Normal Multivariada

Graciela Boente

Definición

- Sea $\boldsymbol{\mu} \in \mathbb{R}^p$ y $\boldsymbol{\Sigma} \in \mathbb{R}^{p \times p}$ simétrica y definida positiva
  Se dice que $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ si su densidad está dada por

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{p}{2}}} \frac{1}{|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \mu)\right\}$$

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- Si $\mathbf{X} \sim N(\mathbf{0}, \mathrm{diag}(\lambda_1, \ldots, \lambda_p))$

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{p}{2}}} \frac{1}{\prod_{j=1}^{p} \lambda_j^{\frac{1}{2}}} \exp\left\{-\frac{1}{2} \sum_{j=1}^{p} \frac{x_j^2}{\lambda_j}\right\}$$

  Por lo tanto, $X_1, \ldots, X_p$ son independientes $X_j \sim N(0, \lambda_j)$.

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$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{p}{2}}} \frac{1}{|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \mu) \right\}$$

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  Por lo tanto, $X_1, \ldots, X_p$ son independientes $X_j \sim N(0, \lambda_j)$.
- En particular, si $\mathbf{X} \sim N(\mathbf{0}, \mathbf{I}_p)$

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{p}{2}}} e^{-\frac{1}{2}\|\mathbf{x}\|^2}$$

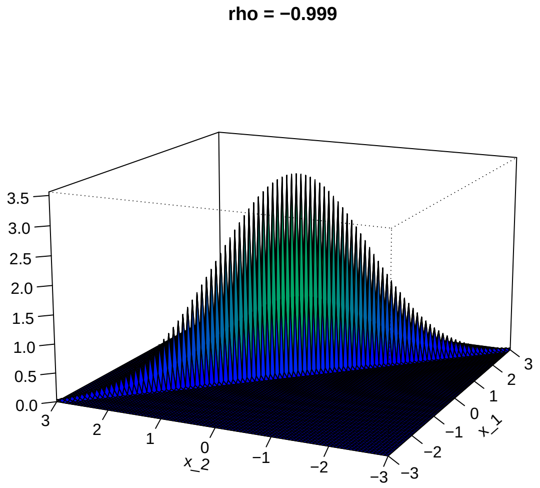
  o sea, $X_1, \ldots, X_p$ son i.i.d. $N(0, 1)$.

Caso p=2

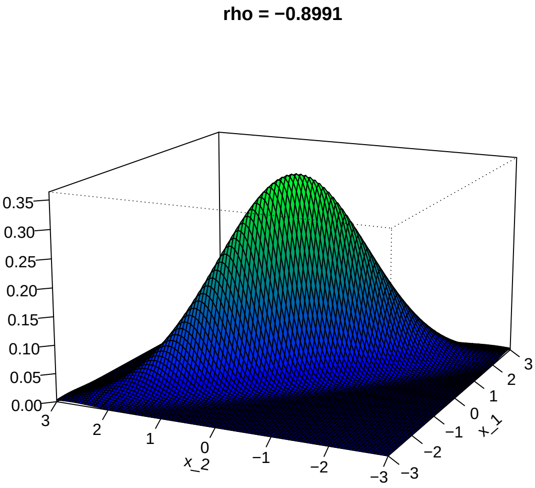
- Sea $\boldsymbol{\mu} \in \mathbb{R}^2$ y $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$ definida positiva $(|\rho| \neq 1)$

$$f(\mathbf{x}) = \frac{1}{2\pi} \frac{1}{\sigma_1\sigma_2(1-\rho^2)^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\left(\frac{x_1-\mu_1}{\sigma_1} \right)^2 \right.\right.$$
$$\left.\left. + \left(\frac{x_2-\mu_2}{\sigma_2} \right)^2 - 2\rho \left(\frac{x_1-\mu_1}{\sigma_1} \right) \left(\frac{x_2-\mu_2}{\sigma_2} \right) \right] \right\}$$

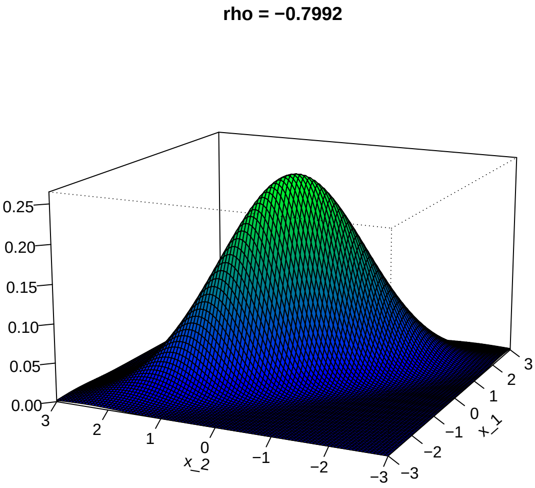
Caso p=2



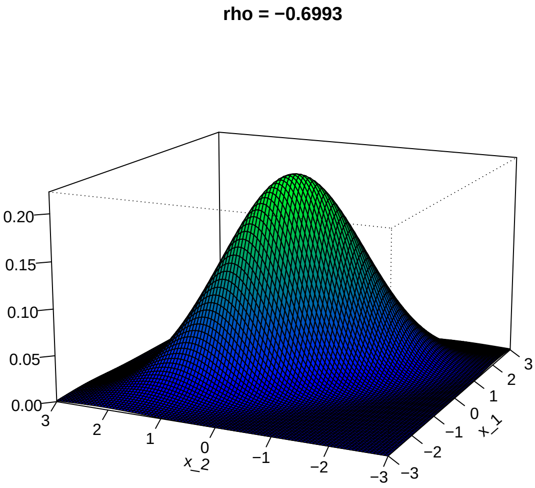
Caso p=2


rho = −0.8991
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
3
2
1
0
−1
−2
−3
x_2
3
2
1
0
−1
−2
−3
x_1

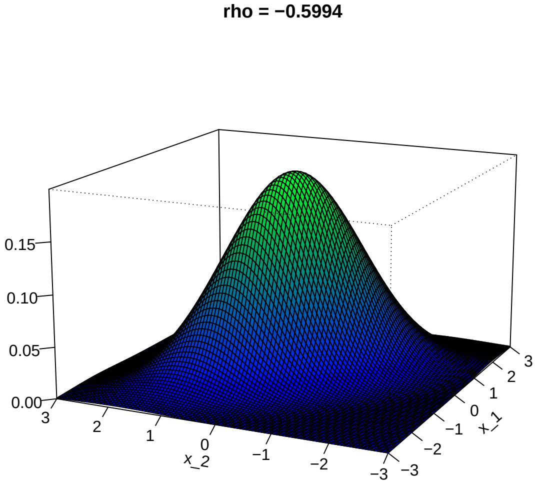
Caso p=2



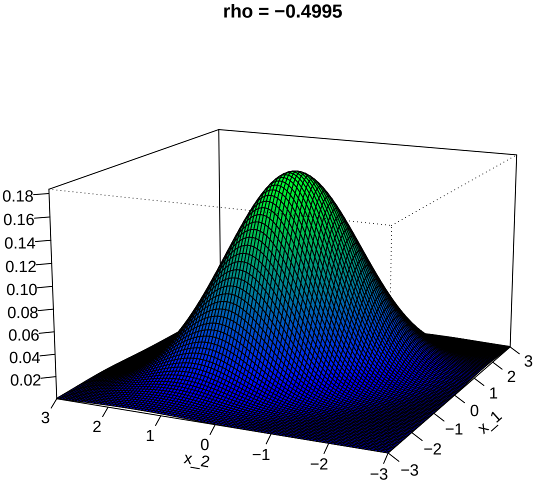
Caso p=2



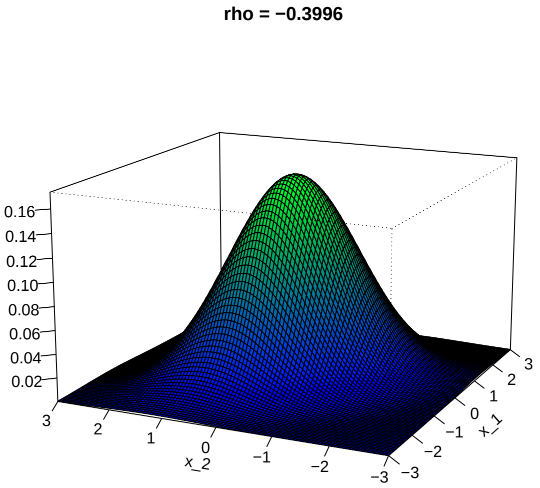
Caso p=2



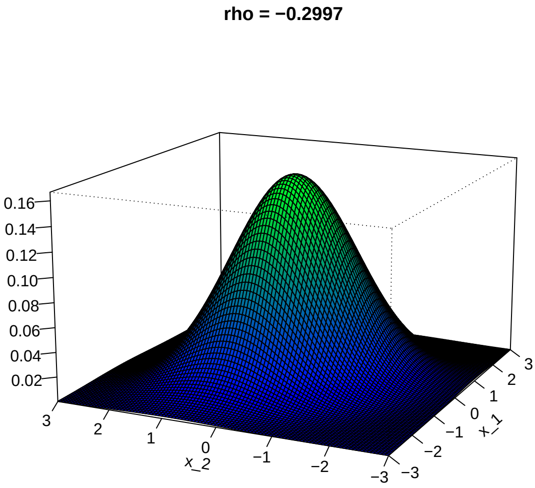
Caso p=2


rho = −0.4995
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
3
2
1
0
−1
−2
−3
x_2
3
2
1
0
−1
−2
−3
x_1

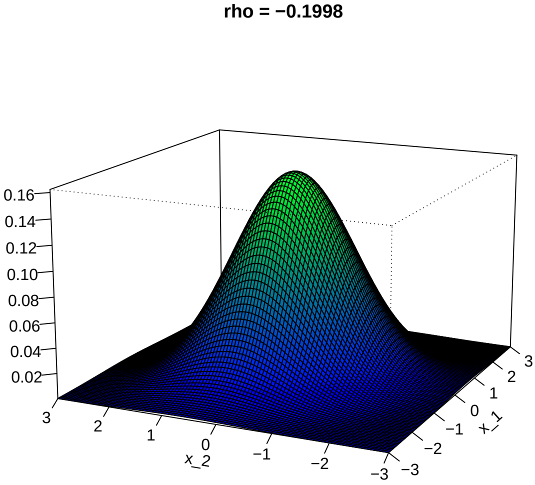
Caso p=2


rho = −0.3996
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
3
2
1
0
x_2
−1
−2
−3
−3
−2
−1
0
1
2
3
x_1

Caso p=2

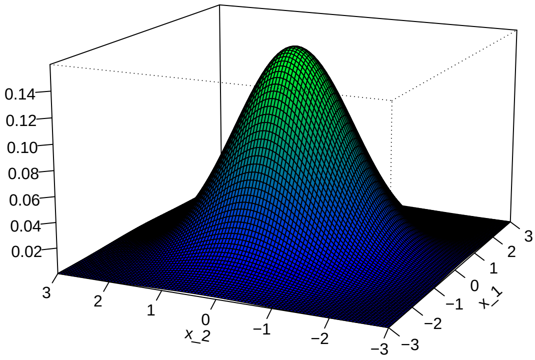


Caso p=2

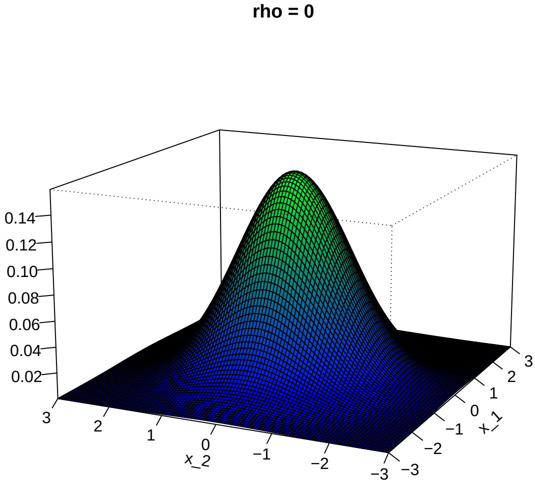

rho = −0.1998
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
3
2
1
0
−1
−2
−3
x_2
3
2
1
0
−1
−2
−3
x_1

Caso p=2

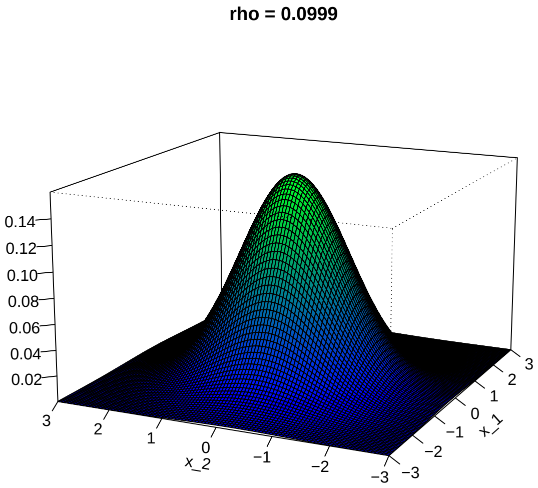
rho = −0.0999



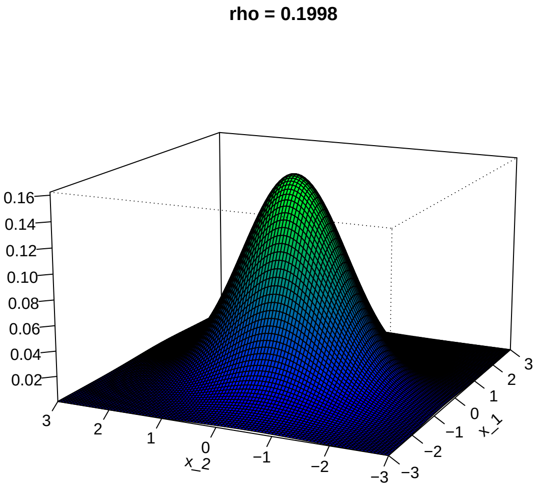
Caso p=2



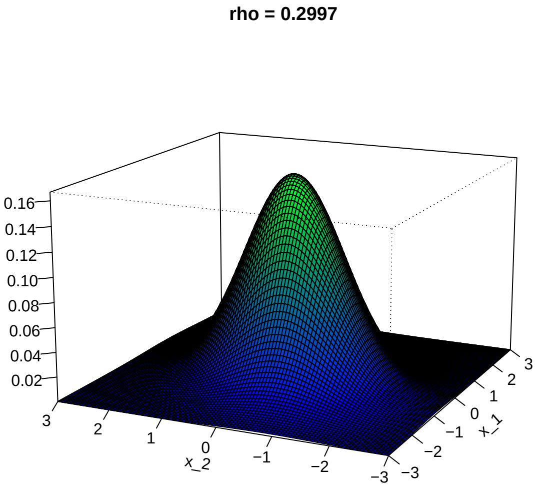
Caso p=2



Caso p=2

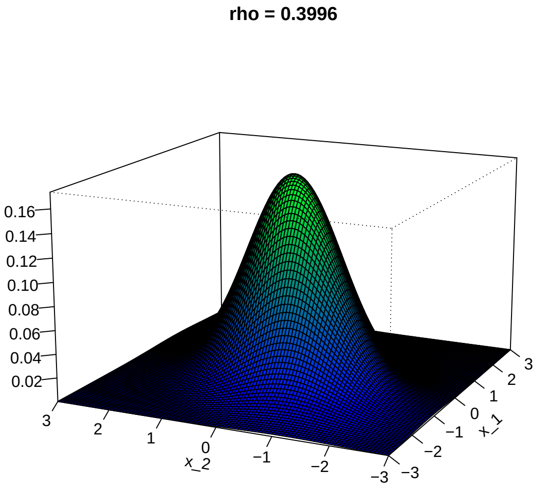

rho = 0.1998
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
3
2
1
0
−1
−2
−3
x_2
3
2
1
0
−1
−2
−3
x_1

Caso p=2



rho = 0.2997
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
3
2
1
0
−1
−2
−3
x_2
3
2
1
0
−1
−2
−3
x_1

Caso p=2


rho = 0.3996
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
3
2
1
0
x_2
−1
−2
−3
−3
−2
−1
0
1
2
3
x_1

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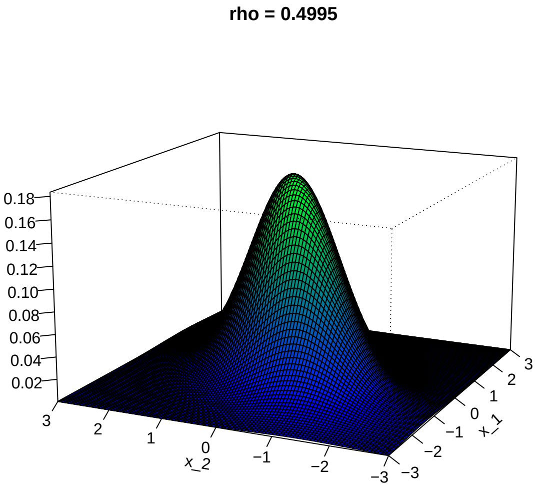
rho = 0.4995
0.18
0.16
0.14
0.12
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3
2
1
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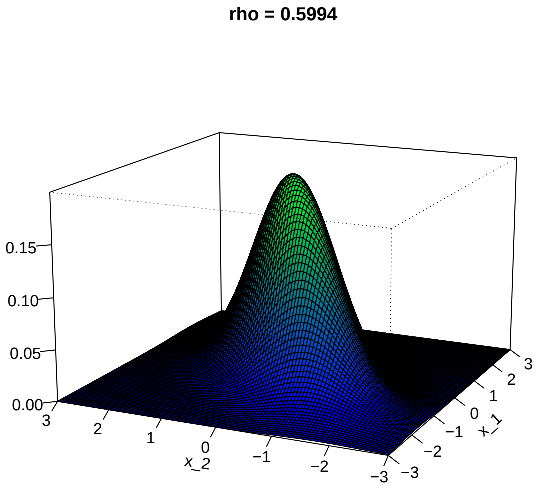
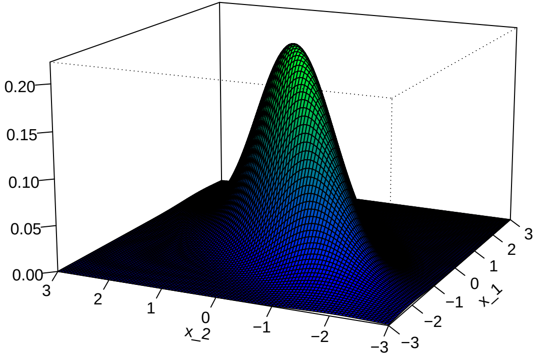
x_2
−1
−2
−3
−3
−2
−1
0
x_1
1
2
3

Caso p=2

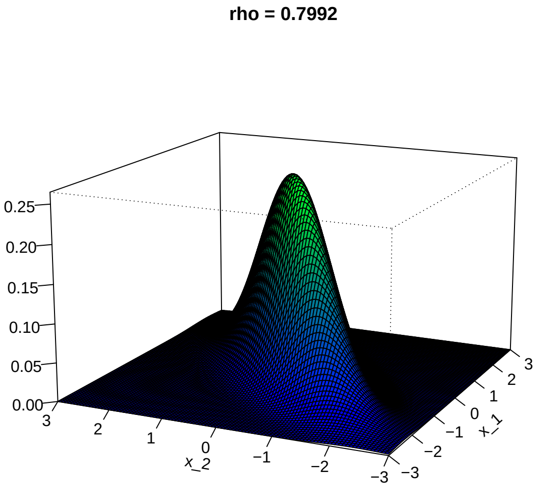


Caso p=2

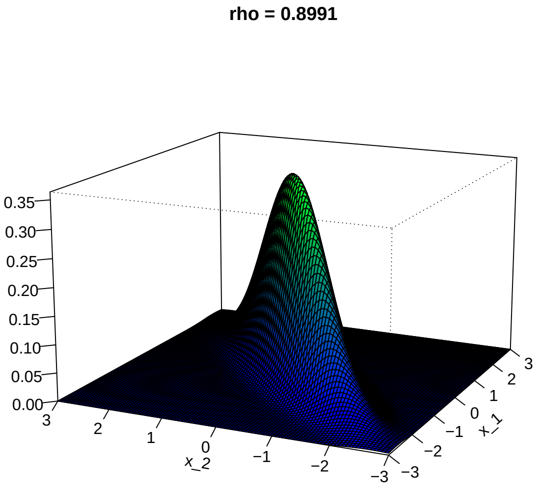
rho = 0.6993



Caso p=2



Caso p=2


rho = 0.8991
0.35
0.30
0.25
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0.10
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3
2
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x_2
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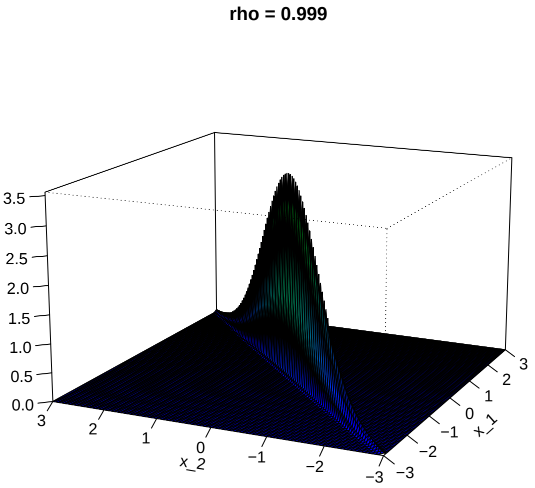
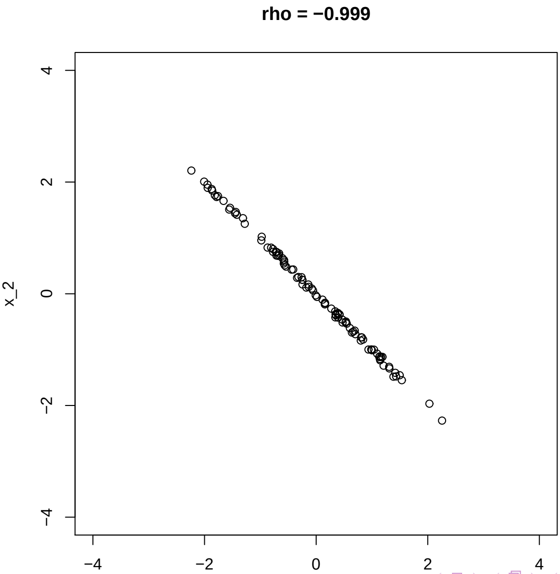
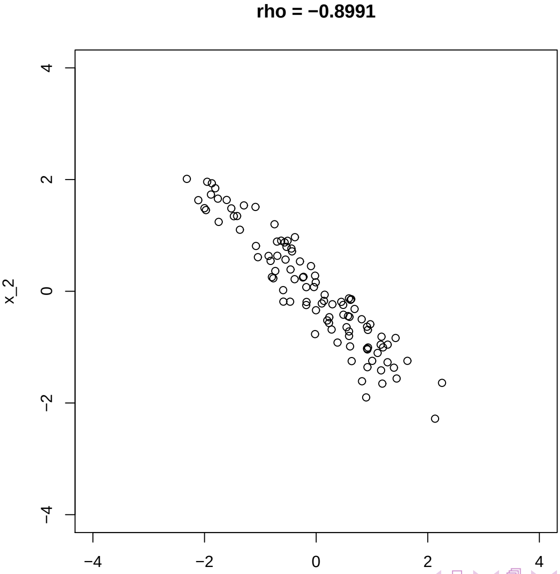
Caso p=2



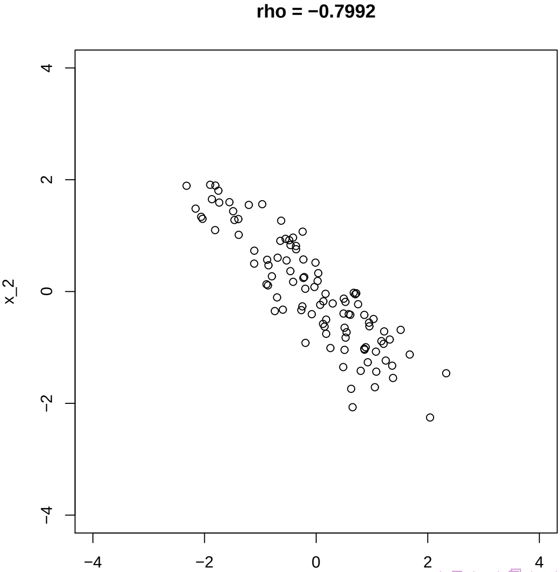
Caso p=2



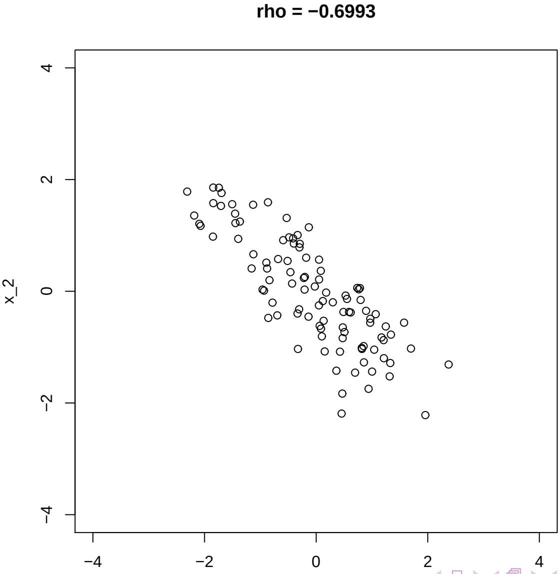
Caso p=2



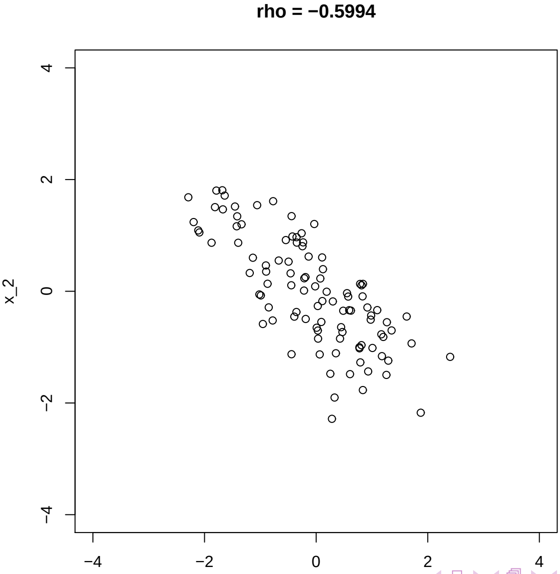
Caso p=2



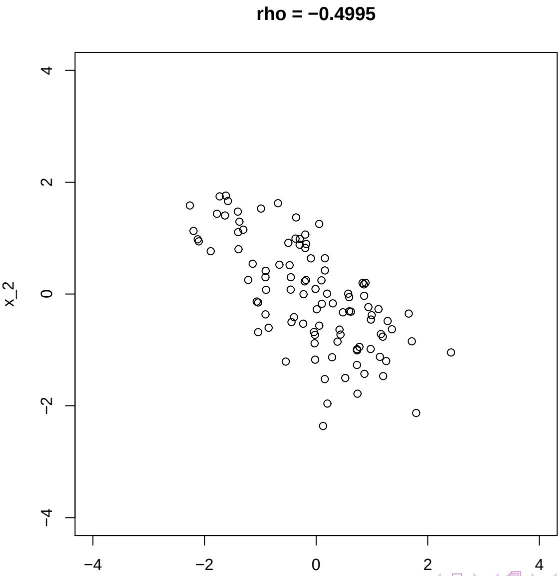
Caso p=2



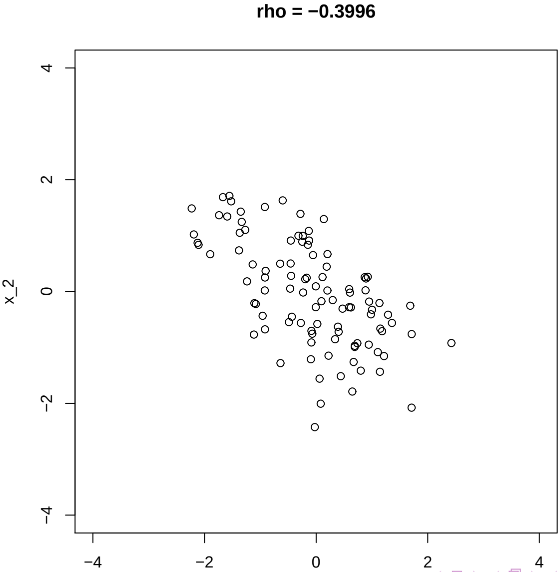
Caso p=2



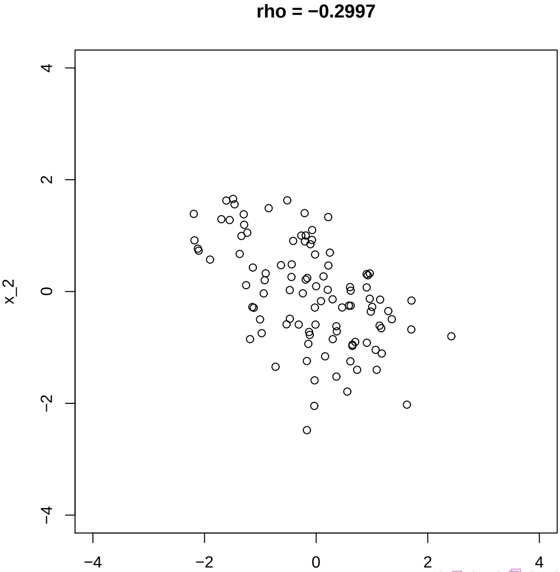
Caso p=2



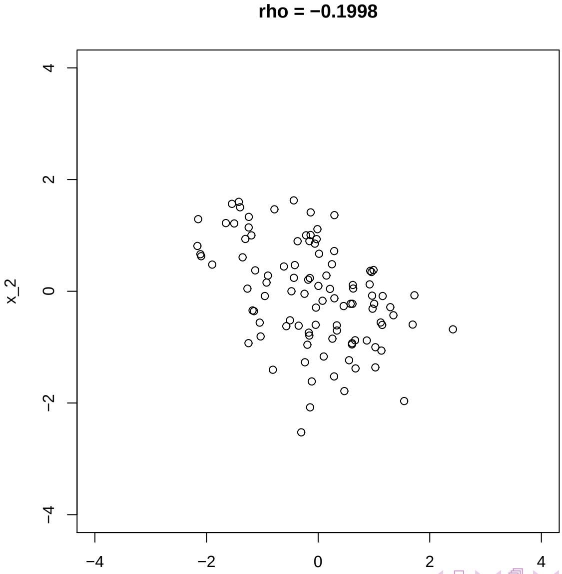
Caso p=2



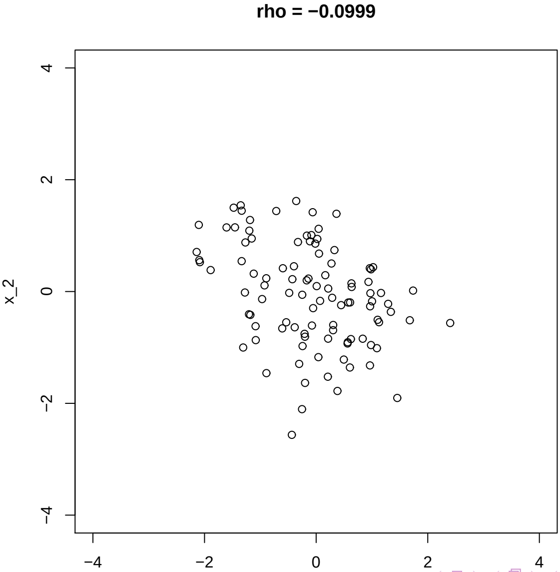
Caso p=2



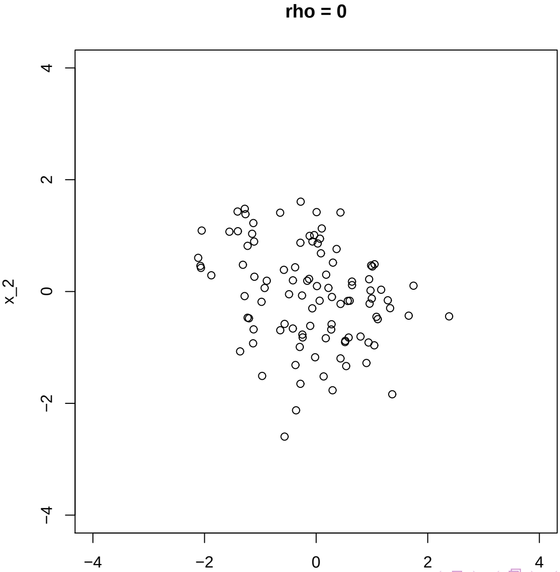
Caso p=2



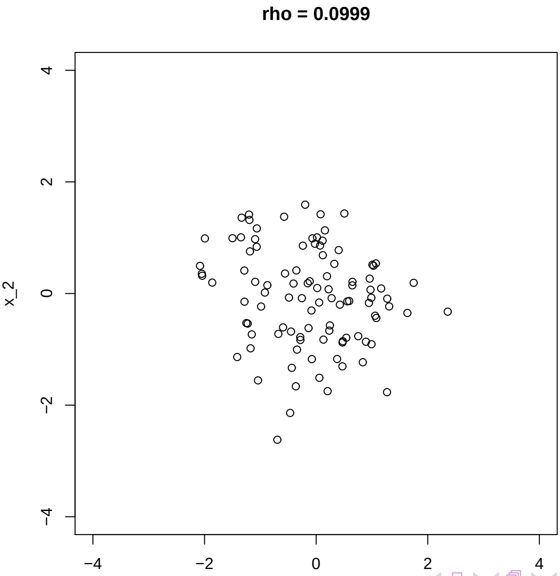
Caso p=2



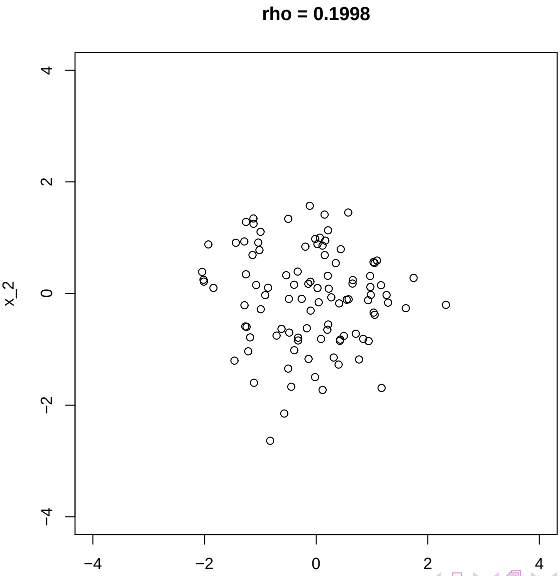
Caso p=2



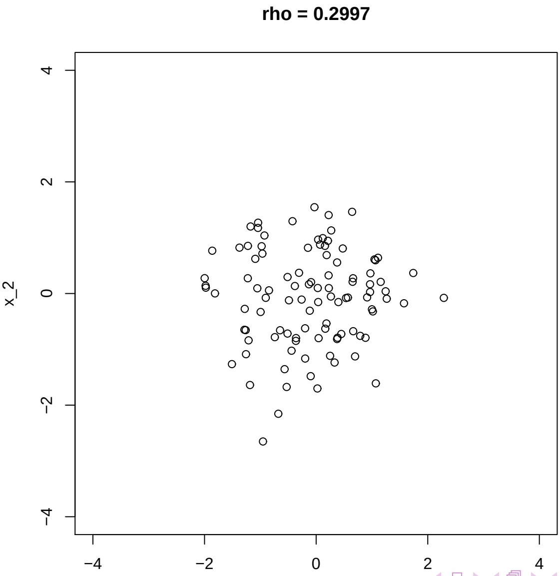
Caso p=2



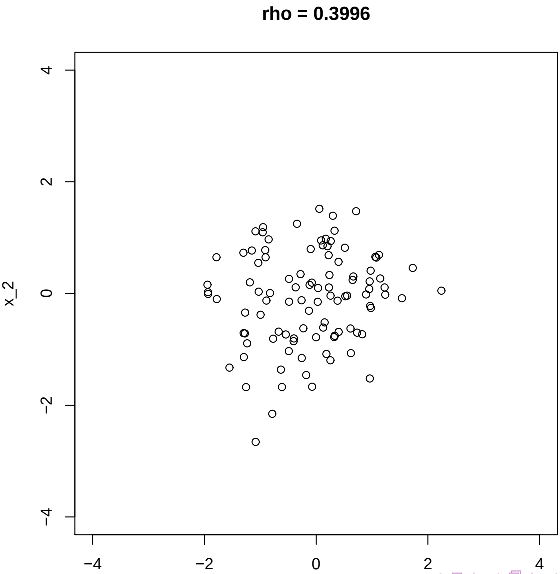
Caso p=2



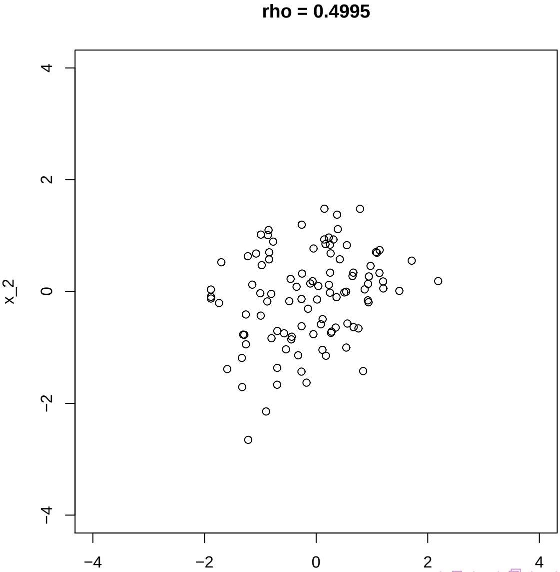
Caso p=2



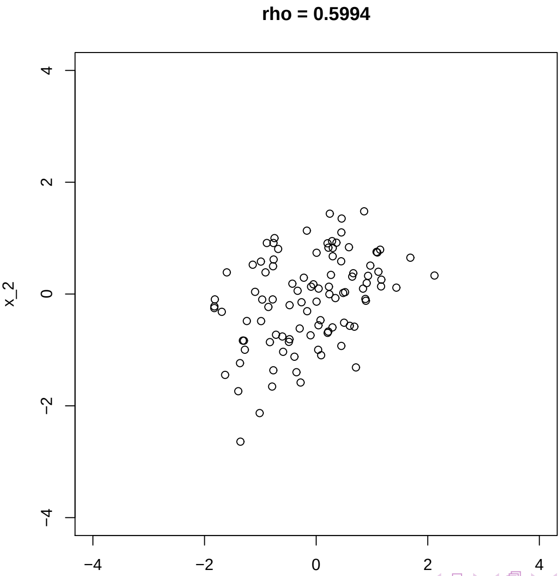
Caso p=2



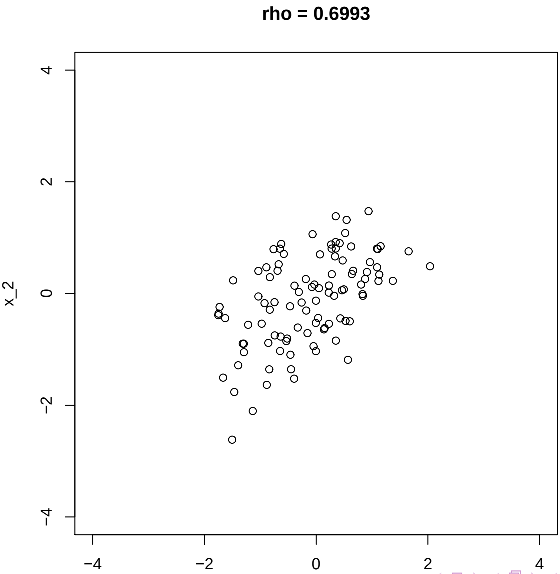
Caso p=2



Caso p=2



Caso p=2



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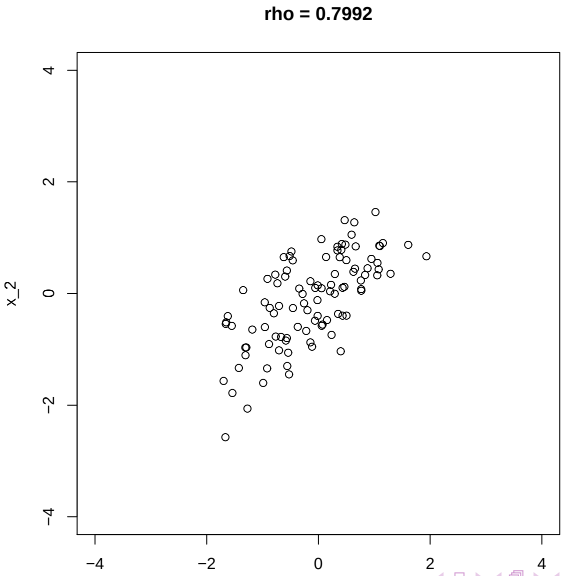
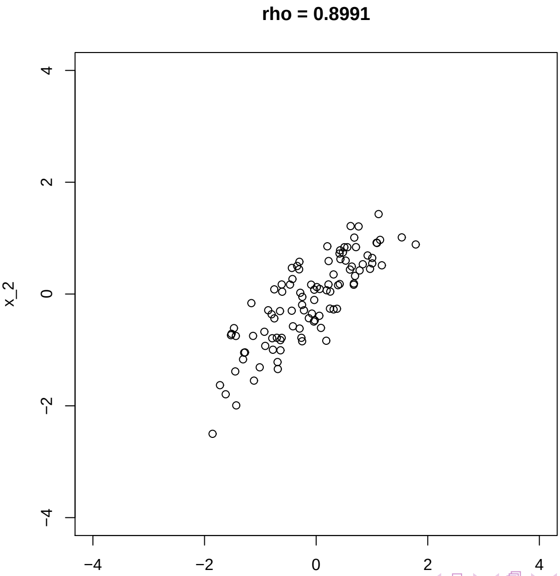
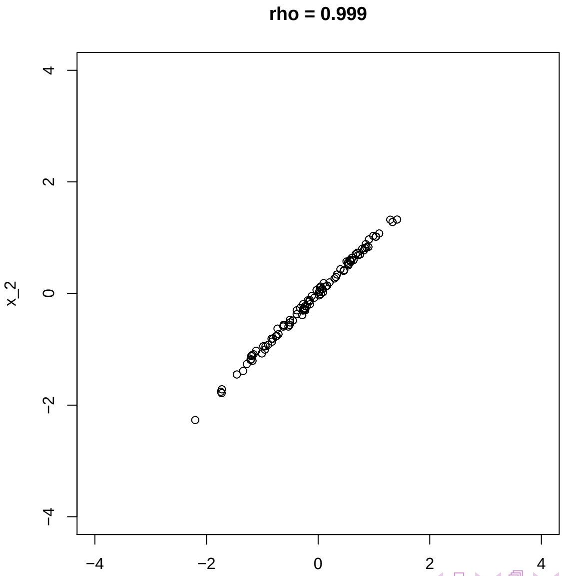


Caso p=2



Caso p=2



Propiedades

- Si $\mathbf{X} \sim N(\mathbf{0}, \mathbf{I}_p) \Longrightarrow \varphi_{\mathbf{X}}(\mathbf{t}) = \exp\{-\frac{1}{2}\|\mathbf{t}\|^2\}$

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- Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \varphi_{\mathbf{X}}(\mathbf{t}) = \exp\{i\mathbf{t}^{\mathrm{T}}\boldsymbol{\mu}\}\exp\{-\frac{1}{2}\mathbf{t}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{t}\}$

Propiedades

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- $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longleftrightarrow \mathbf{a}^{\mathrm{T}}\mathbf{X} \sim N(\mathbf{a}^{\mathrm{T}}\boldsymbol{\mu}, \mathbf{a}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{a})$, $\forall \mathbf{a} \in \mathbb{R}^p$

Propiedades

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- $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longleftrightarrow \mathbf{a}^{\mathrm{T}}\mathbf{X} \sim N(\mathbf{a}^{\mathrm{T}}\boldsymbol{\mu}, \mathbf{a}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{a}), \ \forall \mathbf{a} \in \mathbb{R}^p$
- $\mathbf{X} = (X_1, \ldots, X_p)^{\mathrm{T}} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbb{E}(\mathbf{X}) = \boldsymbol{\mu}$ y $\mathrm{COV}(\mathbf{X}) = (\mathrm{COV}(X_i, X_j))_{1 \leq i,j \leq p} = \boldsymbol{\Sigma}$

Propiedades

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- Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \varphi_{\mathbf{X}}(\mathbf{t}) = \exp\{i\mathbf{t}^{\mathrm{T}}\boldsymbol{\mu}\} \exp\{-\frac{1}{2}\mathbf{t}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{t}\}$
- $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longleftrightarrow \mathbf{a}^{\mathrm{T}}\mathbf{X} \sim N(\mathbf{a}^{\mathrm{T}}\boldsymbol{\mu}, \mathbf{a}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{a}),\ \forall \mathbf{a} \in \mathbb{R}^p$
- $\mathbf{X} = (X_1, \ldots, X_p)^{\mathrm{T}} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbb{E}(\mathbf{X}) = \boldsymbol{\mu}$ y $\mathrm{cov}(\mathbf{X}) = (\mathrm{cov}(X_i, X_j))_{1 \le i,j \le p} = \boldsymbol{\Sigma}$
- Sea $\mathbf{X} = (X_1, \ldots, X_p)^{\mathrm{T}} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, entonces, $X_1, \ldots, X_p$ son independientes $\Longleftrightarrow \boldsymbol{\Sigma}$ es diagonal.

Propiedades

- Sea $\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}$, $\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}$ y $\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$ Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbf{X}_1 \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$ y $\mathbf{X}_2 \sim N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$. Más aún, $\mathbf{X}_1$ y $\mathbf{X}_2$ son independientes $\Longleftrightarrow$ $\boldsymbol{\Sigma}_{21} = 0$.

Propiedades

- Sea $q \leq p$, $\mathbf{A} \in \mathbb{R}^{q \times p}$.
  Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b} \sim N(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathrm{T}})$

Propiedades

- Sea $q \leq p$, $\mathbf{A} \in \mathbb{R}^{q \times p}$.
  Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b} \sim N(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathrm{T}})$

- En particular, si $\mathbf{H} = (\mathbf{h}_1, \ldots, \mathbf{h}_q) \in \mathbb{R}^{p \times q}$, es ortogonal incompleta, o sea, $\mathbf{H}^{\mathrm{T}}\mathbf{H} = \mathbf{I}_q$, entonces
  si $\mathbf{X} \sim N(\mathbf{0}, \mathbf{I}_p)$, $\mathbf{Y} = \mathbf{H}^{\mathrm{T}}\mathbf{X} \sim N(\mathbf{0}, \mathbf{I}_q)$.

Propiedades

- Sea $q \leq p$, $\mathbf{A} \in \mathbb{R}^{q \times p}$.
  Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b} \sim N(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathrm{T}})$

- En particular, si $\mathbf{H} = (\mathbf{h}_1, \ldots, \mathbf{h}_q) \in \mathbb{R}^{p \times q}$, es ortogonal incompleta, o sea, $\mathbf{H}^{\mathrm{T}}\mathbf{H} = \mathbf{I}_q$, entonces
  si $\mathbf{X} \sim N(\mathbf{0}, \mathbf{I}_p)$, $\mathbf{Y} = \mathbf{H}^{\mathrm{T}}\mathbf{X} \sim N(\mathbf{0}, \mathbf{I}_q)$.

- Sea $\boldsymbol{\Sigma} = \mathbf{H}\boldsymbol{\Lambda}\mathbf{H}^{\mathrm{T}}$, con $\mathbf{H}$ ortogonal y $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$,
  $\lambda_1 \geq \ldots, \lambda_p$.
  Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbf{H}^{\mathrm{T}}(\mathbf{X} - \boldsymbol{\mu}) \sim N(\mathbf{0}, \boldsymbol{\Lambda})$.

Propiedades

- Sea $q \leq p$, $\mathbf{A} \in \mathbb{R}^{q \times p}$.
  Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b} \sim N(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathrm{T}})$

- En particular, si $\mathbf{H} = (\mathbf{h}_1, \ldots, \mathbf{h}_q) \in \mathbb{R}^{p \times q}$, es ortogonal incompleta, o sea, $\mathbf{H}^{\mathrm{T}}\mathbf{H} = \mathbf{I}_q$, entonces si $\mathbf{X} \sim N(\mathbf{0}, \mathbf{I}_p)$, $\mathbf{Y} = \mathbf{H}^{\mathrm{T}}\mathbf{X} \sim N(\mathbf{0}, \mathbf{I}_q)$.

- Sea $\boldsymbol{\Sigma} = \mathbf{H}\boldsymbol{\Lambda}\mathbf{H}^{\mathrm{T}}$, con $\mathbf{H}$ ortogonal y $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, $\lambda_1 \geq \ldots, \lambda_p$.
  Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbf{H}^{\mathrm{T}}(\mathbf{X} - \boldsymbol{\mu}) \sim N(\mathbf{0}, \boldsymbol{\Lambda})$.

- Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longleftrightarrow \mathbf{X} = \mathbf{A}\mathbf{Z} + \boldsymbol{\mu}$ con $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I}_p)$ y $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \boldsymbol{\Sigma}$.

Propiedades

- Sea $q \leq p$, $\mathbf{A} \in \mathbb{R}^{q \times p}$.
  Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbf{Y} = \mathbf{AX} + \mathbf{b} \sim N(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathrm{T}})$

- En particular, si $\mathbf{H} = (\mathbf{h}_1, \ldots, \mathbf{h}_q) \in \mathbb{R}^{p \times q}$, es ortogonal incompleta, o sea, $\mathbf{H}^{\mathrm{T}}\mathbf{H} = \mathbf{I}_q$, entonces si $\mathbf{X} \sim N(\mathbf{0}, \mathbf{I}_p)$, $\mathbf{Y} = \mathbf{H}^{\mathrm{T}}\mathbf{X} \sim N(\mathbf{0}, \mathbf{I}_q)$.

- Sea $\boldsymbol{\Sigma} = \mathbf{H}\boldsymbol{\Lambda}\mathbf{H}^{\mathrm{T}}$, con $\mathbf{H}$ ortogonal y $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, $\lambda_1 \geq \ldots, \lambda_p$.
  Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \mathbf{H}^{\mathrm{T}}(\mathbf{X} - \boldsymbol{\mu}) \sim N(\mathbf{0}, \boldsymbol{\Lambda})$.

- Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longleftrightarrow \mathbf{X} = \mathbf{AZ} + \boldsymbol{\mu}$ con $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I}_p)$ y $\mathbf{AA}^{\mathrm{T}} = \boldsymbol{\Sigma}$.

- Si $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow (\mathbf{X} - \boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \mu) \sim \chi^2_p$

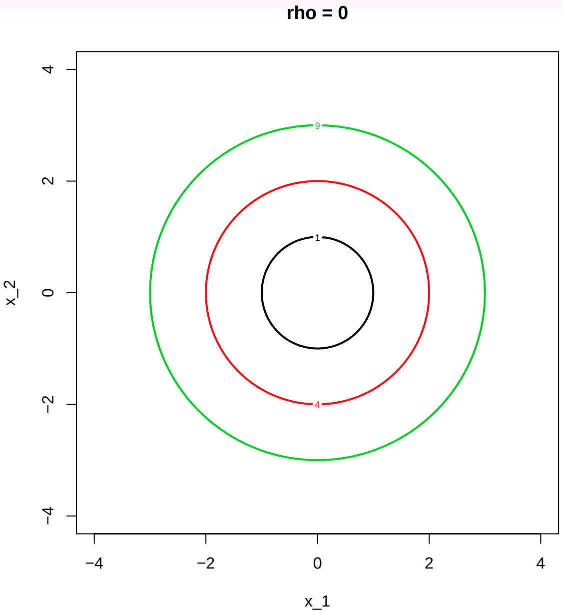
Propiedades

- Las regiones de densidad constante son los elipsoides

$$(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\mu)=c^2$$

- Si $\boldsymbol{\Sigma}=\mathbf{H}\boldsymbol{\Lambda}\mathbf{H}^{\mathrm{T}}$, con $\mathbf{H}$ ortogonal y $\boldsymbol{\Lambda}=\mathrm{diag}(\lambda_1,\ldots,\lambda_p)$, definamos $\mathbf{y}=\mathbf{H}^{\mathrm{T}}\mathbf{x}$ y $\boldsymbol{\nu}=\mathbf{H}^{\mathrm{T}}\boldsymbol{\mu}$.
  Los contornos de densidad constante son los elipsoides centrados en $\boldsymbol{\nu}$ con ejes principales de longitud $2c\lambda_j^{\frac{1}{2}}$ soportados en los autovectores, $\mathbf{h}_1,\ldots,\mathbf{h}_p$, o sea,

$$\sum_{j=1}^{p}\frac{(y_j-\nu_j)^2}{\lambda_j}=c^2$$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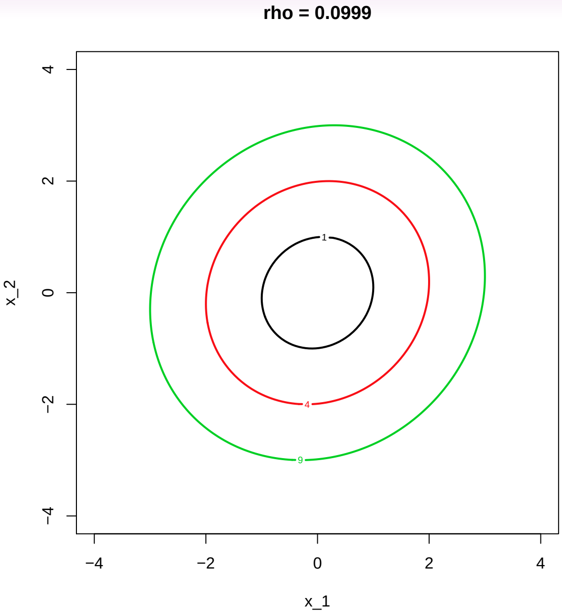
9
4
1
x_1
x_2
−4
−2
0
2
4


rho = 0.0999
x_2
x_1
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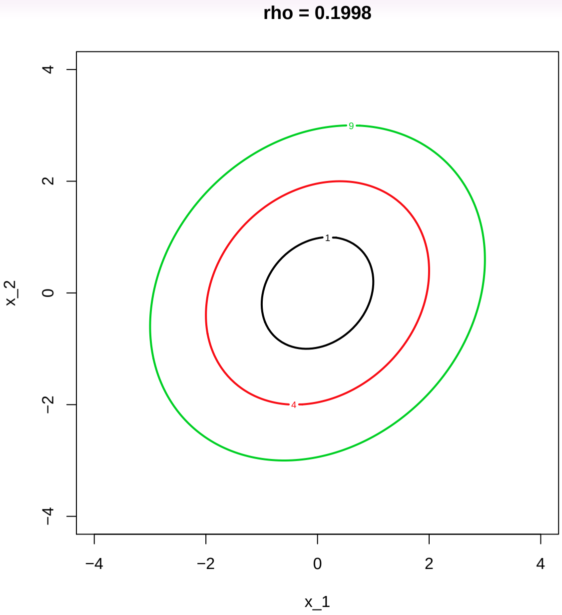
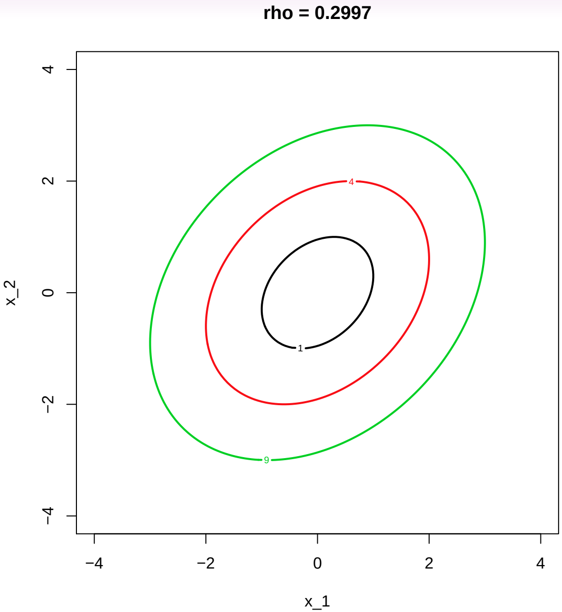
2
0
−2
−4
1
4
9


rho = 0.1998
9
4
1
x_1
x_2
−4
−2
0
2
4


rho = 0.2997
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x_1
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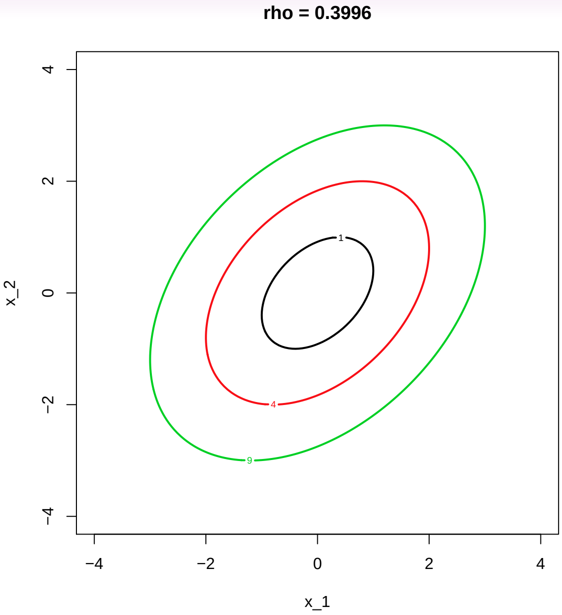
2
0
−2
−4
−4
−2
0
2
4
1
4
9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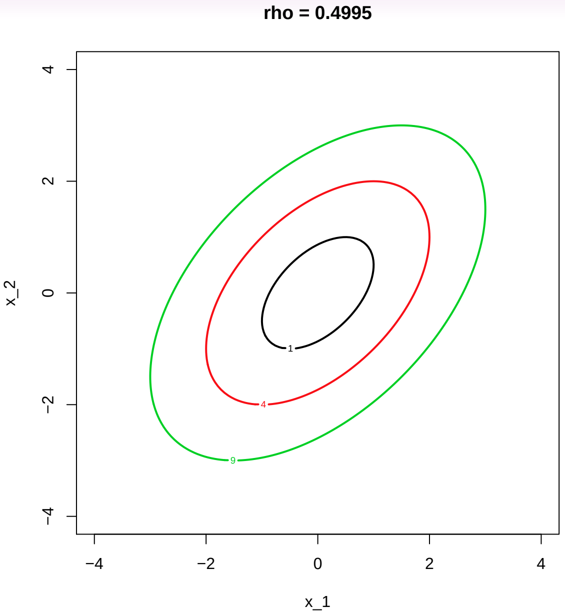
x_1
x_2
−4
−2
0
2
4
1
4
9


rho = 0.4995
x_1
x_2
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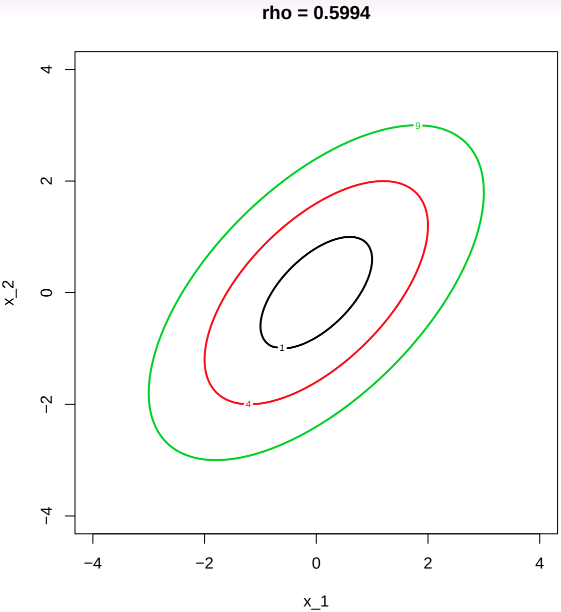
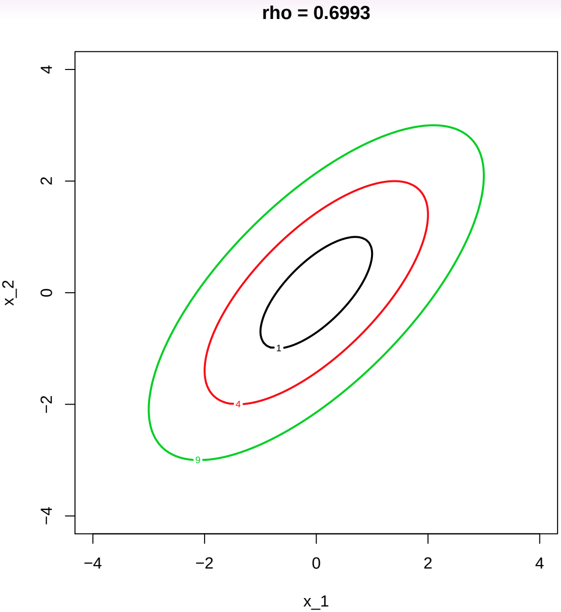
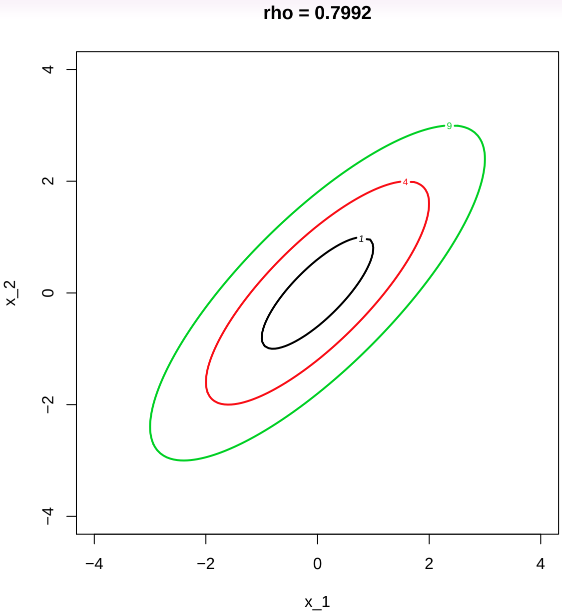
−2
0
2
4
1
4
9


rho = 0.5994
x_1
x_2
−4
−2
0
2
4
9
4
1


rho = 0.6993
x_1
x_2
−4
−2
0
2
4
1
4
9


rho = 0.7992
x_1
x_2
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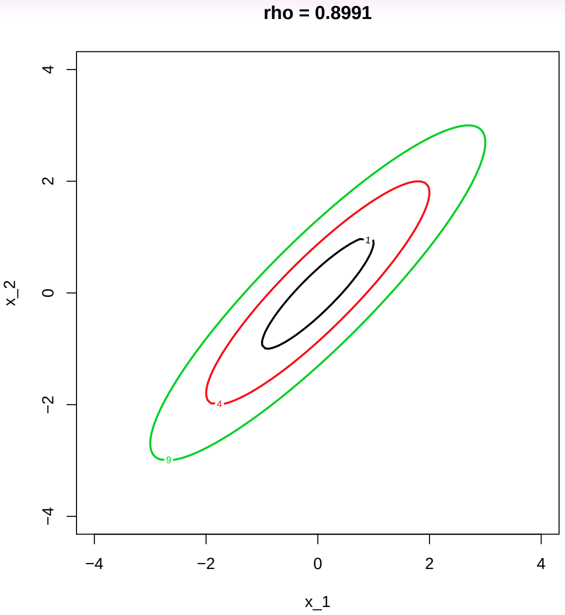
−2
0
2
4
1
4
9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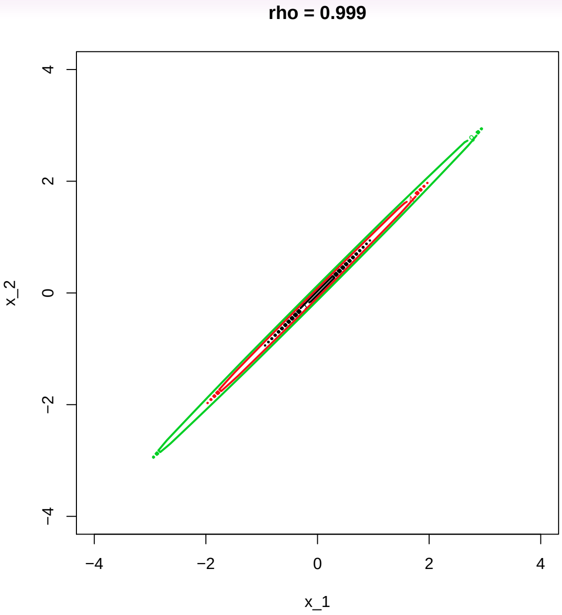
x_2
x_1
4
2
0
−2
−4
−4
−2
0
2
4
1
4
9


rho = 0.999
x_2
x_1
−4
−2
0
2
4
−4
−2
0
2
4