Elementos de Cálculo Numérico

Clase 2

Curso de Verano 2020

Método de Euler

Problema de valores iniciales

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

Recta tangente

$$x = x(t_0) + \dot{x}(t_0)(t - t_0) = x_0 + f(t_0, x_0)(t - t_0)$$

Método de Euler: $x_1 = x_0 + f(t_0, x_0)(t - t_0) = x_0 + f(t_0, x_0)h$

Solución aproximada: $x_n = x_{n-1} + f(t_{n-1}, x_{n-1})h$

con $t_n = t_0 + n\,h$ y $h = T/N$

Error local de truncamiento

Desarrollando en polinomio de Taylor en $t = t_0$

$$x(t_1) = x(t_0) + h\,\dot{x}(t_0) + \frac{h^2}{2}\,\ddot{x}(\tau_1) \qquad \text{para } \tau_1 \in (t_0, t_1)$$

Como $\dot{x}(t) = f(t, x(t))$ por regla de la cadena

$$\ddot{x}(t) = f_t(t, x(t)) + f_x(t, x(t))\,\dot{x}(t)$$

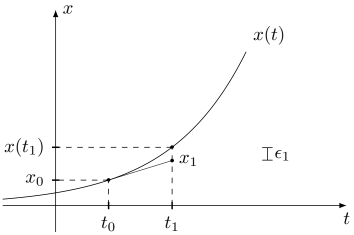
$$\ddot{x}(t) = f_t(t, x(t)) + f_x(t, x(t))\,f(t, x(t))$$

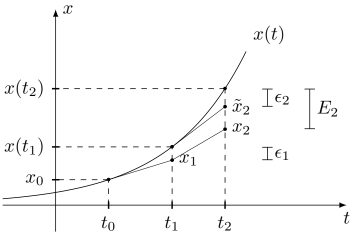
Obtenemos

$$x(t_1) = x_0 + hf(t_0, x_0) + \epsilon(\tau_1) = x_1 + \epsilon(\tau_1)$$

donde $\epsilon(\tau_1) = \dfrac{h^2}{2}(f_t(\tau_1, x(\tau_1)) + f_x(\tau_1, x(\tau_1))\,f(\tau_1, x(\tau_1)))$

Error global





Error global

En general

$|E_n| \leq |\epsilon_n| + (1 + Lh)|E_{n-1}| \quad L = \text{máx}\,|f_x|$

Inductivamente

$$|E_n| \leq |\epsilon_n| + (1 + Lh)\,|\epsilon_{n-1}| + \cdots + (1 + Lh)^{n-1}\,|\epsilon_1|$$

Si definimos $\epsilon_{\max} = \text{máx}\,\{|\epsilon_1|\,, \ldots, |\epsilon_n|\}$

$$|E_n| \leq \left(1 + (1 + Lh) + \cdots + (1 + Lh)^{n-1}\right)\epsilon_{\max}$$

$$|E_n| \leq \frac{(1 + Lh)^n - 1}{Lh}\,\epsilon_{\max} \leq \frac{e^{Lnh} - 1}{Lh}\,\epsilon_{\max}$$

Error global

Como $nh \leq T$, obtenemos

$$E_n \leq \frac{e^{LT} - 1}{L} \frac{\epsilon_{\max}}{h}.$$

Si $\epsilon_{\max} = o(h)$, entonces $E_n = o(1) \quad (E_n \to 0)$

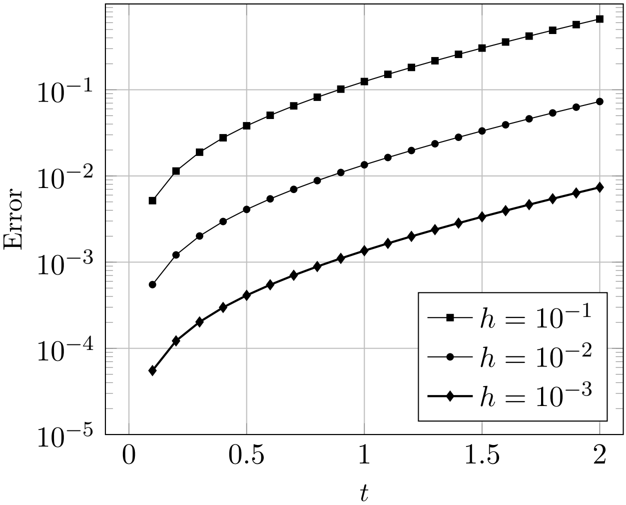
Si $\epsilon_{\max} = O(h^{p+1})$, entonces $E_n = O(h^p)$

Ejemplo

t	$x(t)$	x_n	Error	x_n	Error
0.1	1.105	1.100	5.17×10^{-3}	1.105	5.49×10^{-4}
0.2	1.221	1.210	1.14×10^{-2}	1.220	1.21×10^{-3}
0.3	1.350	1.331	1.89×10^{-2}	1.348	2.01×10^{-3}
0.4	1.492	1.464	2.77×10^{-2}	1.489	2.96×10^{-3}
0.5	1.649	1.611	3.82×10^{-2}	1.645	4.09×10^{-3}
0.6	1.822	1.772	5.06×10^{-2}	1.817	5.42×10^{-3}
0.7	2.014	1.949	6.50×10^{-2}	2.007	6.99×10^{-3}
0.8	2.226	2.144	8.20×10^{-2}	2.217	8.83×10^{-3}
0.9	2.460	2.358	1.02×10^{-1}	2.449	1.10×10^{-2}
1.0	2.718	2.594	1.25×10^{-1}	2.705	1.35×10^{-2}

Tabla: Errores del método de Euler $\dot{x} = x, x(0) = 1$.

Errores



Métodos de Taylor

Existe $\tau \in [t_0, t_1]$

$$x(t_1) = x_0 + h\,\dot{x}(t_0) + \frac{h^2}{2}\,\ddot{x}(t_0) + \frac{h^3}{6}\,\dddot{x}(\tau).$$

Como $\dot{x}(t) = f(t, x(t))$, por regla de la cadena

$$\ddot{x}(t) = f_t(t, x(t)) + f_x(t, x(t))\,\dot{x}(t) \qquad f(t, x(t))$$

Entonces: $\ddot{x}(t_0) = f_t(t_0, x_0) + f_x(t_0, x_0) f(t_0, x_0)$

Podemos escribir

$$x(t_1) = x_0 + h\,f(t_0, x_0) + \frac{h^2}{2}\,(f_t(t_0, x_0) + f_x(t_0, x_0) f(t_0, x_0)) + \frac{h^3}{6}\,\dddot{x}(\tau)$$

Métodos de Taylor

Definimos

$$x_1 = x_0 + h\, f(t_0, x_0) + \frac{h^2}{2}\left(f_t(t_0, x_0) + f_x(t_0, x_0) f(t_0, x_0)\right)$$

Método de segundo orden: $\epsilon_n = O(h^3)$

Error global: $E_n = O(h^2)$

Se necesitan las derivadas de $f(t, x)$

Se puede generalizar a orden mayor