Amenability, interval exchange transformations and groups of dynamical origin

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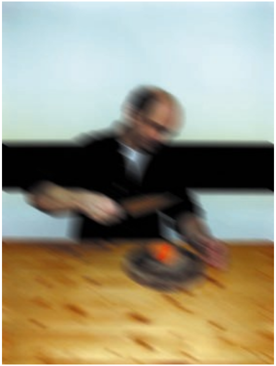
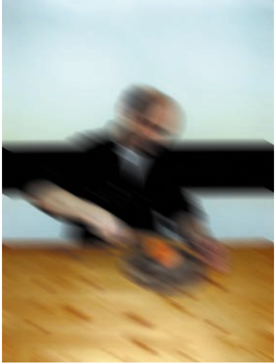
Banach-Tarski Paradox, 1924: There exists a decomposition of a ball into a finite number of non-overlapping pieces, which can be assembeled together into two identical copies of the original ball.





Hausdorff Paradox, 1914: The unit sphere can be decomposed into finitely many pieces in a way that rotating these pieces we can obtain two unit spheres.

Def: An action of a group G on a set X is **paradoxical** if there exist a pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m$ in X and there exist $g_1, \ldots, g_n$, $h_1, \ldots, h_m$ in G such that

$$X = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j.$$

A group G is **paradoxical**, if its left action on itself is paradoxical.

If group G is paradoxical then any free action ($gx = x$ only if $g = e$) is paradoxical.

Hausdorff: The free group on two generators is paradoxical, and SO(3) contains it.

Let $\langle a, b \rangle = \mathbb{F}_2$ be the free non-abelian group, and $\omega(x)$ the set of all reduced words in $\mathbb{F}_2$ that start with x.

$$\mathbb{F}_2 = \{e\} \cup \omega(a) \cup \omega(a^{-1}) \cup \omega(b) \cup \omega(b^{-1}).$$

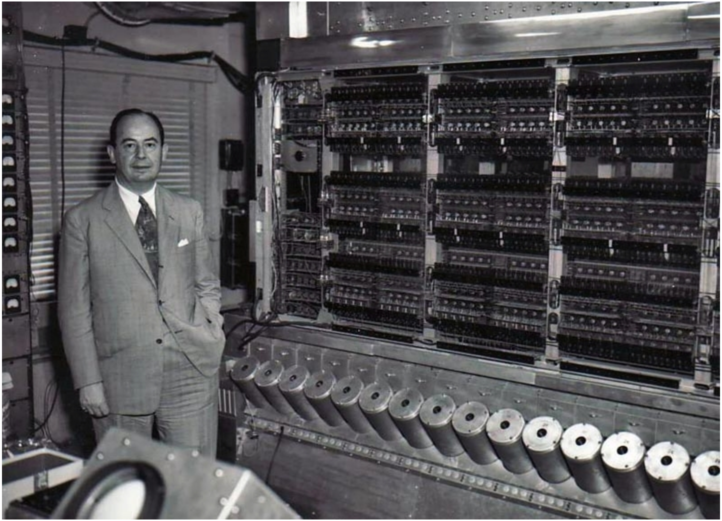
Since $\mathbb{F}_2 \backslash \omega(x) = x\omega(x^{-1})$ for all x in $\{a, a^{-1}, b, b^{-1}\}$ we have a paradoxical decomposition:

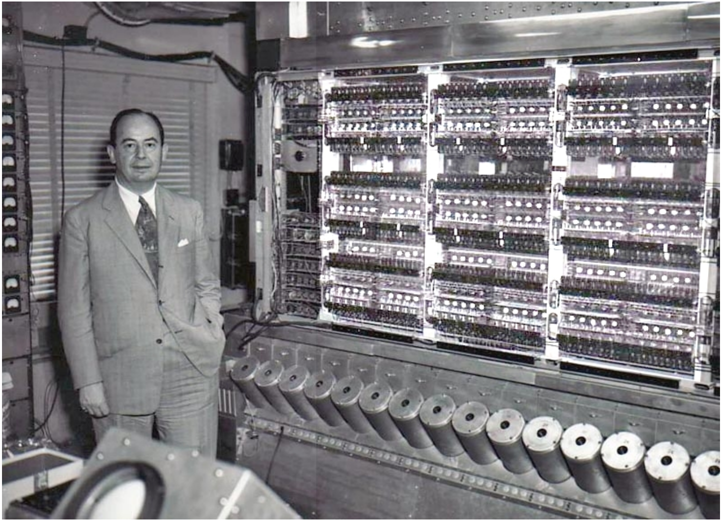
$$\mathbb{F}_2 = \omega(a) \cup a\omega(a^{-1}) = \omega(b) \cup b\omega(b^{-1}).$$

Hausdorff: These are free

$$T^{\pm 1} = \begin{pmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R^{\pm 1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$





Amenability!

Amenable actions

Definition

An action of a discrete group G on a set X is **amenable** if there exists a map $\mu : \mathcal{P}(X) \to [0, 1]$ such that

1. $\mu(X) = 1$, μ is finitely additive
2. $\mu(gE) = \mu(E)$ for all $E \subset X$ and $g \in G$.

Definition

G is **amenable** if the action of G on itself by left multiplication is amenable.

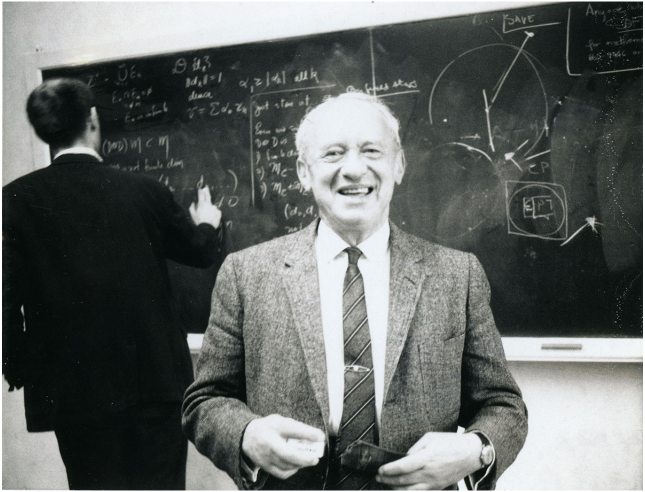
Crucial: If a group is amenable then all its actions are amenable.

Amenable actions are not paradoxical!

$A_1, \ldots, A_n, B_1, \ldots, B_m \subset X$ and $g_1, \ldots, g_n$, $h_1, \ldots, h_m$ in G such that

$$X = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j.$$

$$\begin{aligned}
1 = \mu(X) &\geq \mu(\bigcup_{i=1}^{n} A_i) + \mu(\bigcup_{i=1}^{m} B_j) = \sum_{i=1}^{n} \mu(A_i) + \sum_{j=1}^{m} \mu(B_j) \\
&= \sum_{i=1}^{n} \mu(g_i A_i) + \sum_{j=1}^{m} \mu(h_j B_j) \\
&\geq \mu(\bigcup_{i=1}^{n} g_i A_i) + \mu(\bigcup_{j=1}^{m} h_j B_j) = 2
\end{aligned}$$



Tarski: A group is not amenable if and only if it is paradoxical.

Basic set of equivalent definitions:

The following are equivalent to amenability:

- there exists a G-invariant finitely additive probability measure;
- there exists a G-invariant mean, i.e., $m \in l^{\infty}(G)$ such that $m(\chi_G) = 1$, $m(f) \geq 0$ for every $f \geq 0$;
- *Reiter's condition* For any finite subset $E \subset G$ and $\varepsilon > 0$, there exists $m \in l^1(G)$ with $\|m\|_1 = 1$

  $$\|g.m - m\|_1 \leq \epsilon \text{ for every } g \in E;$$

- *Følner condition*. For any finite subset $E \subset G$ and $\varepsilon > 0$, there exists a finite subset $F \subset G$ such that

  $$|gF \Delta F| \leq \varepsilon |F| \text{ for all } g \in E.$$

Example:

- finite groups;
- $\mathbb{Z}$ is amenable, since $[-n, n]$ is a Følner set.
- Groups of subexponential growth, i.e.,

$$\limsup_n |B_n(S)|^{1/n} = 1.$$

In particular, all abelian groups are amenable.

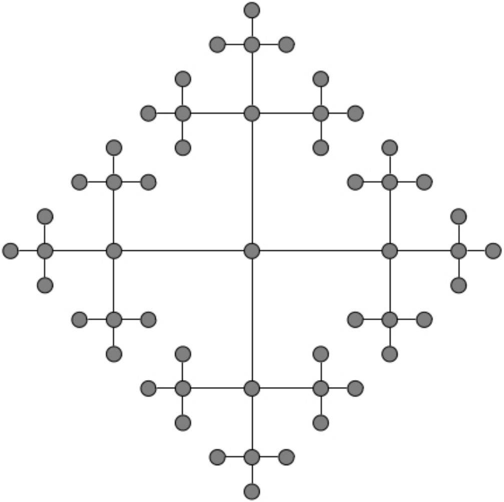
More definitions of non-amenability:

G is finitely generated group.

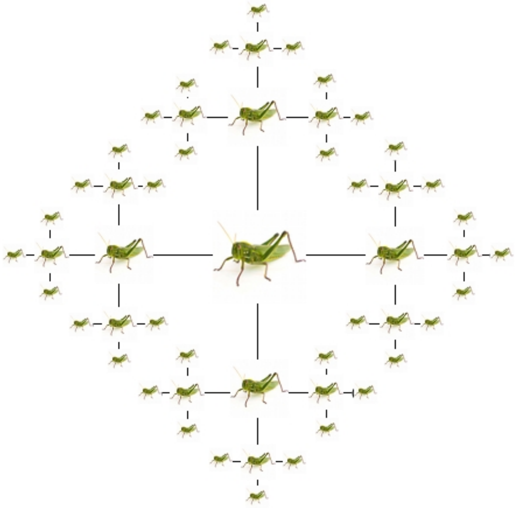
$$N(E) = S \cdot E$$

Gromov's doubling condition: There exists a finite generating set S, such that for every finite set $E \subset G$

$$|N(E)| \geq 2|E|$$



Grasshopper’s condition



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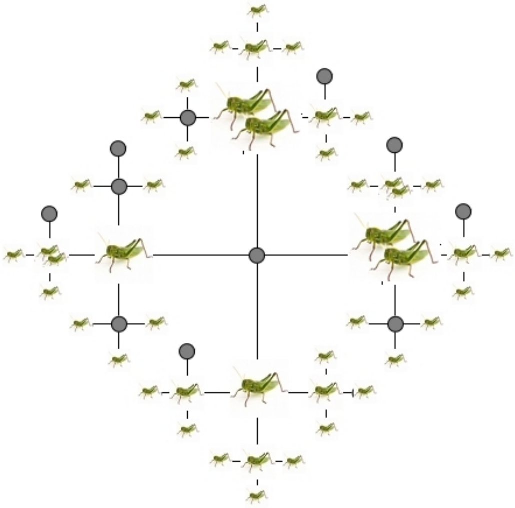

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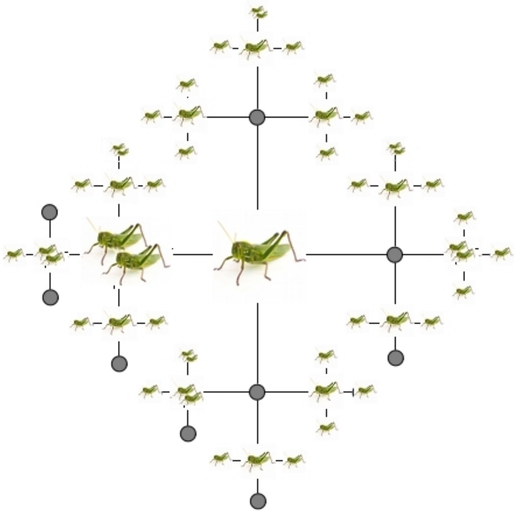

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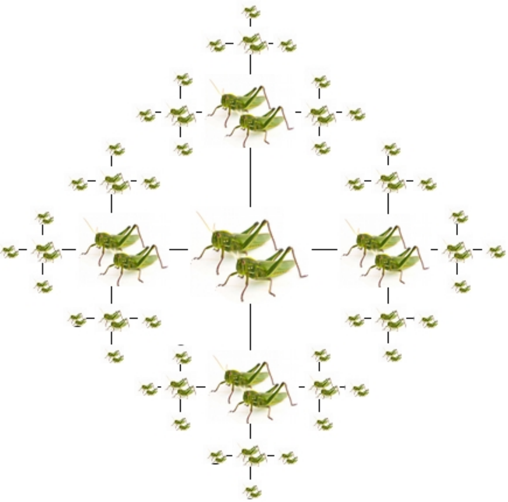


Grasshopper's condition

Elementary amenable groups

Definition

The class of **elementary amenable groups** is the smallest class which contains all finite and abelian groups and closed under taking subgroups, quotients, extensions and direct limits.

von Neumann-Day problem, '57: find non-elementary amenable group.

Properties of elementary amenable groups (mainly due to Chou)

- Finitely generated elementary amenable group has either polynomial or exponential growth;
- Finitely generated torsion groups are not elementary amenable;
- Simple and finitely generated groups are not elementary amenable.

Grigorchuk, '83: Grigorchuk's group of intermediate group

(Grigorchuk, Zuk '02)+(amenability proof of Bartholdi, Virag '05): Basilica group

Juschenko, Monod '12+ Matui 10': the full topological group of Cantor minimal system

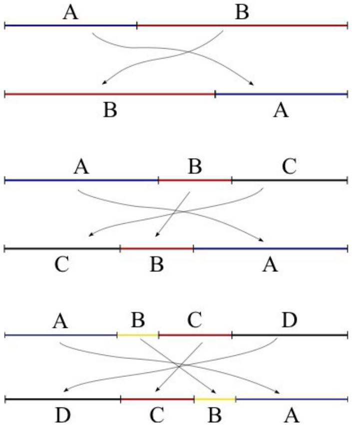
Juschenko, '15: Many subgroups of automorphism group of the tree. Most notable: all branch groups

Juschenko, Matte Bon, Monod, de la Salle '15 + Chorniy, Juschenko, Nekrashaevych, '15: Many subgroups of interval exchange transformation group.

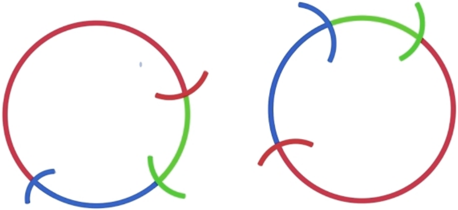
Nekrashevych, '16: subgroups of the full topological groups. In particular, simple Burnside groups, simple groups of intermediate growth

Currently know examples: subgroups of automorphisms of the tree or full topological of minimal $\mathbb{Z}$ or $\mathbb{Z}^2$ actions.

Interval exchange transformation group



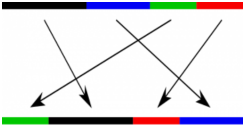
Interval exchange is an orientation preserving left-continuous rearrangement of intervals of $[0, 1)$



We may also think transformations of arcs of $\mathbb{R}/\mathbb{Z}$.

Katok's Conjecture: interval exchange transformation group contains $\mathbb{F}_2$

Difficulty: the growth of orbits of the action on $\mathbb{R}/\mathbb{Z}$ is polynomial.



A permutation σ is called *admissible* if there is no such m with $\sigma(m) = m$ and $\{1, .., m-1\}$ is σ-invariant. Let IET_a be the set of all interval exchanges with admissible underlying permutation.

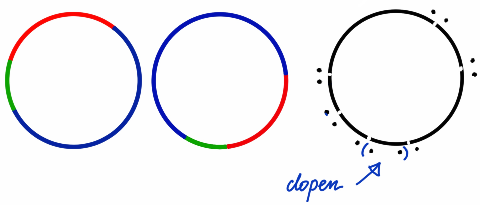
Theorem (Dahmani-Fujiwara-Guirardel, '11)

There is a dense open subset of $IET_a \times IET$ such that the group generated by any pair of it is not free.

Topological full groups

Let G be a group acting by homeomorphisms on a topological space $\mathcal{X}$.

The full topological group of the action, $[[G]]$, is the group of all homeomorphisms h of $\mathcal{X}$ such that for every $x \in \mathcal{X}$ there exists a neighborhood of x such that restriction of h to that neighborhood is equal to restriction of an element of G.



This defines a Cantor space with continuous action.

Let $\Gamma_{\theta_1,\ldots,\theta_n}$ be the group of all iet that piece-wise act as a fixed family of rotations $\mathcal{R}_{\alpha_1}, \ldots, \mathcal{R}_{\alpha_n}$.

$$\Gamma_{\theta_1,\ldots,\theta_n} = [[\mathcal{R}_{\alpha_1}, \ldots, \mathcal{R}_{\alpha_n}]]$$

In particular, we have every finitely generated subgroup of IET is a subgroup of $[[\mathbb{Z}^d \oplus F]]$, for some d and $|F| < \infty$. Moreover, we can assume that the action is minimal.

de Cornulier's conjecture: IET is amenable.

Difficulty:

- IET is not *elementary amenable*;
- For every $d \geq 2$ there are minimal actions of $\mathbb{Z}^d$ with non-amenable $[[\mathbb{Z}^d]]$.

Theorem (Juschenko-Monod, '13, Annals of Math)

For minimal actions of $\mathbb{Z}$ on the Cantor space, the group $[[\mathbb{Z}]]$ is amenable. In particular, Γ_θ for irrational θ is amenable.

Vorobets, ’11: IET$'$ is simple.

Chorniy, Juschenko, Nekrashaevych, ’16:
$\Gamma'_{\theta_1,\ldots,\theta_n} < \mathit{IET}$ is simple and finitely generated.

(Methods are based on a techniques of Matui for $[[\mathbb{Z}]]'$)

The group is not finitely presented, it’s LEF.

Juschenko, Matte Bon, Monod, de la Salle (’15, J. Ergodic Theory and Dyn Syst.):
$\Gamma_{\theta_1,\ldots,\theta_n}$ is amenable provided that the group generated by $\theta_1,\ldots,\theta_n$ is virtually $\mathbb{Z}^2$.

Amenable and faithful actions of non-amenable groups

Let $W(\mathbb{Z}^d)$ be **the wobbling group** of integers, i.e. $W(\mathbb{Z})^d$ consists of all bijections $g : \mathbb{Z}^d \to \mathbb{Z}^d$ such that

$$||g(x) - x|| \text{ is uniformly bounded.}$$

Obviously, the action of $W(\mathbb{Z})$ on $\mathbb{Z}$ admits an invariant mean.

van Douwen: $\mathbb{F}_2 < W(\mathbb{Z})$.

Fact

Assume that the action of G on X is amenable and $Stab_G(x)$ is amenable for every x in X, then G is amenable.

Extensively amenable actions

Fact

Assume that the action of G on X is amenable and $Stab_G(x)$ is amenable for every x in X, then G is amenable.

$G \curvearrowright X$ is **extensively amenable** if one of the following equivalent conditions hold:

- The action of $\bigoplus_X \mathbb{Z}_2 \rtimes G$ on $\bigoplus_X \mathbb{Z}_2$ is amenable;
- The action of G on $\mathcal{P}_f(X)$ admits an G-invariant mean giving the full weight to the collection of sets containing p;

Fact

G is amenable iff $\bigoplus_X \mathbb{Z}_2 \rtimes G$ is amenable.

Theorem (J-Nekrashevych-de la Salle, '14, Invent. Math.)

If $G \curvearrowright X$ recurrent then it is extensively amenable.

If there exists an increasing to X sequence of subsets X_i such that $\sum_i |\partial X_i|^{-1} = \infty$ then the action is recurrent.

Fact

Let F be a functor from the category of sets to the category of amenable groups, which maps each finite set to a finite group. If the action of Γ on X is extensively amenable, then the action of $F(X) \rtimes \Gamma$ on $F(X)$ is extensively amenable.

Example

$\Gamma \curvearrowright X$, we define a functor as $F(X) = \bigoplus_X \mathbb{Z}/2\mathbb{Z}$.

Example

For a set X we define a functor $F(X) = Sym(X)$, the group of finitely supported permutations of a set X, on which Γ acts by conjugation.

Theorem (J-Nekrashevych-de la Salle)

The action of $W(Z^d) \curvearrowright \mathbb{Z}^d$ is extensively amenable for $d = 1, 2$.

$IET(\Lambda)$ the subgroup of all $g \in IET$ so that the angle $gx - x$ is in Λ for every $x \in \mathbb{R}/\mathbb{Z}$.

Define a finitely supported bijection of $\mathbb{R}/\mathbb{Z}$:

$$\tau_g = \hat{g}^{-1} g,$$

where $\hat{g}$ is the right-continuous bijection uniquely defined by g.

$$\begin{aligned} \iota \colon IET &\to Sym_\infty(\mathbb{R}/\mathbb{Z}) \rtimes IET \\ g &\mapsto (\tau_g, g) \end{aligned}$$

Theorem (J-Matte Bon-Monod-de la Salle)

Let Λ be finitely generated subgroup of $\mathbb{R}/\mathbb{Z}$. If $rank(\Lambda) \leq 2$, then $IET(\Lambda)$ is amenable.

Thank you