Construction of the real numbers

In mathematics, there are several ways of defining the real number system as an ordered field. The *synthetic* approach gives a list of axioms for the real numbers as a *complete ordered field*. Under the usual axioms of set theory, one can show that these axioms are categorical, in the sense that there is a model for the axioms, and any two such models are isomorphic. Any one of these models must be explicitly constructed, and most of these models are built using the basic properties of the rational number system as an ordered field.

Synthetic approach

The synthetic approach axiomatically defines the real number system as a complete ordered field. Precisely, this means the following. A *model for the real number system* consists of a set **R**, two distinct elements 0 and 1 of **R**, two binary operations + and * on **R** (called *addition* and *multiplication*, resp.), a binary relation ≤ on **R**, satisfying the following properties.

1. (**R**, +, *) forms a field. In other words,

- For all x, y, and z in **R**, $x + (y + z) = (x + y) + z$ and $x * (y * z) = (x * y) * z$. (associativity of addition and multiplication)
- For all x and y in **R**, $x + y = y + x$ and $x * y = y * x$. (commutativity of addition and multiplication)
- For all x, y, and z in **R**, $x * (y + z) = (x * y) + (x * z)$. (distributivity of multiplication over addition)
- For all x in **R**, $x + 0 = x$. (existence of additive identity)
- 0 is not equal to 1, and for all x in **R**, $x * 1 = x$. (existence of multiplicative identity)
- For every x in **R**, there exists an element $-x$ in **R**, such that $x + (-x) = 0$. (existence of additive inverses)
- For every $x \neq 0$ in **R**, there exists an element x^{-1} in **R**, such that $x * x^{-1} = 1$. (existence of multiplicative inverses)

2. (**R**, ≤) forms a totally ordered set. In other words,

- For all x in **R**, $x \le x$. (reflexivity)
- For all x and y in **R**, if $x \le y$ and $y \le x$, then $x = y$. (antisymmetry)
- For all x, y, and z in **R**, if $x \le y$ and $y \le z$, then $x \le z$. (transitivity)
- For all x and y in **R**, $x \le y$ or $y \le x$. (totalness)

3. The field operations + and * on **R** are compatible with the order ≤. In other words,

- For all x, y and z in **R**, if $x \le y$, then $x + z \le y + z$. (preservation of order under addition)
- For all x and y in **R**, if $0 \le x$ and $0 \le y$, then $0 \le x * y$ (preservation of order under multiplication)

4. The order ≤ is *complete* in the following sense: every non-empty subset of **R** bounded above has a least upper bound. In other words,

- If A is a non-empty subset of **R**, and if A has an upper bound, then A has a least upper bound u, such that for every upper bound v of A, $u \le v$.

The final axiom, defining the order as Dedekind-complete, is most crucial. Without this axiom, we simply have the axioms which define a totally ordered field, and there are many non-isomorphic models which satisfy these axioms. This axiom implies that the Archimedean property applies for this field. Therefore, when the completeness axiom is added, it can be proved that any two models must be isomorphic, and so in this sense, there is only one complete ordered Archimedean field.

When we say that any two models of the above axioms are isomorphic, we mean that for any two models (R, 0_R, 1_R, $+_R$, $*_R$, $\le_R$) and (S, 0_S, 1_S, $+_S$, $*_S$, $\le_S$), there is a bijection $f : R \to S$ preserving both the field operations and the order. Explicitly,

- f is both injective and surjective.

- $f(0_R) = 0_S$ and $f(1_R) = 1_S$.
- For all x and y in R, $f(x +_R y) = f(x) +_S f(y)$ and $f(x *_R y) = f(x) *_S f(y)$.
- For all x and y in R, $x \leq_R y$ if and only if $f(x) \leq_S f(y)$.

Explicit constructions of models

We shall not prove that any models of the axioms are isomorphic. Such a proof can be found in any number of modern analysis or set theory textbooks. We will sketch the basic definitions and properties of a number of constructions, however, because each of these is important for both mathematical and historical reasons. The first three, due to Georg Cantor/Charles Méray, Richard Dedekind and Karl Weierstrass/Otto Stolz all occurred within a few years of each other. Each has advantages and disadvantages. A major motivation in all three cases was the instruction of mathematics students.

Construction from Cauchy sequences

If we have a space where Cauchy sequences are meaningful (such as a `rational' metric space, i.e., a space in which distance is defined and takes rational values, or more generally a uniform space), a standard procedure to force all Cauchy sequences to converge is adding new points to the space (a process called completion). By starting with rational numbers and the metric $d(x,y) = |x - y|$, we can construct the real numbers, as will be detailed below. (A different metric on the rationals could result in the p-adic numbers instead.)

Let R be the set of Cauchy sequences of rational numbers. That is, sequences

$x_1, x_2, x_3, \ldots$ of rational numbers such that for every rational $\varepsilon > 0$, there exists an integer N such that for all natural numbers $m,n > N$, $|x_m - x_n| < \varepsilon$. Here the vertical bars denote the absolute value.

Cauchy sequences (x) and (y) can be added and multiplied as follows:

$$(x_n) + (y_n) = (x_n + y_n)$$
$$(x_n) \times (y_n) = (x_n \times y_n)$$

Two Cauchy sequences are called *equivalent* if and only if the difference between them tends to zero.

Comparison between two cauchy sequences is possible as such : $(x_n) \geq (y_n)$ if and only if x is equivalent to y or there exists an integer N such that $x_n \geq y_n$ for all $n > N$.

This does indeed define an equivalence relation, it is compatible with the operations defined above, and the set **R** of all equivalence classes can be shown to satisfy all the usual axioms of the real numbers. This is remarkable because not all of these axioms necessarily apply to the rational numbers, which are being used to construct the sequences themselves. We can embed the rational numbers into the reals by identifying the rational number r with the equivalence class of the sequence $(r,r,r, \ldots)$.

The only real number axiom that does not follow easily from the definitions is the completeness of $\leq$, i.e. the least upper bound property. It can be proved as follows: Let S be a non-empty subset of **R** and U be an upper bound for S. Substituting a larger value if necessary, we may assume U is rational. Since S is non-empty, there is a rational number L such that $L < s$ for some s in S. Now define sequences of rationals (u_n) and (l_n) as follows:

Set $u_0 = U$ and $l_0 = L$.

For each n consider the number:

$$m_n = (u_n + l_n)/2$$

If m_n is an upper bound for S set:

$$u_{n+1} = m_n \text{ and } l_{n+1} = l_n$$

Otherwise set:

$$l_{n+1} = m_n \text{ and } u_{n+1} = u_n$$

This obviously defines two Cauchy sequences of rationals, and so we have real numbers $l = (l_n)$ and $u = (u_n)$. It is easy to prove, by induction on n that:

u_n is an upper bound for S for all n

and:

l_n is never an upper bound for S for any n

Thus u is an upper bound for S. To see that it is a least upper bound, notice that the limit of $(u_n - l_n)$ is 0, and so $l = u$. Now suppose $b < u = l$ is a smaller upper bound for S. Since (l_n) is monotonic increasing it is easy to see that $b < l_n$ for some n. But l_n is not an upper bound for S and so neither is b. Hence u is a least upper bound for S and ≤ is complete.

A practical and concrete representative for an equivalence class representing a real number is provided by the representation to base b — in practice, b is usually 2 (binary), 8 (octal), 10 (decimal) or 16 (hexadecimal). For example, the number π = 3.14159... corresponds to the Cauchy sequence (3,3.1,3.14,3.141,3.1415,...). Notice that the sequence (0,0.9,0.99,0.999,0.9999,...) is equivalent to the sequence (1,1.0,1.00,1.000,1.0000,...); this shows that 0.999... = 1.

An advantage of this approach is that it does not use the linear order of the rationals, only the metric. Hence it generalizes to other metric spaces.

Construction by Dedekind cuts

A Dedekind cut in an ordered field is a partition of it, (A, B), such that A is nonempty and closed downwards, B is nonempty and closed upwards, and A contains no greatest element. Real numbers can be constructed as Dedekind cuts of rational numbers.

For convenience we may take the lower set A as the representative of any given Dedekind cut (A, B), since A completely determines B. By doing this we may think intuitively of a real number as being represented by the set of all smaller rational numbers. In more detail, a real number r is any subset of the set $\mathbf{Q}$ of rational numbers that fulfills the following conditions:[1]

1. r is not empty
2. $r \neq \mathbf{Q}$
3. r is closed downwards. In other words, for all $x, y \in \mathbf{Q}$ such that $x < y$, if $y \in r$ then $x \in r$
4. r contains no greatest element. In other words, there is no $x \in r$ such that for all $y \in r$, $y \leq x$

- We form the set $\mathbf{R}$ of real numbers as the set of all Dedekind cuts A of $\mathbf{Q}$, and define a total ordering on the real numbers as follows: $x \leq y \Leftrightarrow x \subseteq y$
- We embed the rational numbers into the reals by identifying the rational number q with the set of all smaller rational numbers $\{x \in \mathbf{Q} : x < q\}$.[1] Since the rational numbers are dense, such a set can have no greatest element and thus fulfills the conditions for being a real number laid out above.
- Addition. $A + B := \{a + b : a \in A \land b \in B\}$ [1]
- Subtraction. $A - B := \{a - b : a \in A \land b \in (\mathbf{Q} \setminus B)\}$ where $\mathbf{Q} \setminus B$ denotes the relative complement of B in $\mathbf{Q}$, $\{x : x \in \mathbf{Q} \land x \notin B\}$
- Negation is a special case of subtraction: $-B := \{a - b : a < 0 \land b \in (\mathbf{Q} \setminus B)\}$
- Defining multiplication is less straightforward.[1]
  - if $A, B \geq 0$ then $A \times B := \{a \times b : a \geq 0 \land a \in A \land b \geq 0 \land b \in B\} \cup \{x \in \mathrm{Q} : x < 0\}$
  - if either A or B is negative, we use the identities $A \times B = -(A \times -B) = -(-A \times B) = (-A \times -B)$ to convert A and/or B to positive numbers and then apply the definition above.
- We define division in a similar manner:

- if $A \geq 0$ and $B > 0$ then $A/B := \{a/b : a \in A \wedge b \in (\mathbf{Q} \setminus B)\}$
- if either A or B is negative, we use the identities $A/B = -(A/-B) = -(-A/B) = -A/-B$ to convert A to a non-negative number and/or B to a positive number and then apply the definition above.

- Supremum. If a nonempty set S of real numbers has any upper bound in $\mathbf{R}$, then it has a least upper bound in $\mathbf{R}$ that is equal to $\bigcup S$.[1]

As an example of a Dedekind cut representing an irrational number, we may take the positive square root of 2. This can be defined by the set $A = \{x \in \mathbf{Q} : x < 0 \vee x \times x < 2\}$.[2] It can be seen from the definitions above that A is a real number, and that $A \times A = 2$. However, neither claim is immediate. Showing that A is real requires showing that for any positive rational x with $x \times x < 2$, there is a rational y with $x < y$ and $y \times y < 2$. The choice $y = \frac{2x+2}{x+2}$ works. Then $A \times A \leq 2$ but to show equality requires showing that if r is any rational number less than 2, then there is positive x in A with $r < x \times x$.
An advantage of this construction is that each real number corresponds to a unique cut.

Stevin's construction

It has been known since Simon Stevin[3] that real numbers can be represented by decimals. We can take the infinite decimal expansion to be the definition of a real number, **defining** expansions like 0.9999... and 1.0000... to be equivalent, and define the arithmetical operations formally. This is equivalent to the constructions by Cauchy sequences or Dedekind cuts and incorporates an explicit modulus of convergence. Similarly, another radix can be used. Weierstrass attempted to construct the reals but did not entirely succeed. He pointed out that they need only be thought of as complete aggregates (sets) of units and unit fractions.[4]

This construction has the advantage that it is close to the way we are used to thinking of real numbers and suggests series expansions for functions. A standard approach to show that all models of a complete ordered field are isomorphic is to show that any model is isomorphic to this one because we can systematically build a decimal expansion for each element.

Construction using hyperreal numbers

As in the hyperreal numbers, one constructs the hyperrationals $^*\mathbf{Q}$ from the rational numbers by means of an ultrafilter. Here a hyperrational is by definition a ratio of two hyperintegers. Consider the ring *B* of all limited (i.e. finite) elements in $^*\mathbf{Q}$. Then *B* has a unique maximal ideal *I*, the infinitesimal numbers. The quotient ring *B/I* gives the field **R** of real numbers. Note that *B* is not an internal set in $^*\mathbf{Q}$. Note that this construction uses a non-principal ultrafilter over the set of natural numbers, the existence of which is guaranteed by the axiom of choice.

It turns out that the maximal ideal respects the order on $^*\mathbf{Q}$. Hence the resulting field is an ordered field. Completeness can be proved in a similar way to the construction from the Cauchy sequences.

Construction from surreal numbers

Every ordered field can be embedded in the surreal numbers. The real numbers form a maximal subfield that is Archimedean (meaning that no real number is infinitely large). This embedding is not unique, though it can be chosen in a canonical way.

Construction from the group of integers

A relatively less known construction allows to define real numbers using only the additive group of integers with different versions.[5][6][7] The construction has been formally verified by the IsarMathLib project.[8]

Let an **almost homomorphism** be a map $f : \mathbb{Z} \to \mathbb{Z}$ such that the set $\{f(n+m) - f(m) - f(n) : n, m \in \mathbb{Z}\}$ is finite. We say that two almost homomorphisms f, g are **almost**

equal if the set $\{f(n) - g(n) : n \in \mathbb{Z}\}$ is finite. This defines an equivalence relation on the set of almost homomorphisms. Real nur defined as the equivalence classes of this relation. To add real numbers defined this way we add the almost homomorphisms that represent them. Multiplication of real numbers corresponds to composition of almost homomorphisms. If $[f]$ denotes the real number represented by an almost homomorphism f we say that $0 \leq [f]$ if f is bounded or takes an infinite number of positive values on $\mathbb{Z}^+$. This defines the linear order relation on the set of real numbers constructed this way.

Other constructions

> Few mathematical structures have undergone as many revisions or have been presented in as many guises as the real numbers. Every generation reexamines the reals in the light of its values and mathematical objectives.[9]

A number of constructions have been given.[10] [11] [12][13] [14]

As a reviewer of one noted: "The details are all included, but as usual they are tedious and not too instructive."[15]

References

[1] Pugh, Charles Chapman (2002). *Real Mathematical Analysis* (http://books.google.com/books?id=R_ZetzxFHVwC). New York: Springer. pp. 11–15. ISBN 0-387-95297-7. .

[2] Hersh, Reuben (1997). *What is Mathematics, Really?* (http://books.google.com/books?id=R-qgdx2A5b0C). New York: Oxford University Press US. pp. 274. ISBN 0-19-513087-1. .

[3] Karin Usadi Katz and Mikhail G. Katz (2011) Stevin Numbers and Reality. Foundations of Science. doi:10.1007/s10699-011-9228-9 Online First. (http://www.springerlink.com/content/14365870k6730816/)

[4] V. Dantscher, "Vorlesungen über die Weierstrass'sche Theorie der irrationalen Zahlen" , Teubner (1908)

[5] R.D. Arthan. "The Eudoxus Real Numbers". arXiv:math/0405454.

[6] Norbert A'Campo. "A natural construction for the real numbers". arXiv:math/0301015.

[7] Ross Street (September 2003). "Update on the efficient reals" (http://www.maths.mq.edu.au/~street/reals.pdf). . Retrieved 2010-10-23.

[8] "IsarMathLib" (http://www.nongnu.org/isarmathlib/). .

[9] F. Faltin, N. Metropolis, B. Ross and G.-C. Rota, *The real numbers as a wreath product* (http://dx.doi.org/10.1016/0001-8708(75)90115-2) Advances in Math., 16(1975), 278–304.

[10] N.G. de Bruijn, N. G. Construction of the system of real numbers. (Dutch) Nederl. Akad. Wetensch. Verslag Afd. Natuurk. 86 (1977), no. 9, 121–125.

[11] N. G. de Bruijn,Defining reals without the use of rationals. Nederl. Akad.Wetensch. Proc. Ser. A 79 = Indag. Math. 38 (1976), no. 2, 100–108

also at http://alexandria.tue.nl/repository/freearticles/597556.pdf

[12] Rieger, G. J. A new approach to the real numbers (motivated by continued fractions). Abh. Braunschweig.Wiss. Ges. 33 (1982), 205–217

[13] Knopfmacher, Arnold; Knopfmacher, John Two concrete new constructions of the real numbers. Rocky Mountain J. Math. 18 (1988), no. 4, 813–824.

[14] Knopfmacher, Arnold; Knopfmacher, John A new construction of the real numbers (via infinite products). Nieuw Arch. Wisk. (4) 5 (1987), no. 1, 19–31.

[15] MR693180 (84j:26002) review of A new approach to the real numbers (motivated by continued fractions) by Rieger, G. J.

Article Sources and Contributors

Construction of the real numbers *Source*: http://en.wikipedia.org/w/index.php?oldid=500931034 *Contributors*: Adrianwn, Aetheling, Akvilas, Albmont, Algebraist, AxelBoldt, Bernard Hurley, Brad7777, CRGreathouse, CanisRufus, Charles Matthews, ChrisGualtieri, Ciphergoth, Coppertwig, Count Truthstein, Dysprosia, EmilJ, FilipeS, Frencheigh, Fwappler, Gentlemath, GirasoleDE, Goursat, Grover cleveland, Headbomb, Isaac, Jao, Josh Parris, Joshua121595, KSmrq, Katzmik, Lambiam, Laurusnobilis, Localhost00, Luqui, Micapps.euler, Michael Hardy, Netheril96, NightFalcon90909, Oleg Alexandrov, Patrick, Paul August, Pbubenik, Revolver, Reyk, Rjwilmsi, Salix alba, Shay Falador, Slawekk, StevenJohnston, Tankefugl, Tesseran, TheObtuseAngleOfDoom, Tkuvho, Tschijnmotschau, Waltpohl, Wshun, Wvbailey, Zundark, 44 anonymous edits

License