Probabilidades y estadística (M)
Segundo cuatrimestre 2018
Práctica 4

De la lista de ejercicios de abajo (sacados del libro 'Stochastics' de Georgii), hacer los siguientes:

3.1-3.3, 3.6-3.12, 3.17-3.18, 3.21.

(b) Since $A^c = \bigcup_{m\geq 1}\bigcap_{k\geq m} A_k^c$, we can write

$$\begin{aligned} P(A^c) &\leq \sum_{m\geq 1} P\Big(\bigcap_{k\geq m} A_k^c\Big) = \sum_{m\geq 1} \lim_{n\to\infty} P\Big(\bigcap_{k=m}^{n} A_k^c\Big) \\ &= \sum_{m\geq 1} \lim_{n\to\infty} \prod_{k=m}^{n} [1 - P(A_k)] \\ &\leq \sum_{m\geq 1} \lim_{n\to\infty} \exp\Big[-\sum_{k=m}^{n} P(A_k)\Big] = \sum_{m\geq 1} 0 = 0\,, \end{aligned}$$

provided $\sum_{k\geq 1} P(A_k) = \infty$. Here we have used that the complements $(A_k^c)_{k\geq 1}$ are independent by Corollary (3.20), and that $1 - x \leq e^{-x}$. $\diamond$

The Borel–Cantelli lemma will become important later on in Sections 5.1.3 and 6.4. At this point, we will only present a simple application to number theory.

(3.51) Example. *Divisibility by prime numbers.* For each prime p let A_p be the set containing all multiples of p. Then there is *no* probability measure P on $(\mathbb{N}, \mathscr{P}(\mathbb{N}))$ for which the events A_p are independent with $P(A_p) = 1/p$. Indeed, suppose there were such a P. Since $\sum_{p \text{ prime}} 1/p = \infty$, part (b) of the Borel–Cantelli lemma would then imply that the impossible event

$$A = \{n \in \mathbb{N} : n \text{ is a multiple of infinitely many primes}\}$$

had probability 1 .

Problems

3.1. A shop is equipped with an alarm system which in case of a burglary alerts the police with probability 0.99. During a night without a burglary, a false alarm is set off with probability 0.002 (e.g. by a mouse). The probability of a burglary on a given night is 0.0005. An alarm has just gone off. What is the probability that there is a burglary going on?

3.2. *Prisoners' paradox.* Three prisoners Andy, Bob and Charlie are sentenced to death. By drawing lots, where each had the same chance, one of the prisoners was granted pardon. The prisoner Andy, who has a survival probability of $1/3$, asks the guard, who knows the result, to tell him which of his fellow sufferers has to die. The guard answers 'Bob'. Now Andy calculates: 'Since either me or Charlie are going to survive, I have a chance of 50%.' Would you agree with him? (For the construction of the probability space, suppose that the guard answers 'Bob' or 'Charlie' with equal probability, if he knows that Andy has been granted pardon.)

3.3. You are flying from Munich to Los Angeles and stop over in London and New York. At each airport, including Munich, your suitcase must be loaded onto the plane. During this process, it will get lost with probability p. In Los Angeles you notice that your suitcase hasn't arrived. Find the conditional probability that it got lost in Munich, resp. London, resp. New York. (As always, a complete solution includes a description of the probability model.)

3.4. *Beta-binomial representation of the Pólya distribution.* Consider Pólya's urn model with parameters $a = r/c > 0$ and $b = w/c > 0$. Let R_n be the number of red balls obtained in n draws. Use the recursive formula (2.23) to show that

$$P(R_n = \ell) = \int_0^1 dp\ \beta_{a,b}(p)\ \mathcal{B}_{n,p}(\{\ell\})$$

for all $0 \le \ell \le n$. (Hence, the Pólya model is equivalent to an urn model with replacement where the initial ratio between red and white balls was determined by 'chance' according to a beta distribution.)

3.5. Generalise Pólya's urn model to the case when the balls can take colours from a finite set E (instead of only red and white), and find the distribution of the histogram R_n after n draws; cf. equation (2.7). Can you also generalise the previous Problem 3.4 to this case? (The corresponding generalisation of the beta distribution is called the *Dirichlet distribution.*)

3.6. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $A, B, C \in \mathscr{F}$. Show directly (without using Corollary (3.20) and Theorem (3.24)):

(a) If A, B are independent, then so are $A,\ B^c$.

(b) If A, B, C are independent, then so are $A \cup B,\ C$.

3.7. In number theory, *Euler's φ-function* is defined as the mapping $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\varphi(1) = 1$ and $\varphi(n) =$ the number of integers in $\Omega_n = \{1, \dots, n\}$ that are relatively prime to n, if $n \ge 2$. Show that if $n = p_1^{k_1} \dots p_m^{k_m}$ is the prime factorisation of n into pairwise distinct primes $p_1, \dots, p_m$ with powers $k_i \in \mathbb{N}$, then

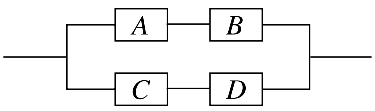
$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_m}\right).$$

Hint: Consider the events $A_i = \{p_i, 2p_i, 3p_i, \dots, n\}$, $1 \le i \le m$.

3.8. Let X be a real-valued random variable on a probability space $(\Omega, \mathscr{F}, P)$. Show that X is independent of itself if and only if X is constant with probability 1, i.e., if there exists a constant $c \in \mathbb{R}$ such that $P(X = c) = 1$. *Hint:* Consider the distribution function of X.

3.9. Let X, Y be independent random variables that are exponentially distributed with parameter $\alpha > 0$. Find the distribution density of $X/(X + Y)$.

3.10. A system consists of four components that are similar but work independently. To operate properly, it is necessary that (A and B) or (C and D) are working.



Let T be the failure time of the complete system, and T_k the failure time of component $k \in \{A, B, C, D\}$. Suppose T_k has the exponential distribution with parameter α. Show that

$$P(T < t) = \left(1 - e^{-2\alpha t}\right)^2.$$

3.11. Consider a fair tetrahedral die, whose faces are numbered by 1, 2, 3, 4, and which is thrown twice. Let X be the sum and Y the maximum of the two respective numbers on the faces falling downside.

(a) Find the joint distribution $P \circ (X, Y)^{-1}$ of X and Y.

(b) Construct two random variables X' and Y' on a suitable probability space $(\Omega', \mathscr{F}', P')$, which have the same distributions as X and Y (i.e., $P \circ X^{-1} = P' \circ X'^{-1}$ and $P \circ Y^{-1} = P' \circ Y'^{-1}$), but so that the distribution of $X' + Y'$ differs from that of $X + Y$.

3.12. Let X, Y be i.i.d. random variables taking values in $\mathbb{Z}_+$. Suppose that either

(a) $P(X = k|X + Y = n) = 1/(n + 1)$ for all $0 \leq k \leq n$, or

(b) $P(X = k|X + Y = n) = \binom{n}{k}2^{-n}$ for all $0 \leq k \leq n$.

Find the distribution of X (and hence of Y).

3.13. *Coin tossing paradox.* Alice suggests the following game to Bob: 'You randomly choose two integers $X, Y \in \mathbb{Z}$ with $X < Y$. Then you toss a fair coin. If it shows tails, you tell me Y, otherwise X. Then I will guess whether the coin showed tails or heads. If my guess is right, you pay me €100, otherwise you get €100 from me.' Should Bob agree to play the game? (After all, he can freely dispose of the distribution β according to which he picks (X, Y), and isn't it clear that the chance of guessing the result of a fair coin toss is at best 50 : 50?) To answer this, consider the following guessing strategy for Alice: Alice picks a random number $Z \in \mathbb{Z}$ with distribution α, where α is any discrete density on $\mathbb{Z}$ with $\alpha(k) > 0$ for every $k \in \mathbb{Z}$. She guesses that the coin was showing tails if the number Bob is announcing is at least Z, otherwise she guesses heads. Set up a stochastic model and find the winning probability for Alice when α and β are given.

3.14. *Dice paradox.* Two dice D_1 and D_2 are labelled as follows.

$$D_1 : 6\,3\,3\,3\,3\,3\,, \qquad D_2 : 5\,5\,5\,2\,2\,2\,.$$

Andy and Beth roll D_1 and D_2 respectively. Whoever gets the higher number wins.

(a) Show that Andy has a higher probability of winning; we write this as $D_1 \succ D_2$.

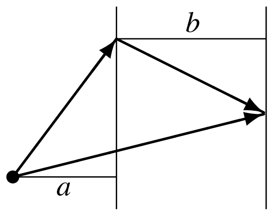
(b) Beth notices this and suggests to Andy: 'I am now going to label a third die. You can then choose an arbitrary die and I will take one of the remaining two.' Can Beth label the third die in such a way that she can always choose a die with a better chance of winning, i.e., so that $D_1 \succ D_2 \succ D_3 \succ D_1$, meaning that the relation $\succ$ is not transitive?

3.15. *Convolutions of gamma and negative binomial distributions.* Show that

(a) for $\alpha, r, s > 0$, we have $\Gamma_{\alpha,r} \star \Gamma_{\alpha,s} = \Gamma_{\alpha,r+s}$;

(b) for $p \in \,]0, 1[$ and $r, s > 0$ we have $\overline{\mathcal{B}}_{r,p} \star \overline{\mathcal{B}}_{s,p} = \overline{\mathcal{B}}_{r+s,p}$. *Hint:* The Pólya distribution provides you with a useful identity for negative binomial coefficients.

3.16. *Convolution of Cauchy distributions (Huygens' principle).* Consider the situation in Problem 2.5 and show that $c_a \star c_b = c_{a+b}$ for $a, b > 0$. In other words, the distribution of light on a straight line at distance $a+b$ from the light source is the same as if every light point on the straight line at distance a is treated as a new light source radiating uniformly in all directions.



Hint: Verify the partial fraction decomposition

$$c_a(y)c_b(x-y)/c_{a+b}(x) = \frac{b}{a+b}\,\frac{x^2+b^2-a^2+2xy}{x^2+(a-b)^2}\,c_a(y) + \frac{a}{a+b}\,\frac{x^2+a^2-b^2+2x(x-y)}{x^2+(a-b)^2}\,c_b(x-y)$$

and use that $\lim_{n\to\infty}\int_{x-n}^{x+n} z\,c_a(z)\,dz = 0$ for all x.

3.17. *Thinning of a Poisson distribution.* Suppose that the number of eggs an insect lays is Poisson distributed with parameter λ. Out of each egg, a larva hatches with probability p, independently of all other eggs. Find the distribution of the number of larvae.

3.18. *Thinning of a Poisson process.* Let $\alpha > 0$, $(L_i)_{i\geq 1}$ be a sequence of i.i.d. random variables that are exponentially distributed with parameter α, and let $T_k = \sum_{i=1}^k L_i$, $k \geq 1$. Furthermore, let $(X_k)_{k\geq 1}$ be a Bernoulli sequence with parameter $p \in \,]0,1[$, which is independent of the L_i. Show that the random variables

$$N_t^X := \sum_{k\geq 1} X_k\, 1_{]0,t]}(T_k)\,, \quad t \geq 0,$$

form a Poisson process with parameter $p\alpha$. In particular, $T_1^X := \inf\{t > 0 : N_t^X \geq 1\}$ is exponentially distributed with parameter $p\alpha$.

3.19. *Telegraph process.* Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\alpha > 0$ and $Z_t = (-1)^{N_t}$. Show that $P(Z_s = Z_t) = (1 + e^{-2\alpha(t-s)})/2$ for $0 \leq s < t$.

3.20. *Bernoulli sequence as a discrete analogue of the Poisson process.*
(a) Let $(X_n)_{n\geq 1}$ be a Bernoulli sequence for $p \in \,]0,1[$ and let

$$T_0 = 0, \quad T_k = \inf\{n > T_{k-1} : X_n = 1\}, \quad L_k = T_k - T_{k-1} - 1$$

for $k \geq 1$. (T_k is the time of the kth success and L_k the waiting time between the $(k-1)$st and the kth success.) Show that the random variables $(L_k)_{k\geq 1}$ are independent and have the geometric distribution with parameter p.

(b) Let $(L_i)_{i\geq 1}$ be an i.i.d. sequence of random variables that are geometrically distributed with parameter $p \in \,]0,1[$. For $k \geq 1$ let $T_k = \sum_{i=1}^k L_i + k$, and for $n \geq 1$ define

$$X_n = \begin{cases} 1 & \text{if } n = T_k \text{ for some } k \geq 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Show that the random variables $(X_n)_{n\geq 1}$ form a Bernoulli sequence with parameter p.

3.21. Let $(N_t)_{t\geq 0}$ be a Poisson process and $0 < s < t$. Find the conditional probability $P(N_s = k | N_t = n)$ for $0 \leq k \leq n$.