About Spetses

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Let us give some examples.

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$$|\mathrm{SO}_{2n+1}(q)| = |\,\mathrm{Sp}_{2n}(q)|\,.$$

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Then we shall come back to the generic properties of $\mathrm{Un}(\mathbb{G})$.

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Let $\mathcal{A}$ be the set of reflecting hyperplanes of W. A root of unity ζ is called regular if there exist $w \in W$ and $x \in V^{\mathrm{reg}} := V \setminus \bigcup_{H \in \mathcal{A}} H$ such that $w(x) = \zeta x$. We then say that w is ζ-regular.

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[Note that $W_1 = W$].

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We shall now introduce the notion of ζ-cyclotomic Hecke algebra.

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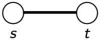

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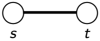

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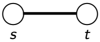

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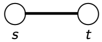
  - Case where $G = \mathrm{GL}_3$, $\zeta = 1$: $W_\zeta = W = \mathfrak{S}_3 \quad \longleftrightarrow$ 

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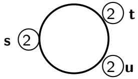
  - For $G = \mathrm{O}_8(q)$, $W = D_4$, $\zeta = i$, $W_i = G(4, 2, 2) \quad \longleftrightarrow$ 

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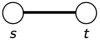

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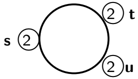

$$\mathcal{H}(W_i) = \left\langle S, T, U \,;\, \begin{Bmatrix} STU = TUS = UST \\ (S - x^2)(S - 1) = 0 \end{Bmatrix} \right\rangle$$

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$$S_\chi \in \mathbb{C}[x^{1/|ZW_\zeta|}, x^{-1/|ZW_\zeta|}] \quad \text{defined by} \quad \tau = \sum_{\chi \in \mathrm{Irr}\mathcal{H}(W_\zeta)} \frac{\chi}{S_\chi}.$$

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M. Broué, G. Malle, J. Michel, Split spetses for primitive reflection groups, *Astérisque* **359** (2014)

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Theorem

2.1. Spetsial 1-cyclotomic Hecke algebras, special groups

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(ii) For each absolutely irreducible character χ of $\mathcal{H}(W)$, $S_\chi \in \mathbb{C}[x, x^{-1}]$.

(iii) W is one of the following groups (Shephard–Todd's notation), called the spetsial groups:

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2.2. Spetsial groups

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- G_4, G_6, G_8, G_{25}, G_{26}, G_{32}.

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with lots of properties (axioms) described below.

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Definition

Let $\zeta \in \boldsymbol{\mu}$.
The ζ-principal series is

$$\mathrm{Un}(\mathbb{G}, \zeta) := \{\rho \in \mathrm{Un}(\mathbb{G}) \mid \mathrm{Deg}_\rho(\zeta) \neq 0\} .$$

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3.3. Rouquier blocks

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- If the representation of W_ζ on V_ζ is rational over some cyclotomic field K, the ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ may be defined over $\mathbb{Z}_K[x, x^{-1}]$.

Definition

The Rouquier blocks of a ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ are the blocks of the algebra

$$\mathbb{Z}_K[x, x^{-1}, ((x^n - 1)^{-1})_{n \geq 1}] \otimes_{\mathbb{Z}[x,x^{-1}]} \mathcal{H}(W_\zeta)\,.$$

- The Rouquier blocks of ζ-cyclotomic Hecke algebras have been classified in all cases (Malle–Rouquier, B.–Kim, Chlouveraki).
- For $\zeta = 1$ and W Coxeter group, Rouquier blocks are nothing but the characters associated with two sided cells (Kazhdan–Lusztig theory).

3.4. Families and Rouquier blocks

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Families

There is a partition

$$\mathrm{Un}(\mathbb{G}) = \bigsqcup_{\mathcal{F} \in \mathrm{Fam}(\mathbb{G})} \mathcal{F}$$

(where the $\mathcal{F}$'s are the families of unipotent characters), hence for all regular ζ,

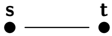
$$\mathrm{Un}(\mathbb{G}, \zeta) = \bigsqcup_{\mathcal{F} \in \mathrm{Fam}(\mathbb{G})} (\mathcal{F} \cap \mathrm{Un}(\mathbb{G}, \zeta)) \,,$$

with the following properties.

1. Through the bijection $\mathrm{Un}(\mathbb{G}, \zeta) \xrightarrow{\sim} \mathrm{Irr}\, \mathcal{H}(W_\zeta)$, the nonempty intersections $\mathcal{F} \cap \mathrm{Un}(\mathbb{G}, \zeta)$ are the Rouquier blocks of $\mathrm{Irr}\, \mathcal{H}(W_\zeta)$.
2. The integers a_ρ (valuation of Deg_ρ) and A_ρ (degree of Deg_ρ) are constant for ρ in a family $\mathcal{F}$.

The Fourier matrices

Let us denote by $\mathbf{B}_2$ the braid group on three brands, generated by two elements $\mathbf{s}$ and $\mathbf{t}$ satisfying the relation



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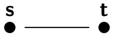

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- the center of $\mathbf{B}_2$ is infinite cyclic and generated by $\mathbf{w}_0^2 = (\mathbf{sts})^2 = (\mathbf{st})^3$,
- the map
$$\mathbf{s} \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \;,\; \mathbf{t} \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$
induces an isomorphism $\mathbf{B}_2/\langle \mathbf{w}_0^4 \rangle \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Z})$.

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There is a complex matrix S with entries indexed by $\mathcal{F} \times \mathcal{F}$, such that for all $\chi_0 \in \mathrm{Irr}(W)$,

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    1. the corresponding row i_0 of S has no zero entry,
    2. (Verlinde type formula) for all $i, j, k \in \mathcal{F}$, the sums $\sum_l S_{l,i} S_{l,j} S^*_{l,k} S^{-1}_{l,i_0}$ are integers.

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All this makes us think of a kind of *modular datum*, and perhaps for the Spets of a kind of *triangulated modular tensor category (?)*.

The Fourier matrix for G_4

		01	02	12		01	34	04	25	13
	1	.	.	.	.	.	.	.	.	.
01	.	$\frac{1+\frac{1}{\sqrt{-3}}}{2}$	$\frac{1-\frac{1}{\sqrt{-3}}}{2}$	$\frac{-1}{\sqrt{-3}}$	.	.	.	.	.	.
02	.	$\frac{1-\frac{1}{\sqrt{-3}}}{2}$	$\frac{1+\frac{1}{\sqrt{-3}}}{2}$	$\frac{1}{\sqrt{-3}}$	.	.	.	.	.	.
12	.	$\frac{-1}{\sqrt{-3}}$	$\frac{1}{\sqrt{-3}}$	$\frac{-1}{\sqrt{-3}}$	.	.	.	.	.	.
	.	.	.	.	1	.	.	.	.	.
01	.	.	.	.	.	$\frac{1}{2\sqrt{-3}}$	$\frac{-1}{2\sqrt{-3}}$	$\frac{1}{2}$	$\frac{-1}{\sqrt{-3}}$	$\frac{1}{2}$
34	.	.	.	.	.	$\frac{-1}{2\sqrt{-3}}$	$\frac{1}{2\sqrt{-3}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{-3}}$	$\frac{1}{2}$
04	.	.	.	.	.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	.	$-\frac{1}{2}$
25	.	.	.	.	.	$\frac{-1}{\sqrt{-3}}$	$\frac{1}{\sqrt{-3}}$	.	$\frac{-1}{\sqrt{-3}}$	.
13	.	.	.	.	.	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	.	$\frac{1}{2}$

Unipotent characters for G_4

Unipotent characters for G_4 ③—③

In red = the Φ'_6–series.

● = the Φ_4–series.

Character	Degree	FakeDegree	Eigenvalue	Family
● $\phi_{1,0}$	● 1	1	1	C_1
$\phi_{2,1}$	$\frac{3-\sqrt{-3}}{6}q\Phi'_3\Phi_4\Phi''_6$	$q\Phi_4$	1	$X_3.01$
$\phi_{2,3}$	$\frac{3+\sqrt{-3}}{6}q\Phi''_3\Phi_4\Phi'_6$	$q^3\Phi_4$	1	$X_3.02$
$Z_3 : 2$	$\frac{\sqrt{-3}}{3}q\Phi_1\Phi_2\Phi_4$	0	ζ_3^2	$X_3.12$
● $\phi_{3,2}$	● $q^2\Phi_3\Phi_6$	$q^2\Phi_3\Phi_6$	1	C_1
$\phi_{1,4}$	$\frac{-\sqrt{-3}}{6}q^4\Phi''_3\Phi_4\Phi''_6$	q^4	1	$X_5.1$
$\phi_{1,8}$	$\frac{\sqrt{-3}}{6}q^4\Phi'_3\Phi_4\Phi'_6$	q^8	1	$X_5.2$
● $\phi_{2,5}$	● $\frac{1}{2}q^4\Phi_2^2\Phi_6$	$q^5\Phi_4$	1	$X_5.3$
$Z_3 : 11$	$\frac{\sqrt{-3}}{3}q^4\Phi_1\Phi_2\Phi_4$	0	ζ_3^2	$X_5.4$
● G_4	● $\frac{1}{2}q^4\Phi_1^2\Phi_3$	0	-1	$X_5.5$

Φ'_3, Φ''_3 (resp. Φ'_6, Φ''_6) are factors of Φ_3 (resp Φ_6) in $\mathbb{Q}(\zeta_3)$