Cox rings of K3 surfaces

Michela Artebani

Departamento de Matemática
Universidad de Concepción

elENA IV (La Falda, Sierras de Córdoba)
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Outline

Cox rings

Let X be a smooth algebraic variety over $\mathbb{C}$ and $L_1, \ldots, L_\rho$ be divisors whose classes freely generate $\mathrm{Pic}(X)$.

The ***Cox ring*** of X is:

$$\mathrm{Cox}(X) := \bigoplus_{a \in \mathbb{Z}^n} H^0(X, a_1 L_1 + \cdots + a_\rho L_\rho)$$

Cox rings

Observation. $\mathrm{Cox}(X)$ is graded by $\mathrm{Pic}(X)$:

$$\mathrm{Cox}(X) \cong \bigoplus_{D \in \mathrm{Pic}(X)} \mathrm{Cox}(X)_D$$

- $\mathrm{Cox}(X)_D = H^0(X, D)$,
- $\mathrm{Cox}(X)_{D_1} \cdot \mathrm{Cox}(X)_{D_2} \subseteq \mathrm{Cox}(X)_{D_1+D_2}$.

Cox rings

Questions.

- When is $\mathrm{Cox}(X)$ finitely generated?
- How do we find its generators and relations?

$$0 \longrightarrow I_X \longrightarrow \mathbb{C}[x_1, \ldots, x_r] \longrightarrow \mathrm{Cox}(X) \longrightarrow 0.$$

Cox rings

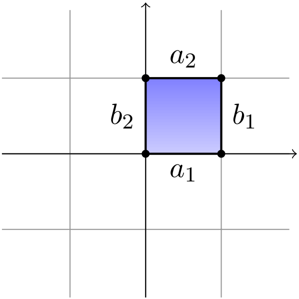
Theorem (D. Cox, 1992). Let X be a toric variety with $E_1, \dots, E_r$ integral invariant divisors, then

$$\mathrm{Cox}(X) = \mathbb{C}[x_1, \dots, x_r]$$

where x_i is a defining section for E_i.

Cox rings

Example: Let $\mathbb{F}_0$ be a quadric surface:



$$\mathrm{Cox}(X) \cong \mathbb{C}[a_1, a_2, b_1, b_2]$$

Cox rings

Examples of smooth surfaces with finitely generated Cox ring:

- toric,
- Del Pezzo,
- blow-up of $\mathbb{P}^2$ at points lying on a conic,
- some $K3$ surfaces.

Cox rings: relations

Theorem (A. Laface, M. Velasco, 2007). Let E_1, E_2 be two generators of $\mathrm{Cox}(X)$ with $E_1 \cdot E_2 = 0$, if the map:

$$\begin{aligned} \bigoplus_{k=1}^{2} H^0(X, D - E_k - E_i) &\longrightarrow H^0(X, D - E_i) \\ (a, b) &\longmapsto ax_1 + bx_2 \end{aligned}$$

is surjective for any $i > 2$ then I_X has no generators in degree D.

Observation. This happens if

$$H^1(X, D - E_1 - E_2 - E_i) = 0 \quad \text{for each } i > 2.$$

K3 surfaces

A K3 surface X is a

- ***simply connected*** compact complex surface

which admits

- a nowhere vanishing ***holomorphic two form*** ω_X.

Examples:

- Quartic surfaces of $\mathbb{P}^3$.
- Double covers of Del Pezzo surfaces branched along $B \in |-2K_Y|$.

Remark: $Pic(X)$ is a lattice.

K3 surfaces

Given an integral divisor $D \subset X$ on a K3 surface, the graded algebra:

$$R(X, D) := \bigoplus_{i \in \mathbb{Z}} H^0(X, iD)$$

is finitely generated by elements of degree $i \leq 3$.

Remark:

- $\mathrm{Cox}(X)$ f.g implies that $R(X, D)$ is f.g.

K3 surfaces

Which K3's have a f.g. Cox ring?

- $\mathrm{NE}(X)$ must be finitely generated.
- A theorem of Kovács (1994) describes the generators of $\mathrm{NE}(X)$:

  either an ample class or rational curves with self-intersections 0 and -2.

Cox rings of K3 surfaces (with J. Hausen and A. Laface)

We deal with two classes of K3 surfaces:

- with $\operatorname{rk}\operatorname{Pic}(X) = 2$,
- admitting a non-symplectic involution.

K3 surfaces with $\rho = 2$

X is a K3 surface with $\operatorname{rk}\operatorname{Pic}(X) = 2$ and cone of curves

$$\mathrm{NE}(X) = \langle E_1, E_2 \rangle.$$

The intersection matrix of $\operatorname{Pic}(X)$ is:

- $\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} = U(k)$ with $k \geq 2$
- $\begin{pmatrix} 0 & k \\ k & -2 \end{pmatrix}$ with $k \geq 1$
- $\begin{pmatrix} -2 & k \\ k & -2 \end{pmatrix}$ with $k \geq 3$

K3 surfaces with $\rho = 2$

Theorem. Let X be a K3 surface with $\mathrm{Pic}(X) \cong U(k)$ and $k \geq 2$, then the degrees of generators and relations of $\mathrm{Cox}(X)$ are:

k	deg. gen.	deg. rel.
2	$(1,0), (0,1), (2,2)$	$(4,4)$
≥ 3	$(1,0), (0,1), (1,1)$	$(2,2)$

K3 surfaces with $\rho = 2$

Example. If X is a K3 surface with $\mathrm{Pic}(X) \cong U(2)$ then

$$\mathrm{Cox}(X) \cong \frac{\mathbb{C}[a_1, a_2, b_1, b_2, c]}{(c^2 - f_{4,4}(a, b))}.$$

K3 surfaces with non-symplectic involution

Proposition. Let (X, σ) be a generic pair of a K3 surface together with a non-symplectic involution and

$$\pi : X \longrightarrow X/ < \sigma >= Y$$

be the induced double cover. Then

- Y is either smooth rational or an Enriques surface,
- $\pi^* \operatorname{Pic}(Y)$ has index 2^r in $\operatorname{Pic}(X)$, where $r-1$ is the number of components of the branch locus of π.

K3 surfaces with non-symplectic involution

Let R be the fixed locus of σ and $B = \pi(R)$.

Lemma. For any divisor D of Y:

$$H^0(X, \pi^* D) \cong \pi^* H^0(X, D) \oplus x_R \cdot \pi^* H^0(X, D - B/2)$$

where x_R is the section defining R.

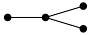
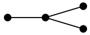
K3 surfaces with non-symplectic involution

Theorem. Let (X, Y, σ) as before. If the branch divisor B of π is irreducible then

$$\mathrm{Cox}(X) \cong \frac{\mathrm{Cox}(Y)[x_R]}{(x_R^2 - x_B)}.$$

Corollary. $\mathrm{Cox}(X)$ is known if Y is a Del Pezzo surface.

K3 surfaces with non-symplectic involution

$\rho(X)$	$\mathrm{Pic}(X)$	Y	Blow-up at:
2	U	$\mathbb{F}_4$	
	$U(2)$	$\mathbb{F}_0$	
	$(2) \oplus A_1$	$\mathbb{P}^2$	1 pt.
3	$U \oplus A_1$	$\mathbb{F}_4$	1 pt.
	$U(2) \oplus A_1$	$\mathbb{F}_0$	1 pt.
4	$U \oplus 2A_1$	$\mathbb{F}_4$	2 pts.
	$U(2) \oplus 2A_1$	$\mathbb{F}_0$	2 pts.
5	$U \oplus 3A_1$	$\mathbb{F}_4$	3 pts.
	$U(2) \oplus 3A_1$	$\mathbb{F}_0$	3 pts.
6	$U \oplus 4A_1$	$\mathbb{F}_4$	4 pts.
	$U(2) \oplus 4A_1$	$\mathbb{F}_0$	4 pts.
	$U \oplus D_4$	$\mathbb{F}_4$	
	$U(2) \oplus D_4$	$\mathbb{F}_0$	

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