Lie theory on real hyperplane arrangements

Marcelo Aguiar, Cornell University
joint with Swapneel Mahajan, IIT Mumbai

XXI Coloquio Latinoamericano de Álgebra
Universidad de Buenos Aires
July 25, 2016

Lie brackets

Given variables x and y, their bracket is

$$[x, y] = xy - yx.$$

Let $x_1, \ldots, x_n$ be n independent variables.
Let Per_n be the space spanned by all monomials

$$x_{\sigma(1)} \cdots x_{\sigma(n)} \quad \text{where} \quad \sigma \in \mathrm{S}_n.$$

Let Lie_n be the subspace of Per_n spanned by all full bracketings of $x_1, \ldots, x_n$. For example, a full bracketing when $n = 3$ is

$$\big[[x_2, x_3], x_1\big] = x_2x_3x_1 - x_1x_2x_3 - x_3x_2x_1 + x_1x_3x_2.$$

Fact. $\dim \mathrm{Lie}_n = (n-1)!$.
Example. Lie_3 is spanned by

$$\big[x_1, [x_2, x_3]\big] \quad \text{and} \quad \big[x_2, [x_1, x_3]\big] \quad \text{(Dynkin basis)}.$$

Free algebras

Let V be a vector space and $L(V)$ the free Lie algebra on V. Then

$$L(V) = \bigoplus_{n \geq 1} \mathrm{Lie}_n \otimes_{\mathrm{S}_n} V^{\otimes n}.$$

The free associative algebra on V is

$$T(V) = \bigoplus_{n \geq 1} \mathrm{Per}_n \otimes_{\mathrm{S}_n} V^{\otimes n} = \bigoplus_{n \geq 1} V^{\otimes n}.$$

The free commutative algebra on V is

$$S(V) = \bigoplus_{n \geq 1} 1_n \otimes_{\mathrm{S}_n} V^{\otimes n} = \bigoplus_{n \geq 1} (V^{\otimes n})_{\mathrm{S}_n},$$

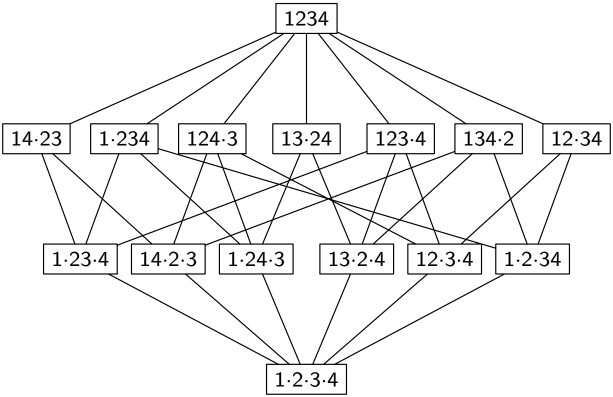
where 1_n is the trivial representation of S_n.
The sequences

$$\{\mathrm{Lie}_n\}_{n \geq 1}, \quad \{\mathrm{Per}_n\}_{n \geq 1}, \quad \{1_n\}_{n \geq 1}$$

are classical operads.

The partition lattice

Let Π_n be the lattice of partitions of $[n]$.



Fact. $\dim H_{\text{top}}(\Pi_n) = (n-1)!$.

Joyal-Klyachko-Stanley

Theorem There is an isomorphism of S_n-modules

$$\mathsf{Lie}_n \cong H_{\mathrm{top}}(\Pi_n) \otimes \epsilon_n$$

where Lie_n = n-linear part of the free Lie algebra,
Π_n = lattice of partitions of $[n]$,
ϵ_n = sign representation of S_n.

- Hanlon (1981), Stanley (1982)
- Klyachko (1974)
- Joyal (1986)
- Barcelo (1990), Barcelo-Bergeron (1990)
- Björner (1982, 1992), Wachs (1994), Björner-Wachs (2005)
- Garsia (1990)

The role of the braid arrangement

Claim: JKS is a statement about the braid arrangement $\mathcal{B}_n$.

$\mathcal{B}_n$ is the collection of hyperplanes $x_i = x_j$ in $\mathbb{R}^n$.
Π_n is the lattice of flats and Per_n is the set of chambers of $\mathcal{B}_n$.

In fact: JKS is a special case of a general result that holds for all real hyperplane arrangements.

Let $\mathcal{A}$ be a real hyperplane arrangement.
We define a space $\mathbf{Lie}(\mathcal{A})$ such that

$$\mathbf{Lie}(\mathcal{A}) \cong H^{\mathrm{top}}\big(\mathbf{\Pi}(\mathcal{A})\big) \otimes \mathbf{O}(\mathcal{A})$$

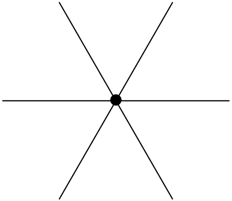
naturally in $\mathcal{A}$, where $\mathbf{\Pi}(\mathcal{A})$ is the lattice of flats and $\mathbf{O}(\mathcal{A})$ is the orientation space of $\mathcal{A}$.

Faces and flats

Let $\mathcal{A}$ be a hyperplane arrangement in a real vector space.

- The hyperplanes in $\mathcal{A}$ split space into a collection $\boldsymbol{\Sigma}(\mathcal{A})$ of convex polyhedral cones called faces.
- The faces of top dimension are called chambers.
  Let $\boldsymbol{\Gamma}(\mathcal{A})$ be the set of chambers.
- The subspaces obtained as intersections of hyperplanes in $\mathcal{A}$ are called flats. Let $\boldsymbol{\Pi}(\mathcal{A})$ be the set of flats.

Example. 3 lines, 13 faces (6 chambers), 5 flats.

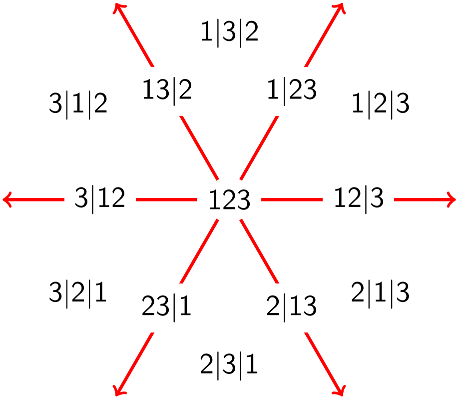


Faces of the braid arrangement

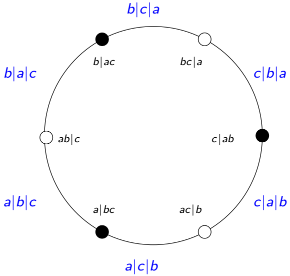
- Faces of $\mathcal{B}_n$ are in bijection with ordered partitions of $[n]$,

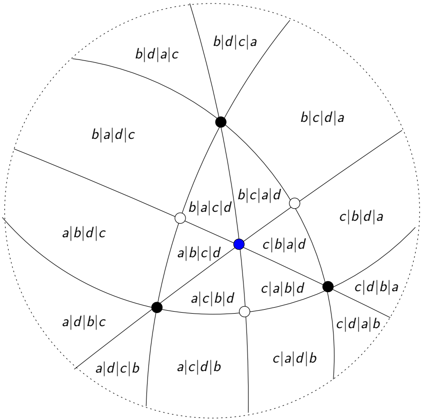
$$\text{e.g. } 1|23 = \{(x_1, x_2, x_3) \mid x_1 > x_2 = x_3\}.$$

- Chambers are in bijection with linear orders on $[n]$ (or permutations in S_n).
- Flats are in bijection with partitions of $[n]$.



The braid arrangements $\mathcal{B}_3$ and $\mathcal{B}_4$

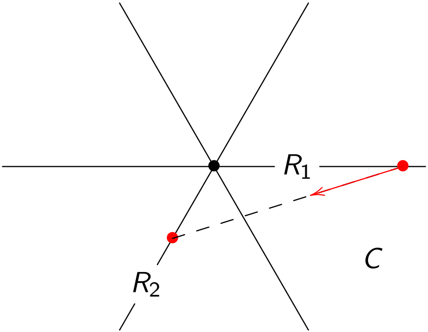




Faces and flats

- The set $\mathbf{\Pi}(\mathcal{A})$ is a lattice.
- The set $\mathbf{\Sigma}(\mathcal{A})$ is a monoid.
- The set $\mathbf{\Gamma}(\mathcal{A})$ is a two-sided ideal in $\mathbf{\Sigma}(\mathcal{A})$.

Example. $R_1 R_2 = C$.



- Bland (1974), Tits (1974), Bidigare-Hanlon-Rockmore (1997).
- Brown-Diaconis (1998), Billera-Brown-Diaconis (1999).

The support map

The support of a face F is the intersection of all the hyperplanes that contain it:

$$\operatorname{supp} F = \bigcap_{\mathrm{H} \supseteq F} \mathrm{H}.$$

It is a flat.

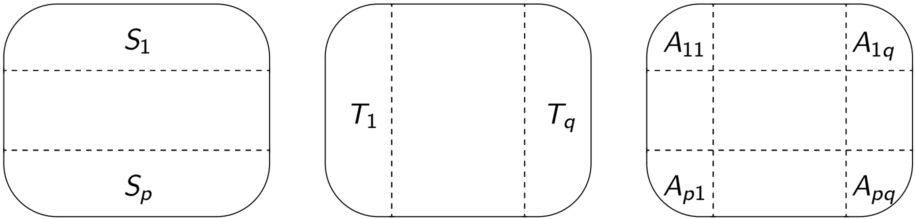
The map $\operatorname{supp} : \mathbf{\Sigma}(\mathcal{A}) \to \mathbf{\Pi}(\mathcal{A})$ is a morphism of monoids:

$$\operatorname{supp}(FG) = \operatorname{supp} F \vee \operatorname{supp} G.$$

Moreover, $\mathbf{\Pi}(\mathcal{A})$ is the abelianization of $\mathbf{\Sigma}(\mathcal{A})$.

The Tits product for the braid arrangement

Let $F = (S_1, \ldots, S_p)$ and $G = (T_1, \ldots, T_q)$ be ordered partitions of $[n]$.



The Tits product of F and G is

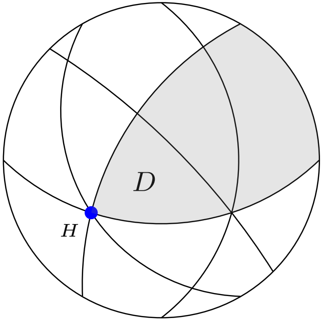
$$FG = (A_{11}, \ldots, A_{1q}, \ldots, A_{p1}, \ldots, A_{pq})$$

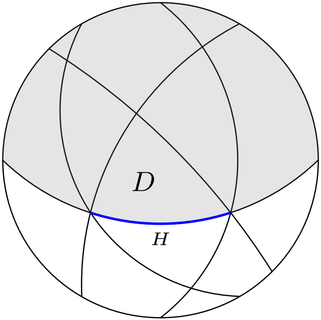
(empty intersections are removed).

Lunes

A face H and a chamber D with $H \leq D$ define a lune

$$\ell(H, D) = \{C \in \mathbf{\Gamma}(\mathcal{A}) \mid HC = D\}$$





Lie and the zero-lune condition

Let $\mathbb{k}$ be a field of characteristic 0.

Definition. $\mathbf{Lie}(\mathcal{A})$ is the subspace of $\mathbb{k}\mathbf{\Gamma}(\mathcal{A})$ consisting of elements

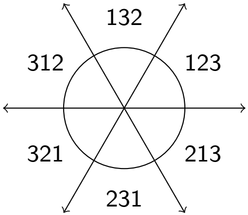
$$\sum_{C \in \mathbf{\Gamma}(\mathcal{A})} a_C \, C \quad \text{such that} \quad \sum_{C \in \ell} a_C = 0$$

for every nontrivial lune ℓ.

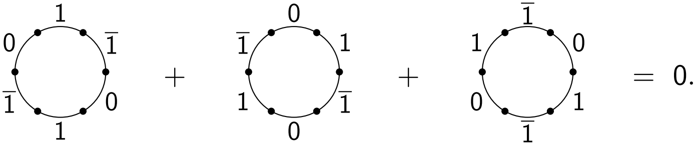
Let $\mathcal{B}_n$ be the braid arrangement.
Then $\mathbb{k}\mathbf{\Gamma}(\mathcal{B}_n) = \mathrm{Per}_n$.
That $\mathbf{Lie}(\mathcal{B}_n) = \mathrm{Lie}_n$ boils down to a classical criterion of Ree.

Zero-lune condition and Jacobi identity

The braid arrangement $\mathcal{A}_3$:



Three Lie elements that sum to 0:

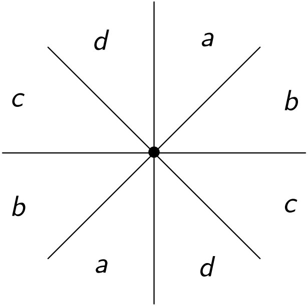


This is the Jacobi identity

$$[[2,3],1] + [[1,2],3] + [[3,1],2] = 0.$$

Example: rank 2 arrangements

Consider an arrangement of 4 lines. Lunes are halfplanes.



Zero-lune condition: $a + b + c + d = 0$.

For an arrangement $\mathcal{D}_n$ of n lines on the plane,

$$\dim \mathbf{Lie}(\mathcal{D}_n) = n - 1.$$

Joyal-Klyachko-Stanley generalized

Theorem. $\mathbf{Lie}(\mathcal{A}) \cong H^{\mathrm{top}}\big(\mathbf{\Pi}(\mathcal{A})\big) \otimes \mathbf{O}(\mathcal{A})$.

Moreover: Dynkin basis $\leftrightarrow$ Björner-Wachs basis.

Corollary. $\dim \mathbf{Lie}(\mathcal{A}) = (-1)^{\mathrm{rank}\mathcal{A}} \mu(\mathbf{\Pi}(\mathcal{A}))$ (Möbius invariant).

$\mathbf{\Pi}(\mathcal{A})$ is a geometric lattice:

$$\mathrm{rank}(X \vee Y) \geq \mathrm{rank}(X) + \mathrm{rank}(Y) - \mathrm{rank}(X \wedge Y).$$

The Dynkin-Specht-Wever Theorem

Let H be a generic hyperplane for $\mathcal{A}$. Define the Dynkin element

$$\theta_{\mathrm{H}} = \sum_{F \colon F \subseteq \mathrm{H}^+} (-1)^{\mathrm{rank}(F)} F.$$

Theorem. θ_{H} is an idempotent in the monoid algebra $\Bbbk\mathbf{\Sigma}(\mathcal{A})$. Moreover,

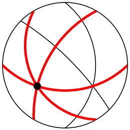
$$\theta_{\mathrm{H}} \Bbbk \mathbf{\Gamma}(\mathcal{A}) = \mathbf{Lie}(\mathcal{A}).$$

(Topology of lunes $\cap$ halfspaces enters in the proof.)

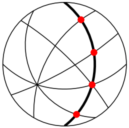
Corollary. The set $\{\theta_{\mathrm{H}} C \mid C \subseteq \mathrm{H}^-\}$ is a basis of $\mathbf{Lie}(\mathcal{A})$. This is the Dynkin basis.

Restriction and contraction

Let X be a flat of $\mathcal{A}$.
The restriction $\mathcal{A}_X$ consists of the hyperplanes H in $\mathcal{A}$ which contain X. The ambient space remains the same.



The contraction $\mathcal{A}^X$ consists of the intersections $\mathrm{H} \cap X$ where H is in $\mathcal{A}$ and does not contain X. The ambient space is X.



Faces under restriction and contraction

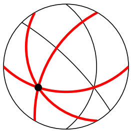
$$\mathbf{\Pi}(\mathcal{A}_X) = \{Y \in \mathbf{\Pi}(\mathcal{A}) \mid Y \geq X\}.$$

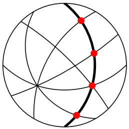
Given a face F with $\operatorname{supp} F = X$, there are canonical bijections

$$\mathbf{\Sigma}(\mathcal{A}_X) \cong \{G \in \mathbf{\Sigma}(\mathcal{A}) \mid G \geq F\},$$
$$\mathbf{\Gamma}(\mathcal{A}_X) \cong \{C \in \mathbf{\Gamma}(\mathcal{A}) \mid C \geq F\}.$$

Let G_F be the face of $\mathcal{A}_X$ corresponding to $G \supseteq F$.

$$\mathbf{\Pi}(\mathcal{A}^X) = \{Y \in \mathbf{\Pi}(\mathcal{A}) \mid Y \leq X\},$$
$$\mathbf{\Sigma}(\mathcal{A}^X) = \{F \in \mathbf{\Sigma}(\mathcal{A}) \mid \operatorname{supp} F \leq X\},$$
$$\mathbf{\Gamma}(\mathcal{A}^X) = \{C \in \mathbf{\Gamma}(\mathcal{A}) \mid \operatorname{supp} F = X\}.$$





Operads

Let $\mathbf{arr}^{\times}$ denote the groupoid of real hyperplane arrangements and their isomorphisms.

A generalized species is a functor

$$\mathbf{P} : \mathbf{arr}^{\times} \to \mathbf{Vec}\,.$$

Thus, $\mathbf{P}$ is a collection of vector spaces $\mathbf{P}(\mathcal{A})$, one space for each isomorphism class of real hyperplane arrangement $\mathcal{A}$.

The category of species is monoidal under substitution:

$$(\mathbf{P} \circ \mathbf{Q})(\mathcal{A}) = \bigoplus_{X \in \mathbf{\Pi}(\mathcal{A})} \mathbf{P}(\mathcal{A}^X) \otimes \mathbf{Q}(\mathcal{A}_X).$$

A generalized operad is a monoid in this category.

This parallels Joyal's approach to classical operads:

$$(\mathcal{B}_n)^X \cong \mathcal{B}_{|X|} \quad \text{and} \quad (\mathcal{B}_n)_X \cong \prod_{S \in X} \mathcal{B}_{|X|}.$$

The trinity of operads

The generalized associative operad is $\mathbf{As} := \Bbbk\mathbf{\Gamma}$:
for any flat X, let

$$\mathbf{As}(\mathcal{A}^X) \otimes \mathbf{As}(\mathcal{A}_X) \to \mathbf{As}(\mathcal{A}), \qquad F \otimes C_F \mapsto C.$$

Lie is the generalized Lie operad. It is a suboperad of **As**:

$$\begin{array}{ccc} \mathbf{As}(\mathcal{A}^X) \otimes \mathbf{As}(\mathcal{A}_X) & \longrightarrow & \mathbf{As}(\mathcal{A}) \\ \uparrow & & \uparrow \\ \mathbf{Lie}(\mathcal{A}^X) \otimes \mathbf{Lie}(\mathcal{A}_X) & \longrightarrow & \mathbf{Lie}(\mathcal{A}) \end{array}$$

For any arrangement $\mathcal{A}$, let $\mathbf{Com}(\mathcal{A}) = \Bbbk$.
For any flat X, let

$$\mathbf{Com}(\mathcal{A}^X) \otimes \mathbf{Com}(\mathcal{A}_X) \to \mathbf{Com}(\mathcal{A}), \qquad \Bbbk \otimes \Bbbk \cong \Bbbk.$$

Com is the generalized commutative operad.

Koszul duality

Theorem.

- **As**, **Lie** and **Com** are Koszul operads.
- $\mathbf{As}^! \cong \mathbf{As}$, $\mathbf{Lie}^! \cong \mathbf{Com}$.

Notes.

- JKS is a consequence of Koszul duality between **Lie** and **Com**. (Classical case: Fresse.)
- Another is the fact that the Tits algebra $\Bbbk\mathbf{\Sigma}(\mathcal{A})$ is (quadratic and) Koszul, with Koszul dual equal to the incidence algebra of the poset $\mathbf{\Pi}(\mathcal{A})$. (Facts known from work of Polo, Saliola).

Associative and commutative

Let $\mathbf{P}$ be a generalized species.
An associative structure on $\mathbf{P}$ is a collection of maps

$$\mu_F : \mathbf{P}(\mathcal{A}_X) \to \mathbf{P}(\mathcal{A}) \quad \text{where} \quad X = \operatorname{supp} F,$$

one for each $\mathcal{A}$ and each $F \in \mathbf{\Sigma}(\mathcal{A})$, subject to:

$$\begin{array}{ccc} \mathbf{P}(\mathcal{A}_O) & \xrightarrow{\mu_O} & \mathbf{P}(\mathcal{A}) \\ \| & \nearrow_{\mathrm{id}} & \\ \mathbf{P}(\mathcal{A}) & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbf{P}(\mathcal{A}_X) & \xrightarrow{\mu_F} & \mathbf{P}(\mathcal{A}) \\ {\scriptstyle \mu_{G_F}} \uparrow & \nearrow_{\mu_G} & \\ \mathbf{P}(\mathcal{A}_Y) & & \end{array}$$

whenever $F \leq G$. Here $X = \operatorname{supp} F$ and $Y = \operatorname{supp} G$.

A commutative structure is an associative structure such that

$$\mu_F = \mu_G \quad \text{whenever} \quad \operatorname{supp} F = \operatorname{supp} G.$$

Hopf and Lie

Let $\mathbf{H}$ be a generalized species.
A Hopf structure on $\mathbf{H}$ consists of two collections of maps

$$\mathbf{H}(\mathcal{A}_X) \underset{\Delta_F}{\overset{\mu_F}{\rightleftarrows}} \mathbf{H}(\mathcal{A})$$

that are associative and coassociative, subject to

$$\begin{array}{ccccc} \mathbf{H}(\mathcal{A}_X) & \xrightarrow{\mu_F} & \mathbf{H}(\mathcal{A}) & \xrightarrow{\Delta_G} & \mathbf{H}(\mathcal{A}_Y) \\ & \searrow^{\Delta_{(FG)_F}} & & \nearrow_{\mu_{(GF)_G}} & \\ & & \mathbf{H}(\mathcal{A}_{X \vee Y}) = \mathbf{H}(\mathcal{A}_{Y \vee X}) & & \end{array}$$

for every pair of faces F and G of $\mathcal{A}$.
Here $X = \operatorname{supp} F$ and $Y = \operatorname{supp} G$.

Lie structures can also be defined.

Primitives

Let $\mathbf{H}$ be a Hopf monoid.
Its primitive part is the generalized species $\mathbf{P}(\mathbf{H})$ defined by

$$\mathbf{P}(\mathbf{H})(\mathcal{A}) = \bigcap_{\substack{F \in \mathbf{\Sigma}(\mathcal{A}) \\ F \neq O}} \ker\big(\Delta_F : \mathbf{H}(\mathcal{A}) \to \mathbf{H}(\mathcal{A}_{\operatorname{supp} F})\big).$$

Proposition. $\mathbf{P}(\mathbf{H})$ is a Lie monoid.

The Hopf monoid of chambers

Consider the generalized species $\Bbbk\mathbf{\Gamma}(\mathcal{A})$ (underlying **As**).
For each face F of $\mathcal{A}$, define

$$\mu_F : \Bbbk\mathbf{\Gamma}(\mathcal{A}_X) \to \Bbbk\mathbf{\Gamma}(\mathcal{A}) \qquad \Delta_F : \Bbbk\mathbf{\Gamma}(\mathcal{A}) \to \Bbbk\mathbf{\Gamma}(\mathcal{A}_X)$$
$$C_F \mapsto C \qquad\qquad C \mapsto (FC)_F.$$

Proposition.

- $\Bbbk\mathbf{\Gamma}$ is a Hopf monoid.
- $\mathbf{Lie} = \mathbf{P}(\Bbbk\mathbf{\Gamma})$.

This generalizes a criterion of Friedrichs for the free Lie algebra:

$$L(V) = P(T(V)).$$

Cartier-Milnor-Moore

Let $\Bbbk$ be a field of characteristic 0.

Theorem. Let $\mathbf{H}$ be a cocommutative Hopf monoid. Then

$$\mathbf{H} \cong \mathbf{Com} \circ \mathbf{P}(\mathbf{H}).$$

In other words,

$$\mathbf{H}(\mathcal{A}) \cong \bigoplus_{X \in \mathbf{\Pi}(\mathcal{A})} \mathbf{P}(\mathbf{H})(\mathcal{A}_X).$$

- Follow Cartier's proof of the classical result.
- Generalize the classical Eulerian idempotents.
- Understand the structure of the algebra $\Bbbk\mathbf{\Sigma}(\mathcal{A})$ and its semisimple quotient $\Bbbk\mathbf{\Pi}(\mathcal{A})$.
- Employ results of Brown-Diaconis and Saliola.

Zaslavsky's formula

This may be obtained as a consequence of CMM.

Friedrichs: $\mathbf{Lie} = \mathbf{P}(\Bbbk\mathbf{\Gamma})$

CMM: $\Bbbk\mathbf{\Gamma}(\mathcal{A}) \cong \bigoplus_{X \in \mathbf{\Pi}(\mathcal{A})} \mathbf{Lie}(\mathcal{A}_X)$

$\Longrightarrow$ $\dim \Bbbk\mathbf{\Gamma}(\mathcal{A}) = \sum_{X \in \mathbf{\Pi}(\mathcal{A})} \dim \mathbf{Lie}(\mathcal{A}_X)$

JKS: $\#\mathbf{\Gamma}(\mathcal{A}) = \sum_{X \in \mathbf{\Pi}(\mathcal{A})} (-1)^{\mathrm{corank} X} \mu(\mathbf{\Pi}(\mathcal{A}_X)).$

The number of chambers is determined by the lattice of flats. (Zaslavsky)

Thank you.