Sum-product estimates in finite fields

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Buenos Aires, 29 July 2016
Given a finite set $A$ we consider the sum-set

$$A + A := \{a_1 + a_2; \ a_1, a_2 \in A\}$$

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$$a_1 + a_1 < a_1 + a_2 < \ldots < a_1 + a_n < a_2 + a_n < a_3 + a_n < \ldots < a_{n-1} + a_n.$$
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Erdős and Szemerédi (1983): if $A \subset \mathbb{Z}$, then

$$\max\{|A + A|, |AA|\} > c_1|A|^{1+c_2}; \quad c_1, c_2 > 0.$$
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The best known result is due to Konyagin+Shkredov (2015) improving on earlier result of Solymosi: if $A \subset \mathbb{R}$, then $\forall \varepsilon > 0$

$$\max\{|A + A|, |AA|\} > c|A|^{4/3+5/9813-\varepsilon}; \quad c = c(\varepsilon) > 0.$$
The main conjecture: for $A \subset \mathbb{R}$,

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Some easy variants. Let $A \subset \mathbb{Z}_+$. We can easily prove that

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The observation is that if $a_0$ is the largest element of $A$, then all the $|A|^2$ numbers

$$x + a_0 y; \quad x, y \in A$$

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x + a_0 y = x_1 + a_0 y_1
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then \( y = y_1 \) and \( x = x_1 \), as otherwise

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a_0 = \frac{x_1 - x}{y - y_1}.
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Impossible, because $|y - y_1| \geq 1$ and $|x_1 - x| < a_0$. 


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Impossible, because $|y - y_1| \geq 1$ and $|x_1 - x| < a_0$. It follows that

$$|A + AA| \geq |A|^2.$$
Another variation.

Let $A \subset \mathbb{R}^+$. We shall prove that 
\[ |AA + AA - AA| \geq |A|^2. \]

Here $AA + AA - AA := \{a_1a_2 + a_3a_4 - a_5a_6; a_i \in A \}$.

Take $a_0$ to be the largest element of $A$, and let $a_1, a_2$ be the closest pair in $A$, i.e. with the smallest $|a_1 - a_2| > 0$.

We claim that all the $|A|^2$ numbers $a_0x + (a_1 - a_2)y; x, y \in A$ are distinct.

Indeed if $a_0x + (a_1 - a_2)y = a_0x_1 + (a_1 - a_2)y_1; x, y, x_1, y_1 \in A$, then
\[ x - x_1 y_1 - y = a_1 - a_2 a_0; \]
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is impossible, because $|x - x_1| \geq |a_1 - a_2|$.

Open Problem: Prove that if $A \subset \mathbb{R}^+$, then 
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In 2014 Balog and Roche-Newton:
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Let $p$ be a prime number, $\mathbb{F}_p$ be the field of residue classes modulo $p$. We shall associate elements of $\mathbb{F}_p$ with $\{0, 1, 2, \ldots, p-1\}$. For instance, if $p = 13$ and if $A \subset \mathbb{F}_p$, say, $A = \{2, 6, 7\}$, then $A + A = \{4, 8, 9, 12, 13, 14\} = \{4, 8, 9, 12, 0, 1\}$. $AA = \{4, 12, 14, 36, 42, 49\} = \{4, 12, 1, 10, 3, 10\} = \{4, 12, 1, 10\}$.

If $|A| \approx |\mathbb{F}_p| = p$, then $|A + A| \approx |A|$, $|AA| \approx |A|$.

It is reasonable to assume that $|A| < p^{1-\epsilon}$ for some $\epsilon > 0$.

Theorem. (Bourgain, Katz, Tao + Konyagin; 2003). Let $A \subset \mathbb{F}_p$ with $|A| < p^{1-\epsilon}$ for some $\epsilon > 0$. Then $\max\{|A + A|, |AA|\} \geq |A|^{1+\delta}$, $\delta = \delta(\epsilon) > 0$. 


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Then

$$\max\{|A + A|, |AA|\} \geq |A|^{1+\delta}, \quad \delta = \delta(\varepsilon) > 0.$$
**Some simple observations**

Let $A, B \subset \mathbb{F}_p$ and $J$ be the number of solutions of the equation $a + b = a_1 + b_1$, $a, a_1 \in A$, $b, b_1 \in B$.

Then $|A + B| \geq |A|^2 |B|^2 J$.

Indeed, if for a given $\lambda \in A + B$ we denote by $T(\lambda)$ the number of representation $a + b = \lambda$, $a \in A$, $b \in B$, then

$T^2(\lambda) \Rightarrow a + b = \lambda = a_1 + b_1; a, a_1 \in A$, $b, b_1 \in B$, $J = \sum_{\lambda \in A + B} T(\lambda)$;

$\sum_{\lambda \in A + B} T(\lambda) = |A||B|$.

By Cauchy-Schwarz inequality we get

$J = \sum_{\lambda \in A + B} T^2(\lambda) \geq \frac{1}{|A + B|} \left( \sum_{\lambda \in A + B} T(\lambda) \right)^2 = |A|^2 |B|^2 |A + B|$.
**Some simple observations**

Let $A, B \subset \mathbb{F}_p$ and $J$ be the number of solutions of the equation

$$a + b = a_1 + b_1, \quad a, a_1 \in A, \quad b, b_1 \in B.$$
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The proof of the sum-product estimate uses a number of observations and ideas.

Let us have a bit closer look at the case \( |A| < p^{1/2} \).
CASE 1. Let

\[ \frac{A - A}{A - A} := \left\{ \frac{a_1 - a_2}{a_3 - a_4} : a_i \in A, \ a_3 \neq a_4 \right\} = \mathbb{F}_p. \]
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Thus, there exist \(a_1, a_2, a_3, a_4 \in A\) such that
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The rest of the proof uses known results from additive combinatorics, such as Plunnecke inequality, Ruzsa triangle inequality.
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RATIONAL TRIGONOMETRIC SUMS

Let $m \geq 2$ be an integer. Consider the trigonometric sum

$$S := \sum x e^{2\pi ix/m},$$

where $x$ runs a system of $N$ integers.

From $|e^{ix}| = 1$ it follows the trivial estimate $|S| \leq N$.

The central problem is to obtain a nontrivial estimate of the form $|S| \leq \Delta N$ with $\Delta = \Delta(N)$ as small as possible.

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Assume that

$$\max_{1 \leq a \leq m-1} \left| \sum_{u} e^{2\pi i au/m} \right| \leq R; \quad \sum_{a=1}^{m-1} \left| \sum_{v} e^{2\pi i av/m} \right| \leq D.$$

Then for any integer $\lambda$ the number $T$ of solutions of the congruence

$$u + v \equiv \lambda \pmod{m}$$

can be represented in the form

$$T = LM \left( 1 + \theta RD \right); \quad |\theta| \leq 1.$$
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Then for any integer $\lambda$ the number $T$ of solutions of the congruence
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T = \frac{LM}{m} \left( 1 + \theta \frac{RD}{LM} \right); \quad |\theta| \leq 1.
\]
The starting point:

\[ \frac{1}{m} \sum_{a=0}^{m-1} e^{2\pi i a x / m} = \begin{cases} 
1, & \text{if } x \equiv 0 \pmod{m}, \\
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Substitute \( x = u + v - \lambda \):

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We sum over \(u = u_1, u_2, \ldots, u_L\) and \(v = v_1, v_2, \ldots, v_M\);

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T = \sum_u \sum_v \frac{1}{m} \sum_{a=0}^{m-1} e^{2\pi i a (u+v-\lambda) / m} = \frac{1}{m} \sum_{a=0}^{m-1} \sum_u \sum_v e^{2\pi i a (u+v-\lambda) / m}.
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\]

Separate \(a = 0\) and obtain

\[
T = \frac{LM}{m} + \text{Error},
\]
where

$$\text{Error} = \frac{1}{m} \sum_{a=1}^{m-1} \left( \sum_{u} e^{2\pi iau/m} \right) \left( \sum_{v} e^{2\pi iav/m} \right) e^{-2\pi i\lambda/m}.$$
where

\[ \text{Error} = \frac{1}{m} \sum_{a=1}^{m-1} \left( \sum_{u} e^{2\pi i u / m} \right) \left( \sum_{v} e^{2\pi i v / m} \right) e^{-2\pi i a \lambda / m}. \]

The conditions of the Lemma imply

\[ |\text{Error}| \leq \frac{1}{m} \sum_{a=1}^{m-1} \left| \sum_{u} e^{2\pi i u / m} \right| \left| \sum_{v} e^{2\pi i v / m} \right| \leq \frac{RD}{m}, \]

and the claim follows.
Lemma. Let $U$, $V \subset \{1, 2, \ldots, p\}$. Then for any $a \in \{1, 2, \ldots, p - 1\}$ the following holds:

$$\left| \sum_{u \in U} \sum_{v \in V} e^{2\pi i auv / p} \right| \leq \sqrt{p |U||V|},$$
Lemma. Let $U, V \subset \{1, 2, \ldots, p\}$. Then for any $a \in \{1, 2, \ldots, p - 1\}$ the following holds:

$$\left| \sum_{u \in U} \sum_{v \in V} e^{2\pi i auv / p} \right| \leq \sqrt{p |U||V|},$$

If, say, $|U||V| > p^{1+\varepsilon}$, then

$$\left| \sum_{u \in U} \sum_{v \in V} e^{2\pi i auv / p} \right| \leq |U||V|p^{-\varepsilon/2}.$$
Lemma. Let $U, V \subset \{1, 2, \ldots, p\}$. Then for any $a \in \{1, 2, \ldots, p - 1\}$ the following holds:

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The proof of the Lemma follows from the Cauchy-Schwarz inequality + Trigonometric identity.
Gauss sums

The Gauss sum:

\[ S_n(a, p) = \sum_{x=0}^{p-1} e^{2\pi i a x^n / p}, \quad (a, p) = 1. \]
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Hardy-Littlewood 1917:

\[ |S_n(a, p)| \leq (n - 1)p^{1/2}. \]

Is nontrivial when \( n < p^{1/2} \).
\[ S_n(a, p) = \sum_{x=0}^{p-1} e^{2\pi iax^n/p}, \quad (a, p) = 1. \]

The problem of obtaining nontrivial estimates for larger values of \( n \) was a topic of investigation of many mathematicians.
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Konyagin (2003): obtained nontrivial estimate for $n \leq p^{3/4-\varepsilon}$. 

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Another view to Gauss sum

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The set

$$H := \{ x^n \pmod{p} : 1 \leq x \leq p - 1 \}$$

is a multiplicative subgroup of $\mathbb{F}_p^*$ of the order $(p - 1)/n$. 

The result of Konyagin applies to the case $|H| > p^{1/4} + \varepsilon$. 

Sum-product estimates in finite fields

M. Z. Garaev
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Another view to Gauss sum

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is a multiplicative subgroup of \( \mathbb{F}_p^* \) of the order \((p - 1)/n\). Each element \( h \in H \) has exactly \( n \) representation in the form

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For this reason one has

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The result of Konyagin applies to the case \(|H| > p^{1/4+\varepsilon}\).
We have the representation

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\sum_{x \in H} e^{2\pi i ax / p} = \frac{1}{|H|^{k-1}} \sum_{x_1 \in H} \ldots \sum_{x_k \in H} e^{2\pi i a x_1 \ldots x_k / p}.
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If we take \( k = 2 \), then we can estimate it using Vinogradov’s bound:

\[ |\sum_{x \in H} e^{2\pi iax/p}| = \frac{1}{|H|} \left| \sum_{x_1 \in H} \sum_{x_2 \in H} e^{2\pi iax_1 x_2 / p} \right| \leq p^{1/2}. \]
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Using the sum-product estimate Bourgain, Glibichuk and Konyagin proved the following result.
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Using the sum-product estimate Bourgain, Glibichuk and Konyagin proved the following result.

**Theorem (BGK).** For any \( \varepsilon > 0 \) there exists a positive integer \( k = k(\varepsilon) \) such that if \( X \subset \mathbb{F}_p \) and \( |X| > p^\varepsilon \), then
\[ \left| \sum_{x_1 \in X} \ldots \sum_{x_k \in X} e^{2\pi iax_1 \ldots x_k / p} \right| < |X|^k p^{-\delta}, \quad \delta = \delta(\varepsilon) > 0. \]
As a corollary it follows that if $H$ is a subgroup of $\mathbb{F}_p^*$ with $|H| > p^\varepsilon$, then for $(a, p) = 1$ we have

$$| \sum_{x \in H} e^{2\pi i ax/p} | < |H|^{1-\delta}; \quad \delta = \delta(\varepsilon) > 0.$$
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In other words, the Gauss sum

$$S_n(a, p) = \sum_{x=0}^{p-1} e^{2\pi i ax^n/p}, \quad (a, p) = 1.$$ 

admits a nontrivial estimate for $n < p^{1-\varepsilon}$, with any small fixed constant $\varepsilon > 0$. 
We know that if $|X| > p^{1/2 + \varepsilon}$ then

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The idea is to use the sum-product estimate and try to substitute this sum with a double sum over sets with larger cardinalities. This trilinear sum already contains product-set:
\[ \{ xy; x \in X, y \in X \} = XX. \]
We know that if $|X| > p^{1/2+\varepsilon}$ then
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Furthermore,
\[ |S|^2 \leq |X|^2 \sum_{y \in X} \sum_{z \in X} \left| \sum_{x_1 \in X} \sum_{x_2 \in X} e^{2\pi i(x_1+x_2)yz} / p \right|. \]
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And here we have the sum-set
\[
X + X = \{x_1 + x_2; x_1 \in X, x_2 \in X\}.
\]
In general, the sum-product estimate eventually reduces the problem of estimating $2k$-linear sum

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To implement into reality one more tool is needed, Balog-Szemeredi-Gowers type estimates.
One recent application, in 2014.

The incomplete $n$-linear Kloosterman sums $M_1 + N_1 \sum x_1 = M_1 + 1 + \ldots + M_n + N_n \sum x_n = M_n + 1 e^{2 \pi i (x_1 \ldots x_n)} / p$; $(a, p) = 1$.

The trivial bound is $\leq N_1 N_2 \ldots N_n$.

The problem is to obtain a nontrivial bound. Consider the case $N_1 = \ldots = N_n$.

Luo in 1999 and Shparlinski in 2007: for large values of $n$ one has a nontrivial bound in the range $N_n > p^n / 4 + \sqrt{n} / 2 + \ldots$.

Now the sum-product estimates eventually leads to the following result

Theorem. (Bourgain & G., 2014).

For $N_1 + N_2 \sum x_1 = M_1 + 1 + \ldots + M_n + N_n \sum x_n = M_n + 1 e^{2 \pi i (x_1 \ldots x_n)} / p$ $\leq \ll N_n p^{\delta / n} + \delta$ for some $\delta = \delta(n) > 0$.

Suffices $N_n > p^{\frac{4}{n}}$. 

\[ \sum_{x \in \mathbb{F}_p} e^{2 \pi i (x_1 \ldots x_n)} / p \leq N_n p^{\delta / n} + \delta \]
One recent application, in 2014.
The incomplete $n$-linear Kloosterman sums

$$\sum_{x_1=M_1+1}^{M_1+N_1} \ldots \sum_{x_n=M_n+1}^{M_n+N_n} e^{2\pi i a(x_1 \ldots x_n)^*/p}; \quad (a, p) = 1$$
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One recent application, in 2014. The incomplete $n$-linear Kloosterman sums

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Now the sum-product estimates eventually leads to the following result

**Theorem. (Bourgain & G., 2014).** For \( N > p^{4/n^2} \), we have

\[ \left| \sum_{x_1=M_1+1}^{M_1+N} \ldots \sum_{x_n=M_n+1}^{M_n+N} e^{2\pi i a(x_1\ldots x_n)^*/p} \right| < N^n p^{-\delta} \]

for some \( \delta = \delta(n) > 0 \).
One recent application, in 2014.
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\sum_{x_1 = M_1 + 1}^{M_1 + N_1} \ldots \sum_{x_n = M_n + 1}^{M_n + N_n} e^{2\pi i a(x_1 \ldots x_n)^*/p}; \quad (a, p) = 1
$$

The trivial bound is $\leq N_1 N_2 \ldots N_n$. The problem is to obtain a nontrivial bound. Consider the case $N_1 = \ldots = N_n = N$.
Luo in 1999 and Shparlinski in 2007: for large values of $n$ one has a nontrivial bound in the range

$$
N^n > p^{n/4 + \sqrt{n}/2 + \ldots}.
$$

Now the sum-product estimates eventually leads to the following result

**Theorem. (Bourgain & G., 2014).** For $N > p^{4/n^2}$, we have

$$
\left| \sum_{x_1 = M_1 + 1}^{M_1 + N} \ldots \sum_{x_n = M_n + 1}^{M_n + N} e^{2\pi i a(x_1 \ldots x_n)^*/p} \right| < N^n p^{-\delta}
$$

for some $\delta = \delta(n) > 0$.
Suffices $N^n > p^{4/n}$. 
Investigations on very short Kloosterman sums started with works of Karatsuba continued by Korolev.

**Theorem.** (Bourgain & G., 2014). The following bound holds:

\[
\max_{(a,p)=1} \left| \sum_{n \leq N} e_p(an^*) \right| \ll \frac{(\log \log p)^3 \log p}{(\log N)^{3/2}} N,
\]

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where the implied constant is absolute.

It follows that if $N = p^\varepsilon$ with $\varepsilon > 0$ fixed, the saving is $O((\log \log p)^3/(\log p)^{1/2})$ and the estimate is nontrivial if $N > \exp((\log p)^{2/3}(\log \log p)^3)$. 
Problem
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Let $A \subset \mathbb{F}_p$. Obtain optimal sum-product estimate.

What is a conjectured bound?
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What is a conjectured bound? One should be careful.

The seemingly reasonable conjecture

$$\max\{|A + A|, |AA|\} \gtrsim \min\{p, |A|^2\}$$

is false.
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For any $N \in [1, p]$ one can construct a subset $A \subset \mathbb{F}_p$ with $|A| = N$ such that

$$\max\{|A + A|, |AA|\} \leq c_1\sqrt{p|A|}.$$
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\]

For instance, if \( |A| \approx p^{1/2} \) then

\[
\max\{|A + A|, |AA|\} \leq c_1 |A|^{3/2} \approx p^{3/4}.
\]
Conjecture.

\[
\max\{|A + A|, |AA|\} \gtrsim \min\{|A|^2, \sqrt{p}|A|\}.
\]
Conjecture.

\[ \max\{|A + A|, |AA|\} \gtrsim \min\{|A|^2, \sqrt{p|A|}\}. \]

**Theorem.** (G., 2008). *If \(|A| > p^{2/3}\), then the conjecture is true:*

\[ \max\{|A + A|, |AA|\} \geq c\sqrt{p|A|}, \quad c > 0. \]
Conjecture.

\[
\max\{|A + A|, |AA|\} \gtrsim \min\{|A|^2, \sqrt{p|A|}\}.
\]

Theorem. (G., 2008). If \(|A| > p^{2/3}\), then the conjecture is true:

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\]

Theorem. (Roche-Newton, Rudnev, Shkredov, 2016). If \(|A| < p^{5/8}\), then

\[
\max\{|A + A|, |AA|\} \geq c|A|^{6/5}, \quad c > 0.
\]