UN MUNDO DE BINOMIOS

ALICIA DICKENSTEIN

Universidad de Buenos Aires

ElENA IV, La Falda, Agosto 6, 2008

PLAN DE LA CHARLA

- **BASICS ON BINOMIALS - How binomials “sit” in the polynomial world and the main “secrets” about binomials**

PLAN DE LA CHARLA

- **BASICS ON BINOMIALS - How binomials “sit” in the polynomial world and the main “secrets” about binomials**
- **COUNTING SOLUTIONS TO BINOMIAL SYSTEMS - with a touch of complexity**

PLAN DE LA CHARLA

- **BASICS ON BINOMIALS - How binomials “sit” in the polynomial world and the main “secrets” about binomials**
- **COUNTING SOLUTIONS TO BINOMIAL SYSTEMS - with a touch of complexity**
- **(Mixed) DISCRIMINANTS (DUALS OF BINOMIAL VARIETIES) and an application to real roots**

PLAN DE LA CHARLA

- **BASICS ON BINOMIALS - How binomials "sit" in the polynomial world and the main "secrets" about binomials**
- **COUNTING SOLUTIONS TO BINOMIAL SYSTEMS - with a touch of complexity**
- **(Mixed) DISCRIMINANTS (DUALS OF BINOMIAL VARIETIES) and an application to real roots**
- **Brief summary of what we won't have time to talk about today (hypergeometric differential equations, mass action kynetics dynamics)**

What is a binomial?

A polynomial with two terms

What is a binomial?

A polynomial with two terms

$$ax^{\alpha} + bx^{\beta} \quad \in k[x_1, \ldots, x_n]$$

$$x = (x_1, \ldots, x_n), \quad \alpha \neq \beta \in \mathbb{N}^n, \quad a, b \in k$$

1. BASICS ON BINOMIALS One step before wilderness

- **Linear systems:** **LINEAR ALGEBRA**

1. BASICS ON BINOMIALS One step before wilderne

- **Linear systems:** LINEAR ALGEBRA
- **Monomials:** COMBINATORICS

1. BASICS ON BINOMIALS One step before wilderne

- **Linear systems:** LINEAR ALGEBRA
- **Monomials:** COMBINATORICS
- **Binomials:** LINEAR ALGEBRA AND COMBINATORICS

1. BASICS ON BINOMIALS One step before wildernes

- **Linear systems:** LINEAR ALGEBRA
- **Monomials:** COMBINATORICS
- **Binomials:** LINEAR ALGEBRA AND COMBINATORICS
- **Trinomials:** THE WHOLE WILD WORLD

1. BASICS ON BINOMIALS One step before wildernes

- **Linear systems:** LINEAR ALGEBRA
- **Monomials:** COMBINATORICS
- **Binomials:** LINEAR ALGEBRA AND COMBINATORICS
- **Trinomials:** THE WHOLE WILD WORLD

Any system is equivalent to a system with at most trinomials

$$m_1 + m_2 + m_3 + m_4 = 0 \Leftrightarrow m_1 + m_2 - z_1 = m_3 + m_4 - z_2 = z_1 + z_2 = 0$$

1. BASICS ON BINOMIALS First main fact

Given any system of binomial equations = any binomial ideal,

$$a_j\, x^{\alpha_j} + b_j\, x_j^{\beta} = 0,\; j = 1, \ldots, r,$$

1. BASICS ON BINOMIALS First main fact

Given any system of binomial equations = any binomial ideal,

$$a_j\, x^{\alpha_j} + b_j\, x_j^{\beta} = 0,\; j = 1, \ldots, r,$$

- If there exists a solution c in the torus

$$T_n = \{(c_1, \ldots, c_n),\; c_i \neq 0,\; i = 1, \ldots n\},$$

in new coordinates $y_i = x_i / c_i$, $i = 1, \ldots, n$ the system looks (up to multiplying by non-zero constants)

$$y^{\alpha_j} - y^{\beta_j} = 0, \quad j = 1, \ldots, r.$$

1. BASICS ON BINOMIALS Second main fact

Given any system of binomial equations = any binomial ideal,

$$a_j\, x^{\alpha_j} + b_j\, x^{\beta_j} = 0,\; j = 1, \ldots, r,$$

1. BASICS ON BINOMIALS Second main fact

Given any system of binomial equations = any binomial ideal,

$$a_j\, x^{\alpha_j} + b_j\, x^{\beta_j} = 0,\; j = 1, \ldots, r,$$

- There exists such a solution $c \in T_n$ if and only if (assuming that not both of a_j and b_j are 0) $a_j, b_j \neq 0$ and for each linear relation

$$\sum_{j=1}^{r} \lambda_j\, (\alpha_j - \beta_j) = 0, \quad \lambda_j \in \mathbb{Z},$$

it holds that

$$\prod_{j=1}^{r} \left(\frac{-b_j}{a_j} \right)^{\lambda_j} = 1.$$

1. BASICS ON BINOMIALS Gale Duality

On the other hand, any sparse polynomial system on the torus T_n is equivalent to a system of binomials in linear forms:

$$f_i = \sum_{j=1}^{N} c_j^i x^{m_j} = 0, \quad i = 1, \ldots r \quad (*)$$

Given $y = (y_1, \ldots, y_N) \in T_N$, there exists $x \in T_n$ such that $y = (x^{m_1}, \ldots, x^{m_N})$ if and only if for any λ in the integer kernel I of the $n \times N$-integer matrix M with columns $m_1, \ldots, m_N$ it holds that

$$y^{\lambda} = 1 \qquad \text{or} \quad y^{\lambda_+} - y^{\lambda_-} = 0 \quad (**)$$

So $(*)$ is equivalent to the system of linear forms and binomials

$$\sum_{i=1}^{N} c_j^i y_j = 0, \quad i = 1, \ldots r, \quad y^{\lambda_+} - y^{\lambda_-} = 0, \lambda \in I \ (***)$$

1. BASICS ON BINOMIALS Gale Duality

$$f_i = \sum_{j=1}^{N} c_j^i x^{m_j} = 0, \quad i = 1, \ldots r \quad (*)$$

$$\sum_{i=1}^{N} c_j^i y_j = 0, \quad i = 1, \ldots r, \quad y^{\lambda_+} - y^{\lambda_-} = 0, \lambda \in I \ (***)$$

If the columns of the matrix V give a basis of the kernel K of the $n \times N$ matrix with entries c_j^i, write any N-tuple $y \in K$ as

$$y = (\langle b_1, t \rangle, \ldots \langle b_s, t \rangle) = \langle b, t \rangle,$$

where $b_1, \ldots, b_s$ are the row vectors of V and $t = (t_1, \ldots, t_{\dim K})$.

1. BASICS ON BINOMIALS Gale Duality

$$f_i = \sum_{j=1}^{N} c_j^i x^{m_j} = 0, \quad i = 1, \dots r \quad (*)$$

$$\sum_{i=1}^{N} c_j^i y_j = 0, \quad i = 1, \dots, r, \quad y^{\lambda_+} - y^{\lambda_-} = 0, \lambda \in I \ (***)$$

If the columns of the matrix V give a basis of the kernel K of the $n \times N$ matrix with entries c_j^i, write any N-tuple $y \in K$ as

$$y = (\langle b_1, t \rangle, \dots \langle b_s, t \rangle) = \langle b, t \rangle,$$

where $b_1, \dots, b_s$ are the row vectors of V and $t = (t_1, \dots, t_{\dim K})$.

Then, finding $x \in T_n$ satisfying $(*)$ is equivalent to finding t with $\langle b, t \rangle \in T^N$ such that for any λ in I (i.e. $\sum_i \lambda_i m_i = 0$),

$$\langle b, t \rangle^{\lambda_+} - \langle b, t \rangle^{\lambda_-} = 0, \quad (****)$$

2. COUNTING SOLUTIONS Third main fact + complexit

Given any square system of n binomial equations in n variables,

$$a_j\, x^{\alpha_j} + b_j\, x^{\beta_j} = 0,\ j = 1, \ldots, n, \quad a_j, b_j \neq 0$$

call $M \in \mathbb{Z}^{n \times n}$ the matrix with rows $\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n$.

2. COUNTING SOLUTIONS Third main fact + complexit

Given any square system of n binomial equations in n variables,

$$a_j\, x^{\alpha_j} + b_j\, x^{\beta_j} = 0,\ j = 1, \ldots, n, \quad a_j, b_j \neq 0$$

call $M \in \mathbb{Z}^{n \times n}$ the matrix with rows $\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n$.

- Set $\delta := |\det(M)|$. When $\delta \neq 0$, the number of solutions in the torus T_n equals $\delta > 0$, independently of the value of the coefficients [BKK].

Can decide in polynomial time if the system has a finite number of solutions. [Cattani-D., JofC'07].

2. COUNTING SOLUTIONS Third main fact + complexit

Given any square system of n binomial equations in n variables,

$$a_j\, x^{\alpha_j} + b_j\, x^{\beta_j} = 0,\ j = 1, \ldots, n, \quad a_j, b_j \neq 0$$

call $M \in \mathbb{Z}^{n \times n}$ the matrix with rows $\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n$.

- Set $\delta := |\det(M)|$. When $\delta \neq 0$, the number of solutions in the torus T_n equals $\delta > 0$, independently of the value of the coefficients [BKK].

Can decide in polynomial time if the system has a finite number of solutions. [Cattani-D., JofC'07].

- When $\delta = 0$, it is possible to decide in polynomial time (in the size of the sparse input) whether for generic coefficients the system has no solutions in the torus.
  Likewise, it is possible to determine in polynomial time whether the zero set of the system in affine space k^n is empty or not [follows from ibid., thanks to J.M Rojas for posing this question] Ex.

2. COUNTING SOLUTIONS Third main fact + complexit

Given any square system of n binomial equations in n variables,

$$a_j\, x^{\alpha_j} + b_j\, x^{\beta_j} = 0,\ j = 1, \ldots, n, \quad a_j, b_j \neq 0$$

call $M \in \mathbb{Z}^{n \times n}$ the matrix with rows $\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n$.

- Set $\delta := |\det(M)|$. When $\delta \neq 0$, the number of solutions in the torus T_n equals $\delta > 0$, independently of the value of the coefficients [BKK].

Can decide in polynomial time if the system has a finite number of solutions. [Cattani-D., JofC'07].

- When $\delta = 0$, it is possible to decide in polynomial time (in the size of the sparse input) whether for generic coefficients the system has no solutions in the torus.
  Likewise, it is possible to determine in polynomial time whether the zero set of the system in affine space k^n is empty or not [follows from ibid., thanks to J.M Rojas for posing this question] Ex.

2. COUNTING SOLUTIONS Complexity

$$a_j\, x^{\alpha_j} + b_j\, x_j^{\beta} = 0,\ j = 1, \ldots, n, \quad a_j, b_j \neq 0$$

- For "generic" exponents the (finite) number of solutions can be computed in polynomial time (precise combinatorial conditions identified through commutative algebra).

However,

- If we allow M to be any integer matrix (even if $\det(M) \neq 0$) counting the number of solutions to a square binomial system with or without multiplicity is #P-complete (thanks to P. Bürgisser).
- We give a "nice" combinatorial formula. The main complexity is based on deciding which are the possible (zero and non zero coordinates) of the solutions.
- In some sense, this problem is "orthogonal" to numerical analysis (pure structure vs. behaviour of coefficients)

2. COUNTING SOLUTIONS Complexity

- Given any bipartite digraph $G = (V, E)$, $V = V_1 \cup V_2$, $E \subset V_1 \times V_2$, $V = \{1, \ldots, n\}$, we define n binomials in n variables defining a complete intersection

$$p_i = x_i - x_i^2, \; i \in V_1 \qquad p_j = x_j - \left(\prod_{(i,j) \in E} x_i \right) x_j^2, \; j \in V_2.$$

- $V(p_1, \ldots, p_n) \subset \{0, 1\}^n$ and its cardinal equals the number of independent sets of G (all roots are simple and determined by its support).
- A universal Gröbner basis of the ideal $\langle p_1, \ldots, p_n \rangle$ equals $x_i - x_i^2 \; (i = 1, \ldots, n)$; $x_j - x_i x_j \; (\forall \, (i, j) \in E)$ [E. Tobis'07]

3. DISCRIMINANTS What is a (mixed) discriminant

Given finite sets $A_1, \dots, A_n \subset \mathbb{Z}^n$ and sparse polynomials $f_1, \dots, f_n$ with these supports,

$$f_i(c^{(i)}, x) \;=\; \sum_{\alpha \in A_i} c_\alpha^{(i)} x^\alpha,$$

there exists (under some conditions) an irreducible integer polynomial Δ_A in the vector of coefficients $\mathbf{c} = (c^{(1)}, \dots, c^{(n)}) \in \mathbb{C}^\ell$ which vanishes whenever there exists $x \in T_n$ which is not a simple zero of $f_1, \dots, f_n$ (where the Jacobian vanishes) [Gelfand-Kapranov-Zelevinsky'94]

3. DISCRIMINANTS What is a (mixed) discriminant

Given finite sets $A_1, \ldots, A_n \subset \mathbb{Z}^n$ and sparse polynomials $f_1, \ldots, f_n$ with these supports,

$$f_i(c^{(i)}, x) = \sum_{\alpha \in A_i} c_\alpha^{(i)} x^\alpha,$$

there exists (under some conditions) an irreducible integer polynomial Δ_A in the vector of coefficients $\mathbf{c} = (c^{(1)}, \ldots, c^{(n)}) \in \mathbb{C}^\ell$ which vanishes whenever there exists $x \in T_n$ which is not a simple zero of $f_1, \ldots, f_n$ (where the Jacobian vanishes) [Gelfand-Kapranov-Zelevinsky'94]

- That is, $\Delta_A = 0$ describes the variety of ill-posed systems, and the distance of a coefficient vector to it is basic for numerical continuation and numerical stability [M. Shub, J.P. Dedieu, C. Beltran, G. Malajovich, ...].

3. DISCRIMINANTS What is a (mixed) discriminant

Given finite sets $A_1, \ldots, A_n \subset \mathbb{Z}^n$ and sparse polynomials $f_1, \ldots, f_n$ with these supports,

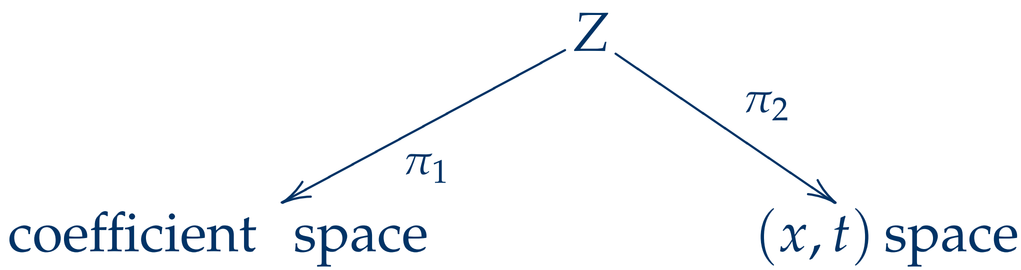
$$f_i(c^{(i)}, x) = \sum_{\alpha \in A_i} c_\alpha^{(i)} x^\alpha,$$

there exists (under some conditions) an irreducible integer polynomial Δ_A in the vector of coefficients $\mathbf{c} = (c^{(1)}, \ldots, c^{(n)}) \in \mathbb{C}^\ell$ which vanishes whenever there exists $x \in T_n$ which is not a simple zero of $f_1, \ldots, f_n$ (where the Jacobian vanishes) [Gelfand-Kapranov-Zelevinsky'94]

- That is, $\Delta_A = 0$ describes the variety of ill-posed systems, and the distance of a coefficient vector to it is basic for numerical continuation and numerical stability [M. Shub, J.P. Dedieu, C. Beltran, G. Malajovich, ...].
- Δ_A is called the mixed discriminant associated to the support sets $A_1, \ldots, A_n$.

3. DISCRIMINANTS What is a (mixed) discriminant

Computing Δ_A is an elimination problem:



where Z is the incidence variety of tuples $(x, t, \mathbf{c})$, $x, t \in T_n$ such that

$$f_1(c^{(1)}, x) = \cdots = f_n(c^{(n)}, x) = 0$$

and moreover

$$\sum_i \frac{\partial}{\partial x_j}(f_i(c^{(1)}, x))t_i = 0, j = 1, \ldots, n.$$

We are interested in the closure of the image $\{\Delta_A = 0\}$ of π_1. But π_2 is much easier to understand and allows us to find a rational parametrization of the discriminant variety

3. MIXED DISCRIMINANTS

- Mixed discriminants are a particular case of general sparse discriminant (aka A-discriminants) and define the dual variety of the toric (binomial) variety associated to the given supports.

3. MIXED DISCRIMINANTS

- Mixed discriminants are a particular case of general sparse discriminant (aka A-discriminants) and define the dual variety of the toric (binomial) variety associated to the given supports.
- When $n = 1$ we recover the well known notion of discriminant of a univariate polynomial of fixed degree.

3. MIXED DISCRIMINANTS

- Mixed discriminants are a particular case of general sparse discriminant (aka A-discriminants) and define the dual variety of the toric (binomial) variety associated to the given supports.
- When $n = 1$ we recover the well known notion of discriminant of a univariate polynomial of fixed degree.
- When $A_i = d_i\Delta_n \cap \mathbb{Z}^n$ are the lattice points of a dilate of the standard n-simplex, f_i is just a generic polynomial with degree d_i.

3. MIXED DISCRIMINANTS

- Mixed discriminants are a particular case of general sparse discriminant (aka A-discriminants) and define the dual variety of the toric (binomial) variety associated to the given supports.
- When $n = 1$ we recover the well known notion of discriminant of a univariate polynomial of fixed degree.
- When $A_i = d_i \Delta_n \cap \mathbb{Z}^n$ are the lattice points of a dilate of the standard n-simplex, f_i is just a generic polynomial with degree d_i.
- The well known numerical unstability of the Wilkinson polynomial

$$W_{20} = \prod_{i=1}^{20} (x + i) = \sum_{j=0}^{20} c_j x^j,$$

can be explained by the fact that its vector of coefficients $c = (20!, \ldots, 1)$ is very close to a singular point of the discriminant variety $\Delta_A = 0$, where $A = \{0, 1, \ldots, 20\}$.

3. MIXED DISCRIMINANT

- Consider the matrix

$$A := \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 6 & 0 & 0 & 0 & 3 & 1 \\ 0 & 3 & 1 & 6 & 0 & 0 \end{pmatrix}.$$

- A is the Cayley matrix associated to 2 planar configurations, and the A-discriminant $\Delta_A(y_1, \ldots, y_6)$ is the *mixed discriminant* of the family of polynomials

$$\begin{cases} h_1(y;t,s) := y_1 t^6 + y_2 s^3 + y_3 s^1 \\ h_2(y;t,s) := y_4 s^6 + y_5 t^3 + y_6 t^1 \end{cases}$$

- $\Delta_A(y) = 0$ whenever there exists a common zero $(s,t) \in (\mathbf{k}^*)^2$ which is not simple.

3. MIXED DISCRIMINANTS   An example

- Consider the matrix

$$A := \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 6 & 0 & 0 & 0 & 3 & 1 \\ 0 & 3 & 1 & 6 & 0 & 0 \end{pmatrix}.$$

- $\Delta_A(y) = 0$ whenever there exists a common zero $(s,t) \in (\mathbf{k}^*)^2$ which is not simple.
- The Horn-Kapranov parametrization of $X_A^* = (\Delta_A(y) = 0)$ is given by

$$y_1 = (-2\lambda_1 + \lambda_2)\, t_1\, t^6, \quad y_2 = (\lambda_1 - 6\lambda_2)\, t_1\, s^3,$$

$$y_3 = (-3\lambda_1 + 6\lambda_2)\, t_1\, s, \quad y_4 = 2\lambda_2\, t_2\, s^6,$$

$$y_5 = (-6\lambda_1 + \lambda_2)\, t_2\, t^3, \quad y_6 = (6\lambda_1 - 3\lambda_2)\, t_2\, t.$$

3. MIXED DISCRIMINANT

...

and $\Delta_A(1, a, -1, 1, b, -1)$ equals

$$82754024941868680778822139064668229594467072 * a^{47} * b^{33} +$$

$$24519711093887016527058411574716512472434688 * a^{46} * b^{39} -$$

$$24519711093887016527058411574716512472434688 * b^{46} * a^{39} +$$

$$236627403090264575474785219707184968001345670463360 * a^{28} * b^{7} +$$

$$17631004810327637966335552676449435712814331054687500 * a^{4} * b^{11} +$$

53 additional monomial terms of comparable size

It is a polynomial of degree 90 with 58 monomials and huge integer coefficients!

3. DISCRIMINANTS Tropical information

- A-discriminants are in general complicated polynomials which carry a lot of combinatorial information.
- In principle, we can compute Δ_A by standard methods in elimination …but in practice we reach the limits of the current computations very easily.
- So, instead, we can try to get a first combinatorial approximation, which can nonetheless give us the information about discrete invariants as dimension and degree (and asymptotics), by computing its Newton polytope $N(\Delta_A)$ or its tropicalization $\tau(X_A^*)$.
- This is obtained from the tropicalization of an homogeneous version of the Horn-Kapranov parametrization, by monomials in linear forms.
- Tropicalization is an operation that turns complex projective varieties into polyhedral fans.

[D.-Feichtner-Sturmfels, JAMS'07]

3. DISCRIMINANTS Tropical Information

- The *tropicalization* $\tau(Y)$ of a variety Y is (as a set)

$$\tau(Y) \;=\; \{w \in \mathbb{R}^n \;:\; \mathrm{in}_w(I_Y) \text{ does not contain a monomial}\,\}\,,$$

where for $w \in \mathbb{R}^n$ and $f = \sum_{c \in C} \gamma_c x^c,\ \gamma_c \neq 0,\ C \subset \mathbb{Z}^n$, define

$$\mathrm{in}_w f \;=\; \sum_{w \cdot c \min} \gamma_c x^c \qquad \text{initial term of } f\,,$$

$$\mathrm{in}_w(I_Y) \;=\; \langle \mathrm{in}_w f \mid f \neq 0 \in I_Y \rangle \;\; \text{initial ideal of } I_Y\,.$$

- ... plus intersection theory information attached to each of the cones in the polyhedral fan $\tau(Y)$ [Sturmfels-Tevelev '07)]
- $\tau(Y)$ can also be defined via valuations [Bieri-Groves'84, Einsidler-Kapranov-Lind'06, Sturmfels-Speyer'06].
- In the hypersurface case, $\tau(\{\Delta_A = 0\})$ is the codimension one skeleton of the normal fan of Δ_A.

3. DISCRIMINANTS Tropical information

The discriminant of a cubic polynomial in 1 variable

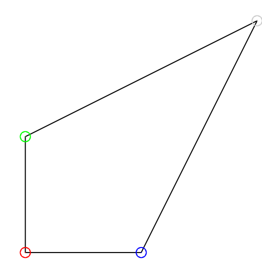
$$A := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \qquad X_A \text{ is the twisted cubic.}$$

$$f_A(x;t) = x_1 t^0 + x_2 t^1 + x_3 t^2 + x_4 t^3$$

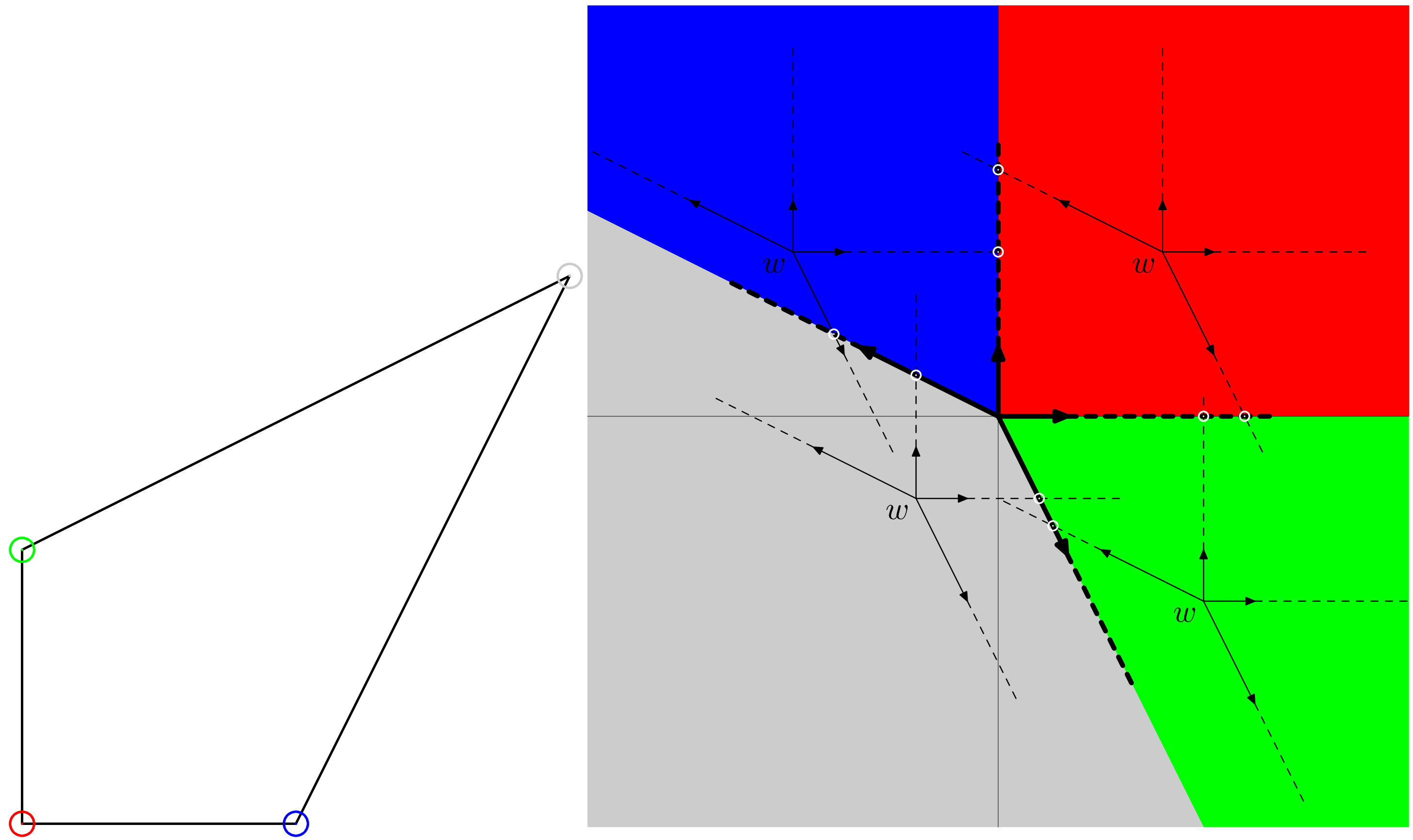
$$\Delta_A = 27\, x_1^2 x_4^2 - 18\, x_1 x_2 x_3 x_4 + 4\, x_1 x_3^3 + 4\, x_2^3 x_4 - x_2^2 x_3^2$$

$$\mathrm{in}_{(-1,-1,-1,0)}(\Delta_A) = 4x_1x_3^3 - x_2^2x_3^2 \qquad (-1,-1,-1,0) \in \tau(X_A^*)$$

$$\mathrm{in}_{(1,0,1,0)}(\Delta_A) = 4x_2^3x_4 \qquad (1,0,1,0) \notin \tau(X_A^*)$$



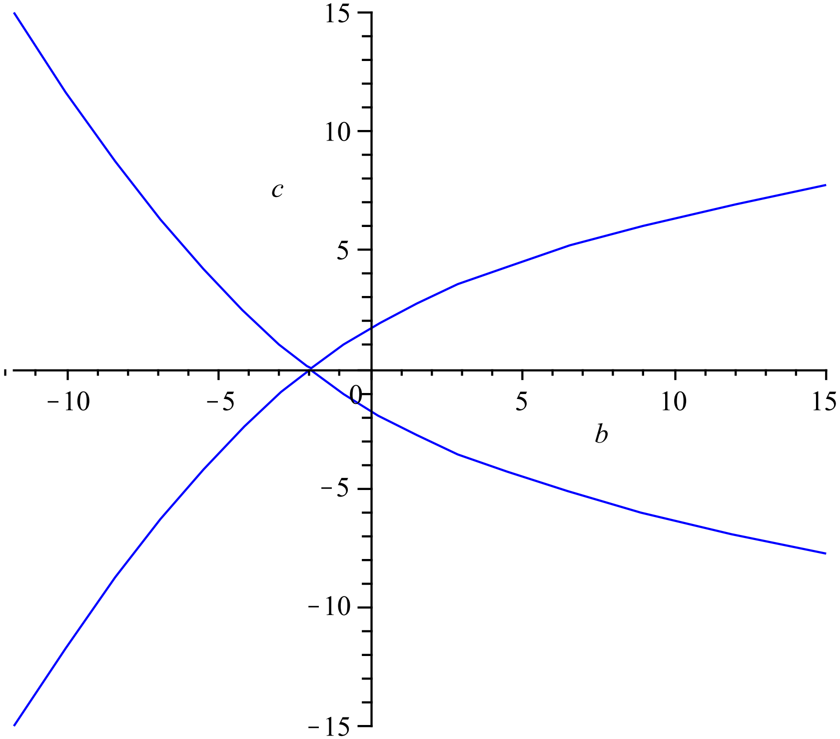
3. DISCRIMINANTS   Tropical information



Newton polygon, tropicalization and extreme monomials of the discriminant of a degree 3 polynomial $\Delta_A = 27\, x_1^2 x_4^2 - 18\, x_1 x_2 x_3 x_4 + 4\, x_1 x_3^3 + 4\, x_2^3 x_4 - x_2^2 x_3^2$

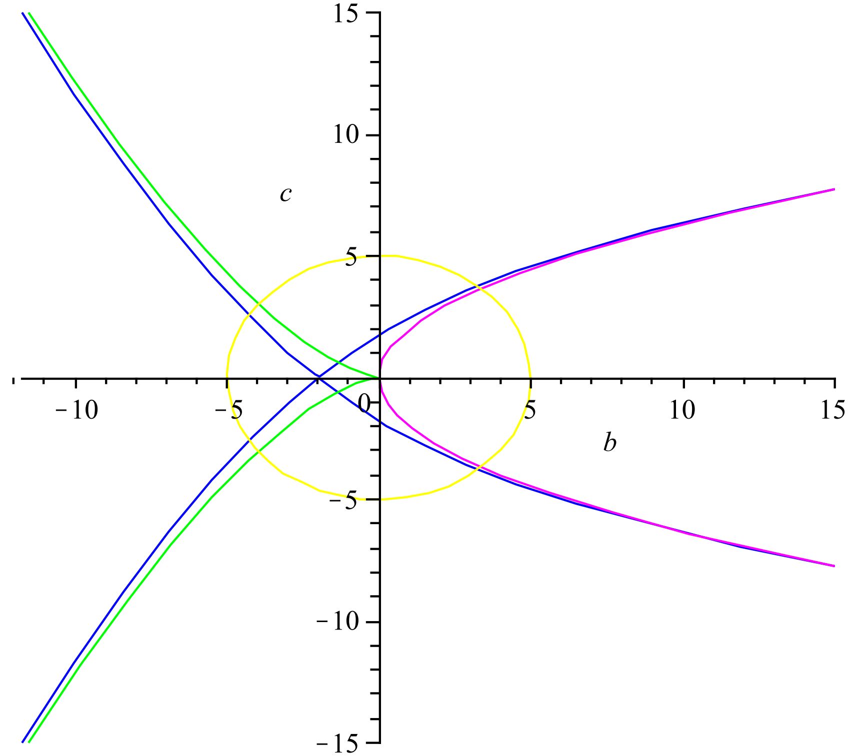
$x_2^3 x_4$, $x_2^2 x_3^2$, $x_1 x_3^3$, $x_1^2 x_4^2$

3. DISCRIMINANTS Tropical information



Discriminant in b, c space of $f := x^4 + b\,x^2 + c\,x + 1$

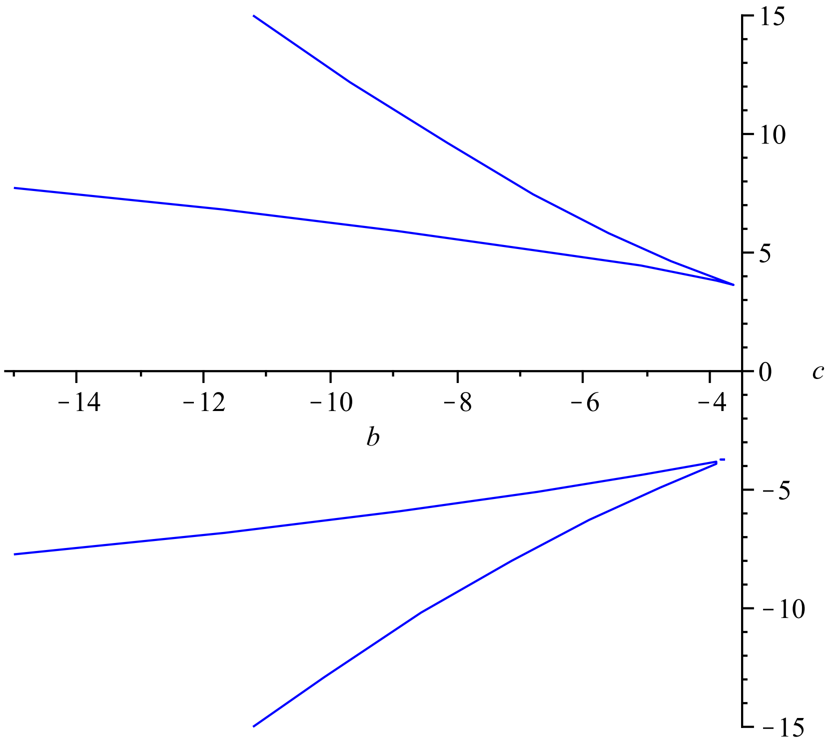
$$-4\,b^3 c^2 - 27\,c^4 + 16\,b^4 - 128\,b^2 + 144\,b\,c^2 + 256 = 0$$



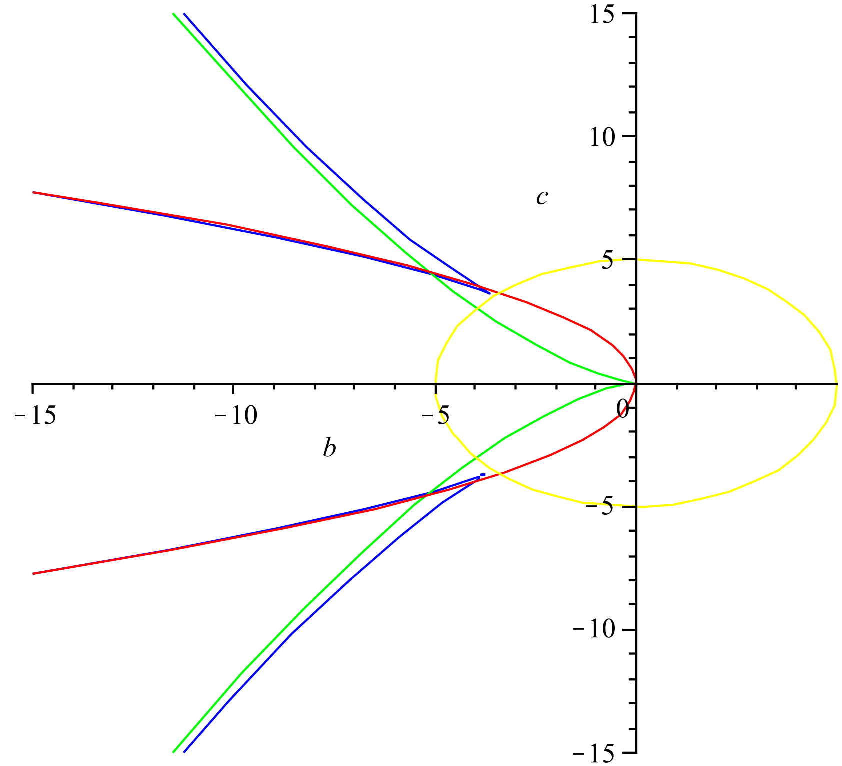
Green: 4*b^3+27*c^2=0, Black= 16*b^4-128*b^2+256=0,
Magenta= -4*c^2+16*b=0

The discriminant of the quartic equation $x^4 + bx^2 + cX + 1$ and its asymptotes

3. DISCRIMINANTS Tropical information



Discriminant in b,c space of $g := x^4 + bx^2 + cx - 1$



Green: 4*b^3+27*c^2=0, Black= 16*b^4-128*b^2+256=0,
Red= -4*c^2116*b=0

The discriminant of the quartic equation $x^4 + bx^2 + cX - 1$ and its asymptotes

3. TWO THEOREMS

- Theorem: The tropical A-discriminant is the Minkowski sum of the tropicalization of the kernel $\mathcal{B}(A)$ and the (classical) row space of the $d \times N$-matrix A.

$$\boxed{\tau(X_A^*) = \{w + vA,\, w \in \mathcal{B}(A), v \in \mathbb{R}^d\}.}$$

3. TWO THEOREMS

- Theorem: The tropical A-discriminant is the Minkowski sum of the tropicalization of the kernel $\mathcal{B}(A)$ and the (classical) row space of the $d \times N$-matrix A.

$$\boxed{\tau(X_A^*) = \{w + vA,\, w \in \mathcal{B}(A), v \in \mathbb{R}^d\}.}$$

- $\mathcal{L}(A)$ = geometric lattice whose elements are the sets of zero-entries of the vectors in kernel(A), ordered by inclusion.
- $\mathcal{C}(A)$ = set of proper maximal chains in $\mathcal{L}(A)$.
- We represent these chains as $(N-d-1)$-element subsets $\sigma = \{\sigma_1, \ldots, \sigma_{N-d-1}\}$ of $\{0,1\}^N$.
- The tropicalization of the kernel of A equals $\mathcal{B}(A) := \tau(\text{kernel}(A)) \quad = \quad \bigcup_{\sigma \in \mathcal{C}(A)} \mathbb{R}_{\geq 0}\, \sigma\,.$
- This tropical linear space is a subset of $\mathbb{R}^N$.

3. TWO THEOREMS

- DATA: $A \in \mathbb{Z}^{d \times N}$ (e.g. A = Cayley matrix of $A_1, \ldots, A_n$, $d = 2n$), $w \in \mathbb{R}^N$ generic.

3. TWO THEOREMS

- DATA: $A \in \mathbb{Z}^{d \times N}$ (e.g. A = Cayley matrix of $A_1, \ldots, A_n$, $d = 2n$), $w \in \mathbb{R}^N$ generic.
- Theorem: The exponent of x_i in the initial monomial $\mathrm{in}_w(\Delta_A)$ equals the number of intersection points of the halfray

$$w + \mathbb{R}_{>0} e_i$$

with the tropical discriminant $\tau(X_A^*)$, counting multiplicities:

$$\deg_{x_i}(\mathrm{in}_w(\Delta_A)) = \sum_{\sigma \in \mathcal{B}(\mathrm{kerA})_{i,w}} \left| \det\left(A^T, \sigma_1, \ldots, \sigma_{N-d-1}, e_i\right) \right|.$$

where $\mathcal{B}(\mathrm{kerA})_{i,w}$ is the subset of $\mathcal{C}(A)$ consisting of all chains such that the row space of the matrix A has non-empty intersection with the cone $\mathbb{R}_{>0}\{\sigma_1, \ldots, \sigma_{N-d-1}, -e_i, -w\}$.

Click Smooth case: [Katz, Kleiman,Holme], [GKZ'94]

3. AN APPLICATION TO COUNTING REAL ROOTS

- Descartes' theorem (1637) for univariate polynomials allows to bound the number of real solutions in terms of the number of monomials independently of the degree.
- e.g. $x^d - a$, $0 \neq a \in \mathbb{R}$ has d complex solutions but at most 2 real solutions (and only one positive).
- A generalization to the multivariate setting is currently an open problem.

3. AN APPLICATION TO COUNTING REAL ROOTS

- Khovanskii (1980): There exists a (huge, non sharp) bound for the number of real solutions of a system of multivariate real polynomials in terms of the number of monomials which are present.
- Better bounds: only a few partial results (Li-Rojas-Wang, Bihan-Sottile, after 2002)
- There exists a (false) conjecture by Koushnirenko, which in particular would imply that the number of positive simple real roots of a system of two trinomials in two variables is at most 4.
- There exists a counterexample by Haas (2002), with polynomials of degree 106 and 5 positive simple real solutions. In fact, 5 is the correct bound.
- "It is hard to find real sparse polynomials systems with many real solutions".

3. AN APPLICATION TO COUNTING REAL ROOTS

We could prove that the two parameter family of real bivariate trinomials

$$H_{(a,b)} := \begin{cases} h_1(x,y) := x^6 + a\,y^3 - y \\ h_2(x,y) := y^6 + b\,x^3 - x \end{cases}$$

gives a far simpler family of counter-examples to Kushnirenko's Conjecture for $a = b = \frac{44}{31}$. [D.-Rojas-Rusek-Shih,MMJ'07]

In fact, the area of the set of points $(a,b) \in \mathbb{R}^2$ such that the system has 5 positive real simple roots is smaller than 5.701×10^{-7}.

This is a dehomogenization of the generic family associated to the configuration

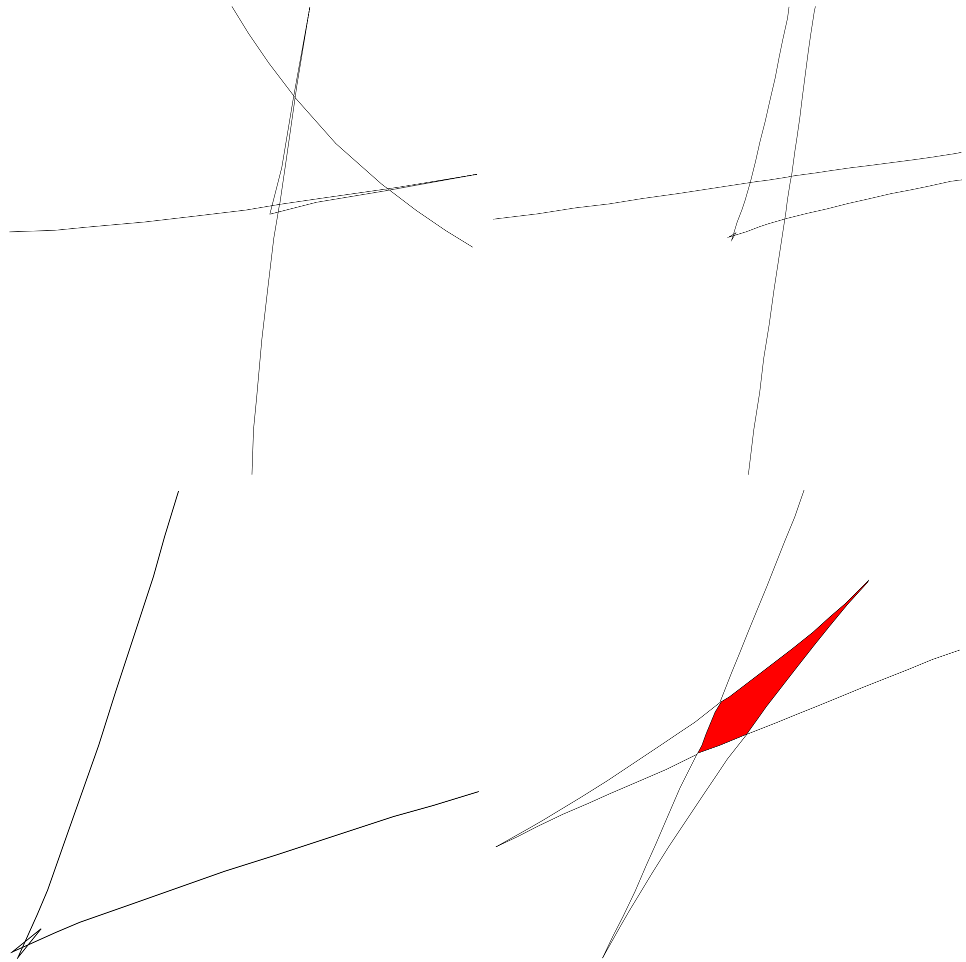
$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 6 & 0 & 0 & 0 & 3 & 1 \\ 0 & 3 & 1 & 6 & 0 & 0 \end{pmatrix}.$$

3. AN APPLICATION TO COUNTING REAL ROOTS

- Two points in the same chamber (connected component) of the complement in $\mathbb{R}^2$ of the zero set of the dehomogenized discriminant $\nabla(a,b) = \Delta(1,a,-1,1,b,-1)$ have the same number of real roots (also the same number of positive real roots in this case).
- In particular, $\nabla_A(\frac{44}{31},\frac{44}{31}) \neq 0$, and this implies that $H_{(44/31,44/31)}$ has no degenerate roots.
- An implicit plot of $(\nabla_A = 0)$ has very poor quality, but instead we can draw it efficiently using the dehomogenized version of the Horn-Kapranov parametrization!

3. AN APPLICATION TO COUNTING REAL ROOTS

Below is a sequence of 4 plots, drawn on a logarithmic scale and successively magnified up to a factor of about 1700, of the real part of the discriminant variety ($\Delta_A = 0$)



3. ANOTHER REAL APPLICATION

- We moreover get an explicit good upper bound on the number of chambers of the complement of the real points in a dehomogenized A-discriminant (not just a mixed discriminant) for general configurations A of codimension two (i.e. $n+3$ general lattice points in $\mathbb{Z}^n$).
- The Horn-Kapranov parametrization of dehomogenizations of A-discriminants gives a (multivalued) inverse of the logarithmic Gauss map.
- We get a bound for the number of chambers smaller than $\frac{26}{5}(n+4)^6$, which is completely independent of the coordinates of A.

SOME OPEN QUESTIONS ABOUT DISCRIMINANTS

- Intrinsic formula for the degree in the singular case

- How to estimate $d(c, \{\Delta_A = 0\})$? i.e., how to assemble the combinatorial description and the numerical aspects (with a condiment of number theory)?

- Discriminantal matrices, i.e. describe $\Delta_A(c) = 0$ as the rank drop of a matrix. In particular, can we then estimate $d(c, \{\Delta_A = 0\})$ with such a matrix?

- Precise description of the singularities of the discriminant locus (Weyman-Zelevinsky'96: hyperdeterminant case; D'Andrea-Chipalkatti'07: univariate case)

WHAT WE MISSED TODAY Hypergeometric functions

- Main “yoga” of hypergeometry:

  Hypergeometric recurrences in (a_α): $\frac{a_{\alpha+e_i}}{a_\alpha}$ rational function of α.

  $$=$$

  Hypergeometric differential equations satisfied by $f = \sum_\alpha a_\alpha x^\alpha$

- Classical hypergeometric series and differential equations

  vs

  Binomial differential equations + Euler operators, and homogeneous Γ-series. The singular locus of the system is described by the vanishing of discriminants.

[Following: Gelfand-Kapranov-Zelevinsky'89,'90 - Saito-Sturmfels-Takayama'00]

WHAT WE MISSED TODAY Hypergeometric functions

- Holonomic rank, (explicit) particular solutions, recurrences with finite support

translated from

Binomial primary decomposition, multiplicities and degrees of components [D.-Matusevich-Sadykov, Adv. Math.'05, D.-Matusevich-Miller, preprint]

- A chemical reaction network consists of n complexes that are comprised of s species.
- Represent reactions by a digraph G with n nodes, one for each complex, labeled by monomials.
- Triangle Example: $s = 2$ species c_1 and c_2, $n = 3$ complexes c_1^2, c_1c_2, c_2^2, with all possible six *reactions* among them.

In this system we have $c_1 + c_2 =$ const (i.e. $dc_1/dt + dc_2/dt = 0$):

$$\begin{aligned} dc_1/dt &= 2\cdot\left(c_1c_2\kappa_{21} + c_2^2\kappa_{31} - c_1^2(\kappa_{12}+\kappa_{13})\right) \\ &\quad + \left(c_1^2\kappa_{12} + c_2^2\kappa_{32} - c_1c_2(\kappa_{21}+\kappa_{23})\right) = \end{aligned}$$

$$\left(\kappa_{21}c_1c_2 - \kappa_{12}c_1^2\right) + 2\cdot\left(\kappa_{31}c_2^2 - \kappa_{13}c_1^2\right) + \left(c_2^2\kappa_{32} - c_1c_2\kappa_{23}\right)$$

- The mathematical foundation for this model of chemical reactions was set by Horn, Jackson and Feinberg (70').
- Dynamics of the concentrations is given by an autonomous system of ODE's of the form $dc/dt = f(c)$, where f is a real polynomial with a lot of combinatorial structure coming from the digraph of reactions, with many unknown parameters (which makes numerical simulations practically unfeasible)
- Binomial equations characterize the "best" models in the rate constant space and give equations for the steady states.
- In these cases, the dynamic behaviour seems to be independent of the chosen constants and there is a (very partially studied) "global attractor conjecture".

THE END

Many thanks for your attention!!

2. COUNTING SOLUTIONS Third main fact + complexit

...Even if it might require polynomials g_i of degree exponential in n to write 1 in terms of the given binomials, as in

$$f_1 := x_1^d,$$

$$f_2 := x_1 x_n^{d-1} - x_2^d,$$

$$\dots$$

$$f_{n-1} := x_{n-2} x_n^{d-1} - x_{n-1}^d,$$

$$f_n := x_{n-1} x_n^{d-1} - 1$$

$$1 = \sum_{i=1}^{n} g_i f_i$$

[among many other examples, thanks to Teresa]