Elementos de Cálculo Numérico

Clase 3

Curso de Verano 2020

Estimación del error

Por el desarrollo de Taylor

si $a = f_t(t_0, x_0) + f_x(t_0, x_0)\, f(t_0, x_0)$

$$\begin{aligned} x(t_1) &= x_0 + hf(t_0, x_0) + h^2/2\, a + O(h^3) \\ x_1 &= x_0 + hf(t_0, x_0) \\ \tilde{x}_1 &= x_0 + hf(t_0, x_0) + h^2/4\, a + O(h^3) \end{aligned}$$

Los errores verifican

$$\begin{aligned} \epsilon_1 = x(t_1) - x_1 &= h^2/2\, a + O(h^3) \\ \tilde{\epsilon}_1 = x(t_1) - \tilde{x}_1 &= h^2/4\, a + O(h^3) \\ \tilde{x}_1 - x_1 = \epsilon_1 - \tilde{\epsilon}_1 &= h^2/4\, a + O(h^3) = 1/2\, \epsilon_1 + O(h^3) \end{aligned}$$

Euler modificado

Por la estimación del error local de trucamiento

$$\begin{aligned} \epsilon_1 &= x(t_1) - x_1 = h^2/2\,a + O(h^3) \\ \tilde{\epsilon}_1 &= x(t_1) - \tilde{x}_1 = h^2/4\,a + O(h^3) \\ \tilde{x}_1 - x_1 &= h^2/4\,a + O(h^3) \end{aligned}$$

Obtenemos $\epsilon_1 = 2(\tilde{x}_1 - x_1) + O(h^3)$

Método de Euler modificado:

$$\bar{x}_1 = x_1 + 2(\tilde{x}_1 - x_1) = 2\tilde{x}_1 - x_1$$

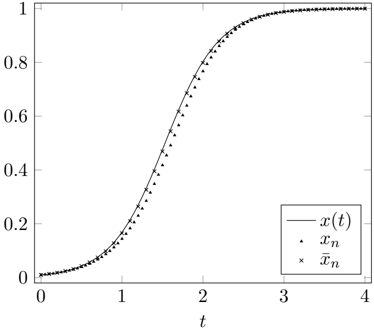
Error de Euler modificado: $\bar{\epsilon}_1 = x(t_1) - \bar{x}_1 = O(h^3)$

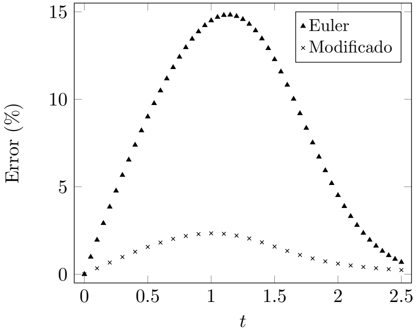
Euler modificado: ejemplo

Problema: $\dot{x}(t) = 3\,x(t) - 3\,x^2(t)$, $x(0) = 0.01$

Solución exacta: $x(t) = 0.01\,e^{3t}/(1 + 0.01\,(e^{3t} - 1))$

Soluciones numéricas y errores para $h = 0.05$





Método Runge-Kutta de segundo orden

$k_1 = f(t_0, x_0)$

$k_2 = f(t_0 + h, x_0 + hk_1)$

Desarrollo de Taylor

$$\begin{aligned} k_2 &= f(t_0, x_0) + h\, f_t(t_0, x_0) + h\, f_x(t_0, x_0)\, k_1 + O(h^2) \\ &= f(t_0, x_0) + h\, (f_t(t_0, x_0) + f_x(t_0, x_0)\, f(t_0, x_0)) + O(h^2) \end{aligned}$$

$(k_1 + k_2)/2 = f(t_0, x_0) + h^2/2\, (f_t(t_0, x_0) + f_x(t_0, x_0)\, f(t_0, x_0))$

Definimos $x_1 = x_0 + (k_1 + k_2)/2$

Error local de truncameinto: $x(t_1) - x_1 = O(h^3)$

Método Runge-Kutta de cuarto orden

$$k_1 = f(t_0, x_0)$$

$$k_2 = f(t_0 + h/2, x_0 + h/2\, k_1)$$

$$k_3 = f(t_0 + h/2, x_0 + h/2\, k_2)$$

$$k_4 = f(t_0 + h, x_0 + h\, k_3)$$

Definimos

$$x_1 = x_0 + \frac{h}{6}(k_1 + 2\, k_2 + 2\, k_3 + k_4)$$

Error local de truncamiento: $x(t_1) - x_1 = O(h^5)$

Método Runge-Kutta de cuarto orden: verificación

Problema: $\dot{x}(t) = x(t), x(0) = 1$

Solución: $x(t) = e^t$

$x(t_1) = x(h) = e^h = 1 + h + h^2/2 + h^3/6 + h^4/24 + O(h^5)$

$k_1 = 1$

$k_2 = 1 + h/2\, k_1 = 1 + h/2$

$k_3 = 1 + h/2\, k_2 = 1 + h/2 + h^2/4$

$k_4 = 1 + h\, k_3 = 1 + h + h^2/2 + h^3/4$

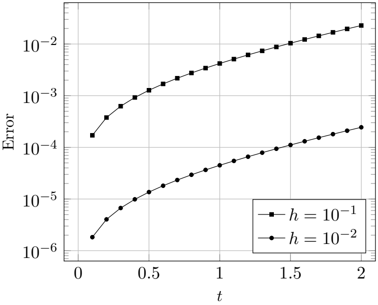
$x_1 = 1 + \frac{h}{6}(k_1 + 2\, k_2 + 2\, k_3 + k_4) = 1 + h + h^2/2 + h^3/6 + h^4/24$

Error local de truncamiento: $x(t_1) - x_1 = O(h^5)$

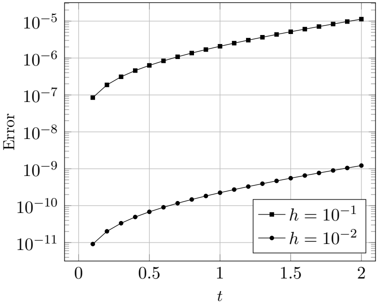
Errores de Runge-Kutta (orden 2 y 4)

Problema: $\dot{x}(t) = x(t), x(0) = 1,\ h = 0.1, 0.01$

Gráfico de errores



(a) R-K de orden 2



(b) R-K de orden 4

Métodos de un paso

Método general: $x_n = x_{n-1} + h\,\Phi(h, t_{n-1}, x_{-1})$

Consistencia:

$$\epsilon(h,t) = x(t+h) - x(t) - h\,\Phi(h,t,x(t)) = O(h^{p+1})$$

Condición de Lipschitz:

$$|\Phi(h,t,x) - \Phi(h,t,\tilde{x})| \leq L|x - \tilde{x}|$$

Convergencia:

$$E_n = |x(t_n) - x_n| \leq \epsilon(h, t_{n-1}) + (1 + L\,h)E_{n-1}$$

$$\max_{0\leq n\leq N} E_n \leq \frac{e^{LT}-1}{L} \max_{1\leq n\leq N} \frac{\epsilon(h,t_{n-1})}{h} = O(h^p)$$

Métodos de un paso

- Método de Euler:

$$\Phi(h,t,x) = f(t,x)$$
$$\epsilon(h,t) = O(h^2),\ L_\Phi = L_f$$

- Método de Euler modificado:

$$\Phi(h,t,x) = f(t+h/2, x+h/2\, f(t,x))$$
$$\epsilon(h,t) = O(h^3),\ L_\Phi = L_f(1+h/2\, L_f)$$

- Runge-Kutta de orden 2:

$$\Phi(h,t,x) = \frac{1}{2}\left(f(t,x) + f(t+h, x+h\, f(t,x))\right)$$
$$\epsilon(h,t) = O(h^3),\ L_\Phi = L_f(1+h/2\, L_f)$$

Métodos de un paso

- Runge-Kutta de orden 4:

$$k_1 = \Phi_1(h,t,x) = f(t,x)$$
$$k_2 = \Phi_2(h,t,x) = f(t+h/2, x+h/2\,\Phi_1(h,t,x))$$
$$k_3 = \Phi_3(h,t,x) = f(t+h/2, x+h/2\,\Phi_2(h,t,x))$$
$$k_4 = \Phi_4(h,t,x) = f(t+h, x+h\,\Phi_3(h,t,x))$$

- Constante de Lipschitz

$$L_{\Phi_1} = L_f$$
$$L_{\Phi_2} = L_f(1+h/2\,L_{\Phi_1}) = L_f(1+h/2\,L_f)$$
$$L_{\Phi_3} = L_f(1+h/2\,L_{\Phi_2}) = L_f(1+h/2\,L_f(1+h/2\,L_f))$$
$$L_{\Phi_4} = L_f(1+h\,L_{\Phi_3}) = L_f(1+h\,L_f(1+h/2\,L_f(1+h/2\,L_f)))$$

Métodos de un paso

- Runge-Kutta de orden 4:

$$\Phi(h,t,x) = \frac{1}{6}(\Phi_1(h,t,x) + 2\,\Phi_1(h,t,x) + 2\,\Phi_1(h,t,x) + \Phi_1(h,t,x))$$

- Constante de Lipschitz

$$\begin{aligned} L_\Phi &= \frac{1}{6}(L_{\Phi_1} + 2\,L_{\Phi_2} + 2\,L_{\Phi_3} + L_{\Phi_4}) \\ &= L_f + \frac{h}{2}L_f^2 + \frac{h^2}{6}L_f^3 + \frac{h^3}{24}L_f^4 \leq \frac{e^{h\,L_f} - 1}{h} \end{aligned}$$