Error para la regla de Simpson

El problema acá es si tenemos $W_3(x) = (x-a)\left(x-\frac{a+b}{2}\right)(x-b)$ cambia de signo. Vamos a usar que Simpson es exacta para grado 3

el problema es que $\left(x-\frac{a+b}{2}\right)$ cambia de signo

<u>Teo</u> (Error de Simpson)

$f \in C^4[a,b] \Rightarrow R(f) = \int_a^b f(x) - S(f)$ es

$$R(f) = -\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\eta)$$

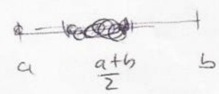
<u>Dem</u>: vemos que era exacta para grado 3.

tenemos el polinomio de Hermite:

$P_3(x_0) = f(x_0)$ $\quad P_3(x_1) = f(x_1)$ $\quad P_3(x_2) = f(x_2)$

$P_3'(x_1) = f'(x_1)$ $\quad$ idea

$W(x) = (x-x_0)(x-x_1)^2(x-x_2)$



$(x-a)(x-b)\left(x-\frac{a+b}{2}\right)^2$

$\Rightarrow \quad f(x) - P_3(x) = \frac{f^{IV}(\xi_x)}{4!} W(x)$

como P_3 interpola a f en $x_0, x_1, x_2 \Rightarrow S(f) = S(P_3) = \int(P_3)$ x ser exacta para grado 3.

$\int f - \int P_3 = \int f - S(P_3) = \int f - S(f) = R(f)$

$$R(f) = \int_a^b \frac{f^{IV}(\xi_x)}{4!}(x-x_0)(x-x_1)^2(x-x_2)$$

$$= \frac{f^{IV}(\eta)}{4!}\int_a^b (x-x_0)(x-x_1)^2(x-x_2)$$

Como $R(f) = I(f) - S(f) = \frac{f^{iv}(\eta)}{4!} \int_a^b (x-x_0)(x-x_1)^2(x-x_2)$

Si tomo $\tilde{f} = (x-x_1)^4$

$\tilde{f}^{iv} = 4!$

$I(\tilde{f}) - S(\tilde{f}) = \int_a^b (x-x_0)(x-x_1)^2(x-x_2)\,dx$

$\int_a^b (x-x_1)^4 = \frac{(x-x_1)^5}{5}\Big/_a^b = \frac{h^5}{5} + \frac{h^5}{5}$

$a \quad \frac{a+b}{2} \quad b$

$= \frac{2}{5} h^5$

$h = \frac{b-a}{2}$

$S(\tilde{f}) = S((x-x_1)^4) = \frac{h}{3} [\tilde{f}(a) + \tilde{f}(b)]$

$= \frac{h}{3} (h^4 + h^4) = \frac{2}{3} h^5$

$\Rightarrow \quad I(\tilde{f}) - S(\tilde{f}) = \frac{2}{5} h^5 - \frac{2}{3} h^5 = \frac{2}{15}(-2) h^5$

$= -\frac{4}{15} h^5$

$R(f) = \frac{f^{iv}(\eta)}{4!} \cdot \frac{-4}{15} h^5 = -\frac{f^{iv}(\eta)}{6 \cdot 15} h^5 = -\frac{f^{iv}(\eta)}{90} h^5$

Ejemplo aproximar $\int_0^1 e^{-x^2}$ x Simpson y estimar error

$\int_0^1 e^{-x^2} \sim \frac{1}{6} (f(0) + 4 f(1/2) + f(1)) = \frac{1}{6} (1 + 4e^{-1/4} + e^{-1})$

$f' = -2 e^{-x^2} x$

$f'' = 4x^2 e^{-x^2} - 2 = e^{-x^2}(4x^2 - 2)$

$f''' = e^{-x^2}[-2x(4x^2-2) + 8x]$

$f^{iv} = e^{-x^2}(4x^2(4x^2-2) - 16x)$

$+$

$f^{iv} = 4(4x^4 - 12x^2 + 3) e^{-x^2}$