Cluster algebras and quantum loop algebras

Bernard Leclerc
Université de Caen

XXI Coloquio Latinoamericano de Álgebra
Buenos Aires, 28/07/2016

Problems of representation theory

Problems of representation theory

- Given a group or an algebra, describe

Problems of representation theory

- Given a group or an algebra, describe
  - irreducible characters,

Problems of representation theory

- Given a group or an algebra, describe
  - irreducible characters,
  - natural bases in irreducible representations,

Problems of representation theory

- Given a group or an algebra, describe
  - irreducible characters,
  - natural bases in irreducible representations,
  - tensor product multiplicities.

Problems of representation theory

- Given a group or an algebra, describe
  - irreducible characters,
  - natural bases in irreducible representations,
  - tensor product multiplicities.

Example:

Finite-dimensional representations of $\mathfrak{sl}_n(\mathbb{C})$:

Problems of representation theory

- Given a group or an algebra, describe
  - irreducible characters,
  - natural bases in irreducible representations,
  - tensor product multiplicities.

Example:

Finite-dimensional representations of $\mathfrak{sl}_n(\mathbb{C})$:

- Schur polynomials,

Problems of representation theory

- Given a group or an algebra, describe
  - irreducible characters,
  - natural bases in irreducible representations,
  - tensor product multiplicities.

Example:

Finite-dimensional representations of $\mathfrak{sl}_n(\mathbb{C})$:

- Schur polynomials,
- Young tableaux,

Problems of representation theory

- Given a group or an algebra, describe
  - irreducible characters,
  - natural bases in irreducible representations,
  - tensor product multiplicities.

Example:

Finite-dimensional representations of $\mathfrak{sl}_n(\mathbb{C})$:

- Schur polynomials,
- Young tableaux,
- Littlewood-Richardson rule.

Classical Lie theory

Classical Lie theory

- $\mathfrak{g}$ semisimple complex Lie algebra.

Classical Lie theory

- $\mathfrak{g}$ semisimple complex Lie algebra.
- $\mathrm{Rep}(\mathfrak{g})$ tensor category of finite-dimensional representations.

Classical Lie theory

- $\mathfrak{g}$ semisimple complex Lie algebra.
- $\mathrm{Rep}(\mathfrak{g})$ tensor category of finite-dimensional representations.
- Weyl's character formula. Steinberg's tensor product formula.

Classical Lie theory

- $\mathfrak{g}$ semisimple complex Lie algebra.
- $\text{Rep}(\mathfrak{g})$ tensor category of finite-dimensional representations.
- Weyl's character formula. Steinberg's tensor product formula.
- Canonical bases and crystal bases (Lusztig, Kashiwara).

Classical Lie theory

- $\mathfrak{g}$ semisimple complex Lie algebra.

- $\mathrm{Rep}(\mathfrak{g})$ tensor category of finite-dimensional representations.

- Weyl's character formula. Steinberg's tensor product formula.

- Canonical bases and crystal bases (Lusztig, Kashiwara).

- Generalizations of Littlewood-Richardson rule (Kashiwara, Littelmann, Berenstein-Zelevinsky, ...)

Classical Lie theory

• $\mathfrak{g}$ semisimple complex Lie algebra.

• $\mathrm{Rep}(\mathfrak{g})$ tensor category of finite-dimensional representations.

• Weyl's character formula. Steinberg's tensor product formula.

• Canonical bases and crystal bases (Lusztig, Kashiwara).

• Generalizations of Littlewood-Richardson rule (Kashiwara, Littelmann, Berenstein-Zelevinsky, ...)

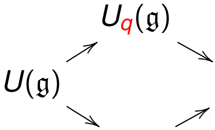
$\rightsquigarrow$ combinatorial understanding of the tensor structure of $\mathrm{Rep}(\mathfrak{g})$.

Quantum groups

Quantum groups

$$U(\mathfrak{g}) \begin{array}{l} \nearrow \qquad\qquad \searrow \\ \\ \searrow \qquad\qquad \nearrow \end{array}$$

Quantum groups



- $U_q(\mathfrak{g})$ quantum enveloping algebra ($q \in \mathbb{C}^*$, not a root of 1).

Quantum groups

$$\begin{array}{ccccc} & & U_q(\mathfrak{g}) & & \\ & \nearrow & & \searrow & \\ U(\mathfrak{g}) & & & & \\ & \searrow & & \nearrow & \\ & & U(L\mathfrak{g}) & & \end{array}$$

- $U_q(\mathfrak{g})$ quantum enveloping algebra ($q \in \mathbb{C}^*$, not a root of 1).
- $L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ loop algebra.

Quantum groups

$$\begin{array}{ccccc} & & U_q(\mathfrak{g}) & & \\ & \nearrow & & \searrow & \\ U(\mathfrak{g}) & & & & U_q(L\mathfrak{g}) \\ & \searrow & & \nearrow & \\ & & U(L\mathfrak{g}) & & \end{array}$$

- $U_q(\mathfrak{g})$ quantum enveloping algebra ($q \in \mathbb{C}^*$, not a root of 1).
- $L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ loop algebra.
- $U_q(L\mathfrak{g})$ quantum loop algebra.

Quantum groups

$$\begin{array}{ccccc} & & U_q(\mathfrak{g}) & & \\ & \nearrow & & \searrow & \\ U(\mathfrak{g}) & & & & U_q(L\mathfrak{g}) \\ & \searrow & & \nearrow & \\ & & U(L\mathfrak{g}) & & \end{array}$$

- $U_q(\mathfrak{g})$ quantum enveloping algebra ($q \in \mathbb{C}^*$, not a root of 1).
- $L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ loop algebra.
- $U_q(L\mathfrak{g})$ quantum loop algebra.

Aim: study the tensor category of finite-dimensional representations of $U_q(L\mathfrak{g})$.

Quantum groups

$$\begin{array}{ccccc} & & U_q(\mathfrak{g}) & & \\ & \nearrow & & \searrow & \\ U(\mathfrak{g}) & & & & U_q(L\mathfrak{g}) \\ & \searrow & & \nearrow & \\ & & U(L\mathfrak{g}) & & \end{array}$$

- $U_q(\mathfrak{g})$ quantum enveloping algebra ($q \in \mathbb{C}^*$, not a root of 1).
- $L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ loop algebra.
- $U_q(L\mathfrak{g})$ quantum loop algebra.

Aim: study the tensor category of finite-dimensional representations of $U_q(L\mathfrak{g})$.

Motivation: mathematical physics,

Quantum groups

$$
\begin{array}{ccccc}
 & & U_q(\mathfrak{g}) & & \\
 & \nearrow & & \searrow & \\
U(\mathfrak{g}) & & & & U_q(L\mathfrak{g}) \\
 & \searrow & & \nearrow & \\
 & & U(L\mathfrak{g}) & &
\end{array}
$$

- $U_q(\mathfrak{g})$ quantum enveloping algebra ($q \in \mathbb{C}^*$, not a root of 1).
- $L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ loop algebra.
- $U_q(L\mathfrak{g})$ quantum loop algebra.

Aim: study the tensor category of finite-dimensional representations of $U_q(L\mathfrak{g})$.

Motivation: mathematical physics, trigonometric solutions of Yang-Baxter equation.

What is known ?

What is known ?

- Classification of irreducibles by highest ℓ-weight (Chari-Pressley):

$$L(\widehat{\lambda}), \qquad \widehat{\lambda} \in \widehat{P}_+ := \bigoplus_{z\in\mathbb{C}^*} \bigoplus_{i\in I} \mathbb{N}(\varpi_i, z).$$

What is known ?

- Classification of irreducibles by highest ℓ-weight (Chari-Pressley):

$$L(\widehat{\lambda}), \qquad \widehat{\lambda} \in \widehat{P}_+ := \bigoplus_{z \in \mathbb{C}^*} \bigoplus_{i \in I} \mathbb{N}(\varpi_i, z).$$

- Kirillov-Reshetikhin modules:

$$W^{(i)}_{k,z} := L\left((\varpi_i, z) + (\varpi_i, zq^2) + \cdots + (\varpi_i, zq^{2k-2})\right).$$

What is known ?

- Classification of irreducibles by highest ℓ-weight (Chari-Pressley):

$$L(\widehat{\lambda}), \qquad \widehat{\lambda} \in \widehat{P}_+ := \bigoplus_{z \in \mathbb{C}^*} \bigoplus_{i \in I} \mathbb{N}(\varpi_i, z).$$

- Kirillov-Reshetikhin modules:

$$W_{k,z}^{(i)} := L\left((\varpi_i, z) + (\varpi_i, zq^2) + \cdots + (\varpi_i, zq^{2k-2})\right).$$

- q-character (Frenkel-Reshetikhin; Frenkel-Mukhin).

What is known ?

- Classification of irreducibles by highest ℓ-weight (Chari-Pressley):

$$L(\widehat{\lambda}), \qquad \widehat{\lambda} \in \widehat{P}_+ := \bigoplus_{z\in\mathbb{C}^*} \bigoplus_{i\in I} \mathbb{N}(\varpi_i, z).$$

- Kirillov-Reshetikhin modules:

$$W^{(i)}_{k,z} := L\left((\varpi_i, z) + (\varpi_i, zq^2) + \cdots + (\varpi_i, zq^{2k-2})\right).$$

- q-character (Frenkel-Reshetikhin; Frenkel-Mukhin).
- Geometric q-character formulas for simply laced $\mathfrak{g}$ (Ginzburg-Vasserot; Nakajima).

What is known ?

- Classification of irreducibles by highest ℓ-weight (Chari-Pressley):

$$L(\widehat{\lambda}), \qquad \widehat{\lambda} \in \widehat{P}_+ := \bigoplus_{z \in \mathbb{C}^*} \bigoplus_{i \in I} \mathbb{N}(\varpi_i, z).$$

- Kirillov-Reshetikhin modules:

$$W^{(i)}_{k,z} := L\Big((\varpi_i, z) + (\varpi_i, zq^2) + \cdots + (\varpi_i, zq^{2k-2})\Big).$$

- q-character (Frenkel-Reshetikhin; Frenkel-Mukhin).
- Geometric q-character formulas for simply laced $\mathfrak{g}$ (Ginzburg-Vasserot; Nakajima).
- T-system (Kuniba-Nakanishi-Suzuki; Nakajima; Hernandez):

$$\left[W^{(i)}_{k,z}\right]\left[W^{(i)}_{k,zq^2}\right] = \left[W^{(i)}_{k+1,z}\right]\left[W^{(i)}_{k-1,zq^2}\right] + \prod_j \left[W^{(j)}_{k,zq}\right]$$

Tensor structure

Tensor structure

- For $\widehat{\lambda} = (\varpi_{i_1}, z_1) + \cdots + (\varpi_{i_k}, z_k) \in \widehat{P}_+$ and $u \in \mathbb{C}^*$, set

$$\widehat{\lambda}[u] := (\varpi_{i_1}, z_1 u) + \cdots + (\varpi_{i_k}, z_k u).$$

Tensor structure

- For $\widehat{\lambda} = (\varpi_{i_1}, z_1) + \cdots + (\varpi_{i_k}, z_k) \in \widehat{P}_+$ and $u \in \mathbb{C}^*$, set

$$\widehat{\lambda}[u] := (\varpi_{i_1}, z_1 u) + \cdots + (\varpi_{i_k}, z_k u).$$

Fact: For any $\widehat{\lambda}, \widehat{\mu} \in \widehat{P}_+$ the tensor product $L(\widehat{\lambda}[u]) \otimes L(\widehat{\mu})$ is simple except for a finite number of $u \in \mathbb{C}^*$.

Tensor structure

- For $\widehat{\lambda} = (\varpi_{i_1}, z_1) + \cdots + (\varpi_{i_k}, z_k) \in \widehat{P}_+$ and $u \in \mathbb{C}^*$, set

$$\widehat{\lambda}[u] := (\varpi_{i_1}, z_1 u) + \cdots + (\varpi_{i_k}, z_k u).$$

Fact: For any $\widehat{\lambda}, \widehat{\mu} \in \widehat{P}_+$ the tensor product $L(\widehat{\lambda}[u]) \otimes L(\widehat{\mu})$ is simple except for a finite number of $u \in \mathbb{C}^*$.

$\rightsquigarrow$ "Many" simples are tensor products of smaller simples.

Tensor structure

- For $\widehat{\lambda} = (\varpi_{i_1}, z_1) + \cdots + (\varpi_{i_k}, z_k) \in \widehat{P}_+$ and $u \in \mathbb{C}^*$, set

$$\widehat{\lambda}[u] := (\varpi_{i_1}, z_1 u) + \cdots + (\varpi_{i_k}, z_k u).$$

Fact: For any $\widehat{\lambda}, \widehat{\mu} \in \widehat{P}_+$ the tensor product $L(\widehat{\lambda}[u]) \otimes L(\widehat{\mu})$ is simple except for a finite number of $u \in \mathbb{C}^*$.

$\rightsquigarrow$ "Many" simples are tensor products of smaller simples.

Problem

- What are the prime simples ?

Tensor structure

- For $\widehat{\lambda} = (\varpi_{i_1}, z_1) + \cdots + (\varpi_{i_k}, z_k) \in \widehat{P}_+$ and $u \in \mathbb{C}^*$, set

$$\widehat{\lambda}[u] := (\varpi_{i_1}, z_1 u) + \cdots + (\varpi_{i_k}, z_k u).$$

Fact: For any $\widehat{\lambda}, \widehat{\mu} \in \widehat{P}_+$ the tensor product $L(\widehat{\lambda}[u]) \otimes L(\widehat{\mu})$ is simple except for a finite number of $u \in \mathbb{C}^*$.

$\rightsquigarrow$ "Many" simples are tensor products of smaller simples.

Problem

- What are the prime simples ?
- What is the prime factorization of an arbitrary simple ?

Tensor structure

- For $\widehat{\lambda} = (\varpi_{i_1}, z_1) + \cdots + (\varpi_{i_k}, z_k) \in \widehat{P}_+$ and $u \in \mathbb{C}^*$, set

$$\widehat{\lambda}[u] := (\varpi_{i_1}, z_1 u) + \cdots + (\varpi_{i_k}, z_k u).$$

Fact: For any $\widehat{\lambda}, \widehat{\mu} \in \widehat{P}_+$ the tensor product $L(\widehat{\lambda}[u]) \otimes L(\widehat{\mu})$ is simple except for a finite number of $u \in \mathbb{C}^*$.

$\rightsquigarrow$ "Many" simples are tensor products of smaller simples.

Problem

- What are the prime simples ?
- What is the prime factorization of an arbitrary simple ?
- Which products of primes are simple ?

Tensor structure

- For $\widehat{\lambda} = (\varpi_{i_1}, z_1) + \cdots + (\varpi_{i_k}, z_k) \in \widehat{P}_+$ and $u \in \mathbb{C}^*$, set

$$\widehat{\lambda}[u] := (\varpi_{i_1}, z_1 u) + \cdots + (\varpi_{i_k}, z_k u).$$

Fact: For any $\widehat{\lambda}, \widehat{\mu} \in \widehat{P}_+$ the tensor product $L(\widehat{\lambda}[u]) \otimes L(\widehat{\mu})$ is simple except for a finite number of $u \in \mathbb{C}^*$.

$\rightsquigarrow$ "Many" simples are tensor products of smaller simples.

Problem

- What are the prime simples ?
- What is the prime factorization of an arbitrary simple ?
- Which products of primes are simple ?

- Chari-Pressley (1991) $\rightsquigarrow$ full answer for $U_q(L\mathfrak{sl}_2)$.

Tensor structure

- For $\widehat{\lambda} = (\varpi_{i_1}, z_1) + \cdots + (\varpi_{i_k}, z_k) \in \widehat{P}_+$ and $u \in \mathbb{C}^*$, set

$$\widehat{\lambda}[u] := (\varpi_{i_1}, z_1 u) + \cdots + (\varpi_{i_k}, z_k u).$$

Fact: For any $\widehat{\lambda}, \widehat{\mu} \in \widehat{P}_+$ the tensor product $L(\widehat{\lambda}[u]) \otimes L(\widehat{\mu})$ is simple except for a finite number of $u \in \mathbb{C}^*$.

$\rightsquigarrow$ "Many" simples are tensor products of smaller simples.

Problem

- What are the prime simples ?
- What is the prime factorization of an arbitrary simple ?
- Which products of primes are simple ?

- Chari-Pressley (1991) $\rightsquigarrow$ full answer for $U_q(L\mathfrak{sl}_2)$.
- Hernandez-L (2010, 2016); Nakajima (2011); Qin (2015):

$\rightsquigarrow$ partial answers and conjectures for $U_q(L\mathfrak{g})$.

Tensor structure

Main message

Tensor structure

Main message

$\mathfrak{g} \quad \rightsquigarrow \quad$ Littlewood-Richardson ring

Tensor structure

Main message

$\mathfrak{g}$	$\rightsquigarrow$	Littlewood-Richardson ring
$U_q(L\mathfrak{g})$	$\rightsquigarrow$	Fomin-Zelevinsky cluster algebra

Cluster algebra

$\mathscr{F} = \mathbb{Q}(x_1, \ldots, x_n)$: ambient field.

Cluster algebra

$\mathscr{F} = \mathbb{Q}(x_1, \ldots, x_n)$: ambient field.

Definition (Seed)

$((y_1, y_2, \ldots, y_n), Q)$, where

Cluster algebra

$\mathscr{F} = \mathbb{Q}(x_1, \ldots, x_n)$: ambient field.

Definition (Seed)

$((y_1, y_2, \ldots, y_n), Q)$, where

- the y_i ("variables") are a free generating set of $\mathscr{F}$;

Cluster algebra

$\mathscr{F} = \mathbb{Q}(x_1, \ldots, x_n)$: ambient field.

Definition (Seed)

$((y_1, y_2, \ldots, y_n), Q)$, where

- the y_i ("variables") are a free generating set of $\mathscr{F}$;
- Q is an oriented graph ("quiver") with n vertices, without loop nor 2-cycle.

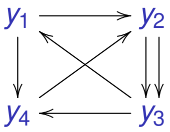
Cluster algebra

$\mathscr{F} = \mathbb{Q}(x_1, \ldots, x_n)$: ambient field.

Definition (Seed)

$((y_1, y_2, \ldots, y_n), Q)$, where

- the y_i ("variables") are a free generating set of $\mathscr{F}$;
- Q is an oriented graph ("quiver") with n vertices, without loop nor 2-cycle.



Mutations μ_k $(1 \leq k \leq n)$

Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of variables)

Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of variables)

$$\mu_k(y_j) = y_j \qquad \text{if } j \neq k;$$

Mutations μ_k $(1 \leq k \leq n)$

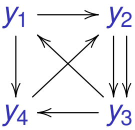
Definition (mutation of variables)

$$\begin{aligned} \mu_k(y_j) &= y_j \qquad \text{if } j \neq k; \\ \mu_k(y_k) &= \frac{\prod\limits_{i \to k} y_i + \prod\limits_{k \to j} y_j}{y_k}. \end{aligned}$$

Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of variables)

$$\begin{aligned} \mu_k(y_j) &= y_j \qquad \text{if } j \neq k; \\ \mu_k(y_k) &= \frac{\prod_{i \to k} y_i + \prod_{k \to j} y_j}{y_k}. \end{aligned}$$

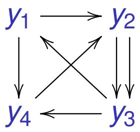


Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of variables)

$$\mu_k(y_j) = y_j \qquad \text{if } j \neq k;$$

$$\mu_k(y_k) = \frac{\prod_{i \to k} y_i + \prod_{k \to j} y_j}{y_k}.$$



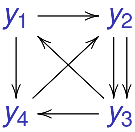
$$\mu_1(y_1) = \frac{y_3 + y_2 y_4}{y_1}$$

Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of variables)

$$\mu_k(y_j) = y_j \qquad \text{if } j \neq k;$$

$$\mu_k(y_k) = \frac{\prod_{i \to k} y_i + \prod_{k \to j} y_j}{y_k}.$$



$$\mu_1(y_1) = \frac{y_3 + y_2 y_4}{y_1}$$

$$\mu_3(y_3) = \frac{y_2^2 + y_1 y_4}{y_3}$$

Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of quiver)

Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of quiver)

(a) For every configuration $i \to k \to j$ add an arrow $i \to j$

Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of quiver)

(a) For every configuration $i \to k \to j$ add an arrow $i \to j$

(b) Erase the 2-cycles possibly created by (a)

Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of quiver)

(a) For every configuration $i \to k \to j$ add an arrow $i \to j$

(b) Erase the 2-cycles possibly created by (a)

(c) Change orientation of each arrow incident to k

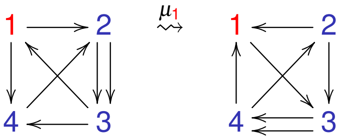
Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of quiver)

(a) For every configuration $i \to k \to j$ add an arrow $i \to j$

(b) Erase the 2-cycles possibly created by (a)

(c) Change orientation of each arrow incident to k



Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of seed)

Mutations μ_k $(1 \leq k \leq n)$

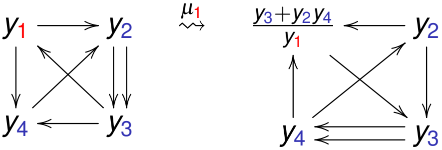
Definition (mutation of seed)

$$\mu_k((y_1, \ldots, y_n), Q) = ((\mu_k(y_1), \ldots, \mu_k(y_n)), \mu_k(Q))$$

Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of seed)

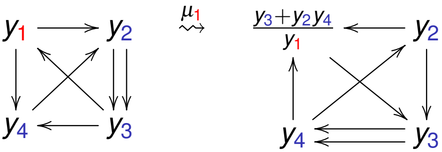
$$\mu_k((y_1,\ldots,y_n),Q) = ((\mu_k(y_1),\ldots,\mu_k(y_n)),\mu_k(Q))$$



Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of seed)

$$\mu_k((y_1,\ldots,y_n),Q) = ((\mu_k(y_1),\ldots,\mu_k(y_n)),\mu_k(Q))$$

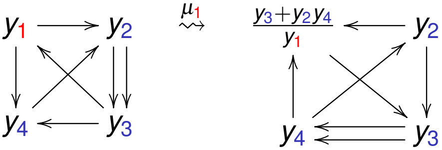


The result is again a seed.

Mutations μ_k $(1 \leq k \leq n)$

Definition (mutation of seed)

$$\mu_k((y_1,\ldots,y_n),Q) = ((\mu_k(y_1),\ldots,\mu_k(y_n)),\mu_k(Q))$$



The result is again a seed. Mutation μ_k is involutive.

Cluster algebra : definition

Cluster algebra : definition

Initial seed: $((x_1, \ldots, x_n), Q)$

Cluster algebra : definition

Initial seed: $((x_1, \ldots, x_n), Q)$

Definition

- cluster: n-tuple $(y_1, \ldots, y_n)$ of a seed obtained from the initial seed via a sequence of mutations

Cluster algebra : definition

Initial seed: $((x_1, \ldots, x_n), Q)$

Definition

- cluster: n-tuple $(y_1, \ldots, y_n)$ of a seed obtained from the initial seed via a sequence of mutations
- cluster variable: element of a cluster

Cluster algebra : definition

Initial seed: $((x_1, \ldots, x_n), Q)$

Definition

- cluster: n-tuple $(y_1, \ldots, y_n)$ of a seed obtained from the initial seed via a sequence of mutations
- cluster variable: element of a cluster
- cluster monomial: product of cluster variables belonging to the same cluster

Cluster algebra : definition

Initial seed: $((x_1, \ldots, x_n), Q)$

Definition

- cluster: n-tuple $(y_1, \ldots, y_n)$ of a seed obtained from the initial seed via a sequence of mutations
- cluster variable: element of a cluster
- cluster monomial: product of cluster variables belonging to the same cluster
- cluster algebra $\mathscr{A}_Q$: subring of $\mathscr{F}$ generated by the cluster variables

Cluster algebra : definition

Initial seed: $((x_1, \ldots, x_n), Q)$

Definition

- cluster: n-tuple $(y_1, \ldots, y_n)$ of a seed obtained from the initial seed via a sequence of mutations
- cluster variable: element of a cluster
- cluster monomial: product of cluster variables belonging to the same cluster
- cluster algebra $\mathscr{A}_Q$: subring of $\mathscr{F}$ generated by the cluster variables

Theorem (Fomin-Zelevinsky, “Laurent phenomenon”)

$$\mathscr{A}_Q \subset \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

A cluster algebra for $U_q(L\mathfrak{g})$

A cluster algebra for $U_q(L\mathfrak{g})$

- $C = (c_{ij} \mid i, j \in I)$, Cartan matrix of $\mathfrak{g}$

A cluster algebra for $U_q(L\mathfrak{g})$

- $C = (c_{ij} \mid i,j \in I)$, Cartan matrix of $\mathfrak{g}$
- $D = \mathrm{diag}(d_i)$, $d_i \in \mathbb{Z}_{>0}$, $\min(d_i) = 1$, such that DC is symmetric

A cluster algebra for $U_q(L\mathfrak{g})$

- $C = (c_{ij} \mid i,j \in I)$, Cartan matrix of $\mathfrak{g}$
- $D = \mathrm{diag}(d_i)$, $d_i \in \mathbb{Z}_{>0}$, $\min(d_i) = 1$, such that DC is symmetric
- Q, quiver with vertex set $V := I \times \mathbb{Z}$, and arrows:

$$(i,r) \to (j,s) \quad \Longleftrightarrow \quad c_{ij} \neq 0 \text{ and } s = r + d_i c_{ij}$$

A cluster algebra for $U_q(L\mathfrak{g})$

- $C = (c_{ij} \mid i,j \in I)$, Cartan matrix of $\mathfrak{g}$
- $D = \mathrm{diag}(d_i)$, $d_i \in \mathbb{Z}_{>0}$, $\min(d_i) = 1$, such that DC is symmetric
- Q, quiver with vertex set $V := I \times \mathbb{Z}$, and arrows:

$$(i,r) \to (j,s) \quad \Longleftrightarrow \quad c_{ij} \neq 0 \text{ and } s = r + d_i c_{ij}$$

- $\mathbf{z} := (z_{i,r} \mid (i,r) \in V)$, indeterminates

A cluster algebra for $U_q(L\mathfrak{g})$

- $C = (c_{ij} \mid i, j \in I)$, Cartan matrix of $\mathfrak{g}$
- $D = \mathrm{diag}(d_i)$, $d_i \in \mathbb{Z}_{>0}$, $\min(d_i) = 1$, such that DC is symmetric
- Q, quiver with vertex set $V := I \times \mathbb{Z}$, and arrows:

$$(i, r) \to (j, s) \quad \Longleftrightarrow \quad c_{ij} \neq 0 \text{ and } s = r + d_i c_{ij}$$

- $\mathbf{z} := (z_{i,r} \mid (i, r) \in V)$, indeterminates

Definition

$\mathscr{A}_Q$, cluster algebra with initial seed $(\mathbf{z}, Q)$

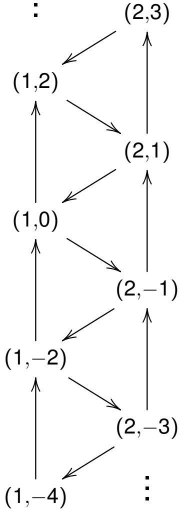
Type A_2

Type A_2

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Type A_2

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$



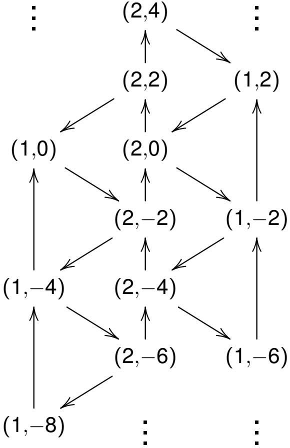
Type B_2

Type B_2

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

Type B_2

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$



Connection between $U_q(L\mathfrak{g})$ and $\mathscr{A}_Q$

Connection between $U_q(L\mathfrak{g})$ and $\mathscr{A}_Q$

- finite-dimensional $U_q(L\mathfrak{g})$-module $M \rightsquigarrow$ q-character $\chi_q(M)$

Connection between $U_q(L\mathfrak{g})$ and $\mathscr{A}_Q$

- finite-dimensional $U_q(L\mathfrak{g})$-module $M \rightsquigarrow$ q-character $\chi_q(M)$: a Laurent polynomial in some variables $Y_{i,a}$ $(i \in I,\ a \in \mathbb{C}^*)$

Connection between $U_q(L\mathfrak{g})$ and $\mathscr{A}_Q$

- finite-dimensional $U_q(L\mathfrak{g})$-module $M \rightsquigarrow q$-character $\chi_q(M)$: a Laurent polynomial in some variables $Y_{i,a}$ $(i \in I,\ a \in \mathbb{C}^*)$
- cluster monomial $x \in \mathscr{A}_Q$

Connection between $U_q(L\mathfrak{g})$ and $\mathscr{A}_Q$

- finite-dimensional $U_q(L\mathfrak{g})$-module $M \rightsquigarrow q$-character $\chi_q(M)$: a Laurent polynomial in some variables $Y_{i,a}$ $(i \in I,\ a \in \mathbb{C}^*)$
- cluster monomial $x \in \mathscr{A}_Q$: a Laurent polynomial in the $z_{i,r}$

Connection between $U_q(L\mathfrak{g})$ and $\mathscr{A}_Q$

- finite-dimensional $U_q(L\mathfrak{g})$-module $M \rightsquigarrow q$-character $\chi_q(M)$: a Laurent polynomial in some variables $Y_{i,a}$ $(i \in I,\ a \in \mathbb{C}^*)$
- cluster monomial $x \in \mathscr{A}_Q$: a Laurent polynomial in the $z_{i,r}$

Main observation

Under the change of variables

$$Y_{i,q^s} = \frac{z_{i,s-d_i}}{z_{i,s+d_i}}$$

the q-characters of many simple finite-dimensional $U_q(L\mathfrak{g})$-modules become equal to certain cluster monomials of $\mathscr{A}_Q$.

Example in type A_2 ($\mathfrak{g} = \mathfrak{sl}_3$)

Example in type A_2 ($\mathfrak{g} = \mathfrak{sl}_3$)

- Fundamental module $L(\varpi_1, q)$:

$$\chi_q(L(\varpi_1, q)) = Y_{1,q} + Y_{1,q^3}^{-1} Y_{2,q^2} + Y_{2,q^4}^{-1}$$

Example in type A_2 ($\mathfrak{g} = \mathfrak{sl}_3$)

- Fundamental module $L(\varpi_1, q)$:

$$\chi_q(L(\varpi_1, q)) = Y_{1,q} + Y_{1,q^3}^{-1} Y_{2,q^2} + Y_{2,q^4}^{-1}$$

- Cluster variable obtained by mutation at $(1,2)$ followed by mutation at $(2,3)$:

$$(\mu_{(2,3)} \circ \mu_{(1,2)})(z_{2,3}) = \frac{z_{1,0}}{z_{1,2}} + \frac{z_{1,4} z_{2,1}}{z_{1,2} z_{2,3}} + \frac{z_{2,5}}{z_{2,3}}$$

Theorem (Hernandez-L 2016)

The "main observation" holds for all Kirillov-Reshetikhin modules. Their q-characters are cluster variables.

Theorem (Hernandez-L 2016)

The "main observation" holds for all Kirillov-Reshetikhin modules. Their q-characters are cluster variables.

$\rightsquigarrow$ Geometric q-character formulas for standard modules when $\mathfrak{g}$ is not simply laced

Theorem (Hernandez-L 2016)

The “main observation” holds for all Kirillov-Reshetikhin modules. Their q-characters are cluster variables.

$\rightsquigarrow$ Geometric q-character formulas for standard modules when $\mathfrak{g}$ is not simply laced

A simple $U_q(L\mathfrak{g})$-module S is called real if $S \otimes S$ is simple.

Theorem (Hernandez-L 2016)

The "main observation" holds for all Kirillov-Reshetikhin modules. Their q-characters are cluster variables.

$\rightsquigarrow$ Geometric q-character formulas for standard modules when $\mathfrak{g}$ is not simply laced

A simple $U_q(L\mathfrak{g})$-module S is called real if $S \otimes S$ is simple.

Conjecture

- The q-characters of all real simple $U_q(L\mathfrak{g})$-modules are equal to certain cluster monomials of $\mathscr{A}_Q$ (under the above change of variables).

Theorem (Hernandez-L 2016)

The “main observation” holds for all Kirillov-Reshetikhin modules. Their q-characters are cluster variables.

$\rightsquigarrow$ Geometric q-character formulas for standard modules when $\mathfrak{g}$ is not simply laced

A simple $U_q(L\mathfrak{g})$-module S is called real if $S \otimes S$ is simple.

Conjecture

- The q-characters of all real simple $U_q(L\mathfrak{g})$-modules are equal to certain cluster monomials of $\mathscr{A}_Q$ (under the above change of variables).
- Factorization of real simple modules into primes corresponds to factorization of cluster monomials into cluster variables.

Note: Some cluster monomials of $\mathscr{A}_Q$ do not correspond to finite-dimensional simple $U_q(L\mathfrak{g})$-modules.

Note: Some cluster monomials of $\mathscr{A}_Q$ do not correspond to finite-dimensional simple $U_q(L\mathfrak{g})$-modules. (e.g. initial cluster variables, one-step mutations, ...).

Note: Some cluster monomials of $\mathscr{A}_Q$ do not correspond to finite-dimensional simple $U_q(L\mathfrak{g})$-modules. (e.g. initial cluster variables, one-step mutations, ...).

What is their meaning in terms of $U_q(L\mathfrak{g})$?

Note: Some cluster monomials of $\mathscr{A}_Q$ do not correspond to finite-dimensional simple $U_q(L\mathfrak{g})$-modules. (e.g. initial cluster variables, one-step mutations, ...).

What is their meaning in terms of $U_q(L\mathfrak{g})$? Maybe one should consider a bigger category than the finite-dimensional modules ?

Note: Some cluster monomials of $\mathscr{A}_Q$ do not correspond to finite-dimensional simple $U_q(L\mathfrak{g})$-modules. (e.g. initial cluster variables, one-step mutations, ...).

What is their meaning in terms of $U_q(L\mathfrak{g})$? Maybe one should consider a bigger category than the finite-dimensional modules ?

Such a category has been introduced by Hernandez and Jimbo (2012), and further studied by Frenkel and Hernandez (2015, 2016).

Note: Some cluster monomials of $\mathscr{A}_Q$ do not correspond to finite-dimensional simple $U_q(L\mathfrak{g})$-modules. (e.g. initial cluster variables, one-step mutations, ...).

What is their meaning in terms of $U_q(L\mathfrak{g})$? Maybe one should consider a bigger category than the finite-dimensional modules ?

Such a category has been introduced by Hernandez and Jimbo (2012), and further studied by Frenkel and Hernandez (2015, 2016).

Conjecture (Hernandez-L 2016)

All cluster monomials of $\mathscr{A}_Q$ correspond to real simple modules in the category of Hernandez and Jimbo.

Note: Some cluster monomials of $\mathscr{A}_Q$ do not correspond to finite-dimensional simple $U_q(L\mathfrak{g})$-modules. (e.g. initial cluster variables, one-step mutations, ...).

What is their meaning in terms of $U_q(L\mathfrak{g})$? Maybe one should consider a bigger category than the finite-dimensional modules ?

Such a category has been introduced by Hernandez and Jimbo (2012), and further studied by Frenkel and Hernandez (2015, 2016).

Conjecture (Hernandez-L 2016)

All cluster monomials of $\mathscr{A}_Q$ correspond to real simple modules in the category of Hernandez and Jimbo.

The conjecture is true in type A_1.

Note: Some cluster monomials of $\mathscr{A}_Q$ do not correspond to finite-dimensional simple $U_q(L\mathfrak{g})$-modules. (e.g. initial cluster variables, one-step mutations, ...).

What is their meaning in terms of $U_q(L\mathfrak{g})$? Maybe one should consider a bigger category than the finite-dimensional modules ?

Such a category has been introduced by Hernandez and Jimbo (2012), and further studied by Frenkel and Hernandez (2015, 2016).

Conjecture (Hernandez-L 2016)

All cluster monomials of $\mathscr{A}_Q$ correspond to real simple modules in the category of Hernandez and Jimbo.

The conjecture is true in type A_1. The conjecture is true for one-step mutations.

Note: Some cluster monomials of $\mathscr{A}_Q$ do not correspond to finite-dimensional simple $U_q(L\mathfrak{g})$-modules. (e.g. initial cluster variables, one-step mutations, ...).

What is their meaning in terms of $U_q(L\mathfrak{g})$? Maybe one should consider a bigger category than the finite-dimensional modules ?

Such a category has been introduced by Hernandez and Jimbo (2012), and further studied by Frenkel and Hernandez (2015, 2016).

Conjecture (Hernandez-L 2016)

All cluster monomials of $\mathscr{A}_Q$ correspond to real simple modules in the category of Hernandez and Jimbo.

The conjecture is true in type A_1. The conjecture is true for one-step mutations.

Theorem (Hernandez-Frenkel 2016)

The relations given by one-step mutations yield the proof of the Bethe Ansatz equations for integrable models associated with $U_q(L\mathfrak{g})$.