**Teorema.** *Sean* $\mathbf{x}_{i,1}, \ldots, \mathbf{x}_{i,n_i} \in \mathbb{R}^p$, $1 \leq i \leq k$ *independientes tales que* $\mathbf{x}_{i,j} \sim N((\boldsymbol{\mu}_i, \boldsymbol{\Sigma})$. *Definamos*

$$\mathbf{H} = \sum_{i=1}^{k} n_i (\overline{\mathbf{x}}_i - \overline{\mathbf{x}})(\overline{\mathbf{x}}_i - \overline{\mathbf{x}})^{\mathrm{T}}$$

*donde*

$$\overline{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{k} n_i \overline{\mathbf{x}}_i \,.$$

*Si* $\boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_k$, *entonces* $\mathbf{H} \sim \mathcal{W}(\boldsymbol{\Sigma}, p, k-1)$.

Demostración. Sea $\mathbf{v}_i = \sqrt{n_i}\,((\overline{\mathbf{x}}_i - \boldsymbol{\mu}_i)$, $1 \leq i \leq k$ e indiquemos por $\mathbf{V}^{\mathrm{T}} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$, es decir, la matriz $\mathbf{V}$ tiene como filas a $\mathbf{v}_1^{\mathrm{T}}, \ldots, \mathbf{v}_k^{\mathrm{T}}$. Luego, $\mathbf{v}_i$ son independientes, $\mathbf{v}_i \sim N(\mathbf{0}, \boldsymbol{\Sigma})$.

Consideremos el vector

$$\mathbf{a} = \left( \sqrt{\frac{n_1}{n}}, \ldots, \sqrt{\frac{n_k}{n}} \right)^{\mathrm{T}} \in \mathbb{R}^k$$

y observemos que $\|\mathbf{a}\| = 1$ y

$$\begin{aligned}\mathbf{V}^{\mathrm{T}}\mathbf{a} &= \sum_{i=1}^{k} \sqrt{\frac{n_i}{n}} \mathbf{v}_i = \sum_{i=1}^{k} \sqrt{\frac{n_i}{n}} \sqrt{n_i}\,(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_i) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} n_i\,(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_i) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} n_i\,\overline{\mathbf{x}}_i - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} n_i \boldsymbol{\mu}_i \,.\end{aligned}$$

Si $\boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_k$, usando que $n\,\overline{\mathbf{x}} = \sum_{i=1}^{n} n_i\,\overline{\mathbf{x}}_i$ obtenemos que

$$\mathbf{V}^{\mathrm{T}}\mathbf{a} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} n_i\,\overline{\mathbf{x}}_i - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} n_i \boldsymbol{\mu} = \sqrt{n}\,(\overline{\mathbf{x}} - \boldsymbol{\mu}) \,.$$

Sea $\mathbf{P} \in \mathbb{R}^{k \times k}$ ortogonal tal que $\mathbf{P} = (\mathbf{a}_1, \ldots, \mathbf{a}_k)$ y $\mathbf{a}_k = \mathbf{a}$. Definamos $\mathbf{Y}^{\mathrm{T}} = (\mathbf{y}_1, \ldots, \mathbf{y}_k) = \mathbf{V}^{\mathrm{T}}\mathbf{P}$, por lo tanto, $\mathbf{y}_k = \mathbf{V}^{\mathrm{T}}\mathbf{a}_k = \mathbf{V}^{\mathrm{T}}\mathbf{a} = \sqrt{n}\,(\overline{\mathbf{x}} - \boldsymbol{\mu})$.

Usando que $\mathbf{v}_i \sim N(\mathbf{0}, \boldsymbol{\Sigma})$, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ son independientes y $\mathbf{a}_1, \ldots, \mathbf{a}_k$ son ortonormales, concluímos que $\mathbf{V}^{\mathrm{T}}\mathbf{a}_j \sim N(\mathbf{0}, \|\mathbf{a}_j\|^2\,\boldsymbol{\Sigma}) = N(\mathbf{0}, \boldsymbol{\Sigma})$ independientes entre sí, es decir, $\mathbf{y}_1, \ldots, \mathbf{y}_k$ son i.i.d. $\mathbf{y}_j \sim N(\mathbf{0}, \boldsymbol{\Sigma})$.

Usando que $\mathbf{P}$ es una matriz ortogonal deducimos que

$$\mathbf{Y}^{\mathrm{T}}\mathbf{Y} = \mathbf{V}^{\mathrm{T}}\mathbf{P}\;\mathbf{P}^{\mathrm{T}}\mathbf{V} = \mathbf{V}\mathbf{V}^{\mathrm{T}} \,. \tag{1}$$

Por otra parte, usando que $\mathbf{v}_i = \sqrt{n_i}\,((\overline{\mathbf{x}}_i - \boldsymbol{\mu}_i)$ y que $\mathbf{y}_k = \sqrt{n}\,(\overline{\mathbf{x}} - \boldsymbol{\mu})$ obtenemos que

$$\begin{aligned}
\mathbf{H} &= \sum_{i=1}^{k} n_i(\overline{\mathbf{x}}_i - \overline{\mathbf{x}})(\overline{\mathbf{x}}_i - \overline{\mathbf{x}})^{\mathrm{T}} = \sum_{i=1}^{k} n_i \left\{(\overline{\mathbf{x}}_i - \boldsymbol{\mu}) + (\boldsymbol{\mu} - \overline{\mathbf{x}})\right\} \left\{(\overline{\mathbf{x}}_i - \boldsymbol{\mu}) + (\boldsymbol{\mu} - \overline{\mathbf{x}})\right\}^{\mathrm{T}} \\
&= \sum_{i=1}^{k} n_i(\overline{\mathbf{x}}_i - \boldsymbol{\mu})(\overline{\mathbf{x}}_i - \boldsymbol{\mu})^{\mathrm{T}} - n\,(\overline{\mathbf{x}} - \boldsymbol{\mu})\,(\overline{\mathbf{x}} - \boldsymbol{\mu})^{\mathrm{T}} \\
&= \sum_{i=1}^{k} \mathbf{v}_i\mathbf{v}_i^{\mathrm{T}} - \mathbf{y}_k\mathbf{y}_k^{\mathrm{T}}\ ,
\end{aligned}$$

de donde, usando (1) deducimos que

$$\mathbf{H} = \sum_{i=1}^{k} \mathbf{v}_i\mathbf{v}_i^{\mathrm{T}} - \mathbf{y}_k\mathbf{y}_k^{\mathrm{T}} = \mathbf{V}^{\mathrm{T}}\mathbf{V} - \mathbf{y}_k\mathbf{y}_k^{\mathrm{T}} = \mathbf{Y}^{\mathrm{T}}\mathbf{Y} - \mathbf{y}_k\mathbf{y}_k^{\mathrm{T}} = \sum_{i=1}^{k} \mathbf{y}_i\mathbf{y}_i^{\mathrm{T}} - \mathbf{y}_k\mathbf{y}_k^{\mathrm{T}} = \sum_{i=1}^{k-1} \mathbf{y}_i\mathbf{y}_i^{\mathrm{T}}\ .$$

Como $\mathbf{y}_1, \ldots, \mathbf{y}_{k-1}$ son independientes, $\mathbf{y}_j \sim N(\mathbf{0}, \boldsymbol{\Sigma})$, obtenemos que

$$\mathbf{H} = \sum_{i=1}^{k-1} \mathbf{y}_i\mathbf{y}_i^{\mathrm{T}} \sim \mathcal{W}(\boldsymbol{\Sigma}, p, k-1)$$

lo que concluye la demostración. $\square$