Norma y traza:

Ej: 1) Hallar, si existe, $\alpha \in \overline{\mathbb{Q}}$ tq $m_{\alpha,\mathbb{Q}}$ tenga grado 3, $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha) = 1$, $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha+1) = 1$, y $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha-1) = -7$.

Digamos. $m_{\alpha,\mathbb{Q}} = x^3 + ax^2 + bx + c$

$$\Rightarrow \quad N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha) = (-1)^3 c = -c \Rightarrow \boxed{c = -1}$$

Ahora: • $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha+1) = (\alpha+1)(\alpha_2+1)(\alpha_3+1)$

$$= -(-1-\alpha)(-1-\alpha_2)(-1-\alpha_3)$$

$$= -m_{\alpha,\mathbb{Q}}(-1)$$

$$= -(-1+a-b-1)$$

$$= 2-a+b.$$

• $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha-1) = -m_{\alpha,\mathbb{Q}}(1)$

$$= -(1+a+b-1)$$

$$= -a-b.$$

$$\Rightarrow \begin{cases} 2-a+b = 1 \\ -a-b = -7 \end{cases} \Rightarrow \boxed{a = 4, \quad b = 3}$$

Queda entonces: $m_{\alpha,\mathbb{Q}} = x^3 + 4x^2 + 3x - 1$, que es efectivamente irreducible, entonces existe tal α y es cualquier raíz de ese polinomio.

2) Sea $K = \mathbb{Q}(\underbrace{\sqrt[n]{a}}_{=\alpha})$, con $[K:\mathbb{Q}] = n$. Sea $\mathbb{Q} \subseteq E \subseteq K$ tq $[E:\mathbb{Q}] = d$. Probar que:

$E = \mathbb{Q}(\sqrt[d]{a})$ $\quad (a \in \mathbb{Q})$

Dem.

$$\begin{tikzcd}
K \arrow[d, dash] \arrow[dd, dash, bend right, "n"'] \\
E \arrow[d, dash, "d"] \\
\mathbb{Q}
\end{tikzcd}$$

Calculemos $N_{K/E}(\alpha) := \prod_{\sigma : K \to_E \overline{E}} \sigma(\alpha)$

Pero: $m_{\alpha, E} \mid x^n - a$

$\Rightarrow N_{K/E}(\alpha) = \pm \prod_{i \in I} \alpha \cdot \zeta_n^i$,

con $I \subseteq \{0, \dots, n-1\}$, $|I| = n/d$

O sea. $N_{K/E}(\alpha) = \pm \alpha^{\frac{n}{d}} \zeta_n^{\ell} = \pm \sqrt[d]{a}\, \zeta_n^{\ell}$

Pero $N_{K/E}(\alpha) \in E$, luego $\ell = 0$

En particular $\sqrt[d]{a} \in E$

Tenemos:

$$\begin{tikzcd}
K \arrow[d, dash] \arrow[dddd, dash, bend right, "n"'] \\
E \arrow[d, dash] \arrow[ddd, dash, bend left, "d"] \\
\mathbb{Q}(\sqrt[d]{a}) \arrow[d, dash] \\
\mathbb{Q}
\end{tikzcd}$$

Bvq $[\mathbb{Q}(\sqrt[d]{a}) : \mathbb{Q}] = d$

O sea: $m_{\sqrt[d]{a}, \mathbb{Q}} = x^d - a$

Veamos entonces q' $x^d - a$ es irreducible:

En efecto: si $x^d - a = g(x)h(x)$

$$\Rightarrow \quad x^n - a = g(x^{\frac{m}{d}})\, h(x^{\frac{m}{d}})$$

Pero $x^n - a$ es irred: $\underline{\underline{\text{Abs}}}$.

3) Sea E/K de grado m y sea $d \mid m$ tq. $(d : \frac{m}{d}) = 1$. Sean $\alpha \in E$ y $a \in K$ tq. $N_{E/K}(\alpha) = a^d$. Sea F una subextensión de grado $\frac{m}{d}$. Probar que existe $\beta \in F$ tq.

$$N_{F/K}(\beta) = a.$$

Dem: Supongamos $r d + s\frac{m}{d} = 1$

$$\Rightarrow \quad a = a^{rd + s(\frac{m}{d})} = (a^d)^r \cdot (a^s)^{\frac{m}{d}}$$

$$\begin{array}{c} E \\ d \,\big|\, \\ F \\ \frac{m}{d} \,\big|\, \\ K \end{array}$$

$$= N_{E/K}(\alpha)^r \cdot N_{F/K}(a^s)$$

$$= N_{E/K}(\alpha^r) \cdot N_{F/K}(a^s)$$

$$= N_{F/K}(N_{E/F}(\alpha^r)) \cdot N_{F/K}(a^s)$$

$$= N_{F/K}\Big(\underbrace{N_{E/F}(\alpha^r) \cdot a^s}_{=:\beta}\Big)$$

Extensiones trascendentes:

Def. • $\{t_1, \ldots, t_n\}$ son alg indep'tes sobre K si

$$f(t_1, \ldots, t_n) = 0, \quad f \in K[x_1, \ldots, x_n] \Rightarrow f = 0$$

• T un conjunto es alg indep'te sobre K si

$\forall \{t_1, \ldots, t_n\} \subseteq T$, $\{t_1, \ldots, t_n\}$ es alg. indep'te sobre K

Def. Dada E/K, $B \subseteq E$ es base de trascendencia, si B es un conjunto alg indep'te maximal

Obs. B es base de trasc de E/K

$\Leftrightarrow$ • B es alg indep'te sobre K
• $E/K(B)$ es algebraica

Teo: Siempre existe una tal base (que puede ser $\emptyset$) y dos bases tienen igual cardinal

Def. $\operatorname{trdeg}_{E/K} := \# B$, con B base de trasc

Prop. trdeg es aditivo por torres

Ej 1)

$$\begin{tikzcd}[column sep=small]
\mathbb{C}(x,y) \arrow[d, dash] \arrow[dr, dash, "\text{alg } (\text{trdeg}=0)"] & \\
\mathbb{R} \arrow[r, dash, "\text{trdeg}=2"'] & \mathbb{R}(x,y)
\end{tikzcd}$$

- $\{x,y\}$ es alg indep'te sobre $\mathbb{R}$
- $\mathbb{C}(x,y)/\mathbb{R}(x,y)$ alg

$\Rightarrow \text{trdeg}_{\mathbb{C}(x,y)/\mathbb{R}} = 2$

2)

$$\begin{tikzcd}[column sep=small]
\mathbb{Q}(x,\sqrt{2},\sqrt{x+y}) \arrow[d, dash, "\Rightarrow \text{trdeg}=1"'] \arrow[dr, dash, "\text{alg } (\text{trdeg}=0)"] & \\
\mathbb{Q}(x) \arrow[r, dash, "\text{trdeg}=1"'] & \mathbb{Q}(x)(y)
\end{tikzcd}$$

3) $f = X^3 + 2xy\,X + \sqrt{y} \in \mathbb{Q}(x,\sqrt{y})[X]$

α raíz de f. Calculemos trdeg de

$\mathbb{Q}(x,y,\alpha)/\mathbb{Q}$:

$$\begin{tikzcd}[column sep=small]
& \mathbb{Q}(x,\sqrt{y},\alpha) \arrow[dl, dash, "\text{trdeg}=0"'] \arrow[dr, dash, "\text{trdeg}=0"] & \\
\mathbb{Q}(x,y,\alpha) \arrow[dr, dash, "\Rightarrow \text{trdeg}=2"'] & & \mathbb{Q}(x,\sqrt{y}) \arrow[dl, dash, "\text{trdeg}=2"] \\
& \mathbb{Q} &
\end{tikzcd}$$

Notar que: $\{x, \sqrt{y}\}$ es base de trasc y

$\{x, y\}$ también (de $\mathbb{Q}(x,\sqrt{y},\alpha)/\mathbb{Q}$)

Pues

$$\begin{tikzcd}
& \mathbb{Q}(x,\sqrt{y},\alpha) \arrow[dl, dash] \arrow[dr, dash, "\text{alg}"] & \\
\mathbb{Q}(x,y,\alpha) \arrow[d, dash] & & \mathbb{Q}(x,\sqrt{y}) \arrow[dll, dash, "\text{alg}"] \\
\mathbb{Q}(x,y) \arrow[d, dash] \arrow[uur, dash, bend left=60, "\Rightarrow \text{alg}"] & & \\
\mathbb{Q} & &
\end{tikzcd}$$

Observar que no necesariamente

$$[E : k(B_1)] = [E : k(B_2)]$$

Acá tenemos:

$$\begin{tikzcd}
& \mathbb{Q}(x,\sqrt{y},\alpha) \arrow[dl, dash, "1 \text{ ó } 2"'] \arrow[dr, dash, "\boxed{3}"] & \\
\mathbb{Q}(x,y,\alpha) \arrow[d, dash] & & \mathbb{Q}(x,\sqrt{y}) \arrow[ddl, dash] \\
\mathbb{Q}(x,y) \arrow[dr, dash] \arrow[uur, dash, bend left=60] & & \\
& \mathbb{Q} &
\end{tikzcd}$$