First step: extend $\phi$ to a map $\mathcal{V} \dashrightarrow \mathbb{P}^3$ for a suitable compactification $\mathcal{V}$ of $\mathbb{A}^2$ (i.e. homogenize the map).

# Toric embeddings

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  where $(i,j) \in \mathrm{N}'(f) \cap \mathbb{Z}^2$
- Actually: Any polytope $Q$ with $\mathrm{N}(f) \subseteq d \cdot Q$ for some $d$ will work as well...

# Toric embeddings

- $\phi$ factorizes through $\mathscr{T}$ in the following way

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- New homogeneous parametrization $\psi = (g_1 : g_2 : g_3 : g_4)$ with $g_i \in A = \mathbb{K}[X_0, \ldots, X_m]/I(\mathscr{T})$ and $deg(g_i) = d$.

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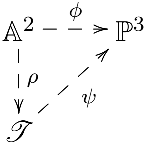


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- $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are special cases.
- Main difficulty: working over the affine normal semigroup ring $A$ instead of a polynomial ring

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- The canonical module $\omega_A$ of $A$ is the ideal generated by the monomials that correspond to points in the interior of $C$.
- The local cohomology of $A$ is

$$H^i_{\mathfrak{m}}(A) = \begin{cases} 0 & \text{if } i \neq 3 \\ \omega_A^\vee & \text{if } i = 3 \end{cases}$$

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- It is a bi-graded complex of $A[\underline{T}]$-modules constructed by means of the Koszul complex $(K_\bullet(\underline{g}, A), d_\bullet)$.
- For any given degree $\nu$ in the $X_i$ it induces a graded complex $(\mathcal{Z}_\bullet)_\nu$ of $\mathbb{K}[\underline{T}]$-modules

$$0 \to (\mathcal{Z}_3)_\nu \xrightarrow{\bar{e}_3} (\mathcal{Z}_2)_\nu \xrightarrow{\bar{e}_2} (\mathcal{Z}_1)_\nu \xrightarrow{\bar{e}_1} (\mathcal{Z}_0)_\nu$$

and $\bar{e}_1$ is the matrix $M_\nu$.

- Theorem

  Suppose that there are only finitely many isolated base points and that $V(I)$ is a local complete intersection, $I = (g_1, \ldots, g_4)$. If $\nu_0$ is an integer such that

$$H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu = 0 \ \text{ for all } \nu \geq \nu_0$$

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  - The proof follows the theory for $\mathbb{P}^2$, which has to be translated to our case by working with $A = \mathbb{K}[X_0, \ldots, X_m]/I(\mathscr{T})$ instead of $\mathbb{K}[X_0, X_1, X_2]$.

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  - This is rather technical and requires tools from homological algebra (blow-up algebras, local cohomology, determinants of complexes, etc.)
  - Question: What is the lowest possible $\nu_0$?

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  - In some cases, the bound can be lowered (depending on the base points).
- Corollary
  Suppose that there are only finitely many isolated base points and that $V(I)$ is a local complete intersection. Then for all $\nu \geq 2d$ the first matrix $M_\nu$ of $(\mathcal{Z}_\bullet)_\nu$ is a matrix representation of $\mathscr{S}$.

- Very sparse parametrization:
  $(f_1, f_2, f_3, f_4) = (st^6 + 2, st^5 - 3st^3, st^4 + 5s^2t^6, 2 + s^2t^6)$

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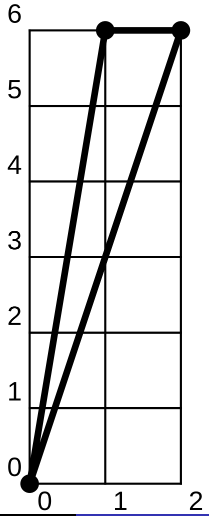
- 

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- For $\nu_0 = 2d = 2$ the matrix $M_{\nu_0}$ is a matrix representation of size $17 \times 34$.

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- Over $\mathbb{P}^1 \times \mathbb{P}^1$, we obtain $A = \mathbb{K}[x_0, \ldots, x_7]/J$ and for $\nu_0 = 2$ the $21 \times 34$-matrix $M_{\nu_0}$ represents a multiple of $F_{\mathscr{S}}$ of **degree 9**.

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- Over $\mathbb{P}^2$, we obtain $A = \mathbb{K}[x_0, x_1, x_2]$ and for $\nu_0 = 6$ the $28 \times 35$-matrix $M_{\nu_0}$ represents a multiple of $F_{\mathscr{S}}$ of **degree 21**.

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- This shows that our method really is a generalization of the previous methods.

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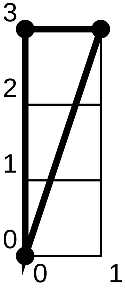
- Previous example with polytope $Q$:



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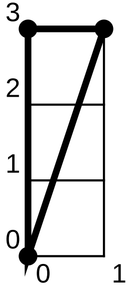


- $\mathrm{N}(f) \subset 2 \cdot Q$, so the parametrization factorizes through the toric variety associated to $Q$.

# Is $\mathrm{N}'(f)$ always the optimal choice?

- New parametrization defined by $(g_1, g_2, g_3, g_4) =$

$$(2X_0^2 + X_3X_4, -3X_0X_4 + X_2X_4, X_1X_4 + 5X_4^2, 2X_0^2 + X_4^2)$$

over the coordinate ring $A = \mathbb{K}[X_0, \dots, X_4]/J$ with $J = (X_2^2 - X_1X_3, X_1X_2 - X_0X_3, X_1^2 - X_0X_2)$.

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- For $\nu_0 = 2$: matrix representation of size $12 \times 19$, compared to $17 \times 34$ for $\mathrm{N}'(f)$.
- **Philosophy:** compromise between two criteria:
  - polytope should be as small as possible (higher degree $d$)
  - polytope should respect the sparseness of the parametrization (similar to Newton polytope)

Thank you for your attention!