Álgebra 1

Práctica 7: Polinomios

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Ej. 1.

Factorizar $f = x^{16} - 97x^8 + 1296$ en $\mathbb{Q}[x],\ \mathbb{R}[x],\ \mathbb{C}[x]$

Solución: • Raíces de f: $u = x^8 \implies f = u^2 - 97u + 1296 = 0$

$$\implies \left(u - \frac{97}{2}\right)^2 - \frac{97^2}{4} + 1296 = 0 \Leftrightarrow \left(u - \frac{97}{2}\right)^2 = \frac{97^2}{4} - 1296$$

$$= \frac{1}{4}\left[(16+81)^2 - 4\cdot 16 \cdot 81\right] = \frac{1}{4}(16-81)^2$$

$$\Leftrightarrow u - \frac{(16+81)}{2} = \pm\frac{1}{2}(16-81)$$

$$\Leftrightarrow u = 16 \text{ ó } u = 81 \Leftrightarrow x^8 = 16 \text{ ó } x^8 = 81$$

$$\Rightarrow f = (x^8 - 16)(x^8 - 81)$$

$$\Leftrightarrow \left(\frac{x}{\sqrt{2}}\right)^8 = 1 \text{ ó } \left(\frac{x}{\sqrt{3}}\right)^8 = 1$$

$\Longrightarrow$ Raíces (f):

$$= \{x \in \mathbb{C} : x = \sqrt{2}\,\omega : \omega \in G_8\} \cup \{x \in \mathbb{C} : x = \sqrt{3}\,\omega : \omega \in G_8\}$$

$$\omega \in G_8 \Leftrightarrow \omega^8 = 1 \Longrightarrow G_8 = \{\omega_k = e^{\frac{2k}{8}\pi i}\} \ \textit{pues}$$

$$\omega \in G_8 \Rightarrow 8 \cdot arg(\omega) = arg(1) = 2k\pi \ \Rightarrow arg(\omega) = \frac{2k}{8}\pi : 0 \leq k \leq 7$$

$$\Longrightarrow arg(\omega) \in \left\{\theta_k = \frac{k}{4}\pi : \ 0 \leq k \leq 7\right\}$$

$$= \left\{\frac{1}{4}\pi, \ \frac{1}{2}\pi, \ \frac{3}{4}\pi, \ \pi, \ \frac{5}{4}\pi, \ \frac{3}{2}\pi, \ \frac{7}{4}\pi\right\}$$

$$\Rightarrow G_8 = \{1, \ -1, \ i, \ -i\} \cup \left\{\textit{Primitivas} : \ \frac{\pm 1 \pm i}{\sqrt{2}}\right\}$$

$\Longrightarrow$ Raíces (f):

$$= \{x \in \mathbb{C} : x = \sqrt{2}\,\omega : \omega \in G_8\} \cup \{x \in \mathbb{C} : x = \sqrt{3}\,\omega : \omega \in G_8\}$$

$$= \left\{\pm\sqrt{2},\ \pm\sqrt{2}i,\ \pm 1 \pm i\right\} \cup \left\{\pm\sqrt{3},\ \pm\sqrt{3}i,\ \sqrt{3}\left(\frac{\pm 1 \pm i}{\sqrt{2}}\right)\right\}$$

Factorización

$$\Rightarrow f = (x^8 - 16)(x^8 - 81)$$

$$= (x^4 - 4)(x^4 + 4)(x^4 - 9)(x^4 + 9)$$

$$= (x^2 - 2)(x^2 + 2)(x^4 + 4)(x^2 - 3)(x^2 + 3)(x^4 + 9) \quad \in \mathbb{Q}[x]$$

$$= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)(x^4 + 4)(x - \sqrt{3})(x + \sqrt{3})(x^2 + 3)(x^4 + 9)$$

$$\in \mathbb{R}[x]$$

$$= (x - \sqrt{2})(x + \sqrt{2})(x + \sqrt{2}i)(x - \sqrt{2}i)(x - (1+i))(x - (1-i))(x + (1+i))(x + (1-i))$$

$$(x - \sqrt{3})(x + \sqrt{3})(x + \sqrt{3}i)(x - \sqrt{3}i)\left(x - \sqrt{\frac{3}{2}}(1+i)\right)\left(x - \sqrt{\frac{3}{2}}(1-i)\right)\left(x + \sqrt{\frac{3}{2}}(1+i)\right)\left(x + \sqrt{\frac{3}{2}}(1-i)\right)$$

$$\in \mathbb{C}[x] \quad \checkmark$$

$$\Rightarrow f = (x^8 - 16)(x^8 - 81)$$

$$= (x - \sqrt{2})(x + \sqrt{2})(x + \sqrt{2}i)(x - \sqrt{2}i)$$

$$(x - (1+i))(x - (1-i))(x + (1+i))(x + (1-i))$$

$$(x - \sqrt{3})(x + \sqrt{3})(x + \sqrt{3}i)(x - \sqrt{3}i)$$

$$\left(x - \sqrt{\frac{3}{2}}(1+i)\right)\left(x - \sqrt{\frac{3}{2}}(1-i)\right)\left(x + \sqrt{\frac{3}{2}}(1+i)\right)\left(x + \sqrt{\frac{3}{2}}(1-i)\right)$$

$\in \mathbb{C}[x]$ ✓

Ej. 2.

Hallar $\mathbf{c} \in \mathbb{R}$ tal que $f = x^6 + 8x^4 - 14x^3 - 19x^2 - 126x + \mathbf{c}$ tenga una raíz imaginaria pura $\neq 0$.
Factorizar f en $\mathbb{Q}[x]$, $\mathbb{R}[x]$, $\mathbb{C}[x]$ sabiendo, además, que f tiene una raíz de la forma $1 + \sqrt{p}$, con p primo.

Solución: Evaluemos $f(ai)$: $a \in \mathbb{R}$

$$\Rightarrow 0 = f(ai) = (ai)^6 + 8(ai)^4 - 14(ai)^3 - 19(ai)^2 - 126(ai) + \mathbf{c}$$

$$= -a^6 + 8a^4 + 14a^3 i + 19a^2 - 126ai + \mathbf{c}$$

$$= -a^6 + 8a^4 + 19a^2 + \mathbf{c} + ai\left(14a^2 - 126\right)$$

$$\Rightarrow \begin{cases} 0 = Re\ f(ai) & = -a^6 + 8a^4 + 19a^2 + \mathbf{c} \\ 0 = Im\ f(ai) & \Rightarrow 0 = 14a^2 - 126 = 14\left(a^2 - 9\right) \Leftrightarrow a^2 = 9 \end{cases}$$

$$\Rightarrow \boxed{\pm 3i \text{ son raices de } f}$$

$$\Rightarrow 0 = Re\, f(ai) = -a^6 + 8a^4 + 19a^2$$

$$= -9^3 + 8 \cdot 9^2 + 19 \cdot 9 + \mathbf{c} \Rightarrow \mathbf{c} = -90$$

- $f = x^6 + 8x^4 - 14x^3 - 19x^2 - 126x - 90$

- $\pm 3i$ *son raices de f* $\Rightarrow (x-3i)(x+3i) = (x^2+9) \Big| f \in \mathbb{R}[x]$

$$\Rightarrow f = (x^2+9)(x^4 - x^2 - 14x - 10)$$

Cálculos Auxiliares

x^6	$+8x^4$	$-14x^3$	$-19x^2$	$-126x$	-90	x^2+9
$-x^6$	$-9x^4$					x^4
	$-x^4$	$-14x^3$	$-19x^2$	$-126x$	-90	
	$+x^4$		$+9x^2$			$-x^2$
		$-14x^3$	$-10x^2$	$-126x$	-90	
		$+14x^3$		$+126x$		$-14x$
			$-10x^2$		-90	
			$+10x^2$		$+90$	-10
					0	

$$\Rightarrow f = x^6 + 8x^4 - 14x^3 - 19x^2 - 126x - 90$$

$$\boxed{= (x^2+9)(x^4 - x^2 - 14x - 10)}$$

- $f = (x^2+9)(x^4 - x^2 - 14x - 10)$

- $1+\sqrt{p}$ es raíz de $f \Leftrightarrow$ es raíz de $(x^4 - x^2 - 14x - 10)$

Evaluemos en $1+\sqrt{p}$:

$$0 = (1+\sqrt{p})^4 - (1+\sqrt{p})^2 - 14(1+\sqrt{p}) - 10$$

$$= [1+4\sqrt{p}+6p+4p\sqrt{p}+p^2] - [1+2\sqrt{p}+p] - 14 - 14\sqrt{p} - 10$$

$$= \underbrace{[1+6p+p^2-1-p-24]}_{(*)\in\mathbb{Z}} + \sqrt{p}\underbrace{[4p-12]}_{debe\ ser\ =0}$$

$$\Rightarrow p = 3 \ \Rightarrow (*) = 0$$

$$\Rightarrow \boxed{1\pm\sqrt{3} \text{ son raíces de } f \in \mathbb{Q}[x]}$$

Raíces complejas

$$\Rightarrow 1 \pm \sqrt{3} \text{ son raíces de } f \;\Rightarrow \underbrace{(x-(1+\sqrt{3}))(x-(1-\sqrt{3}))}_{=(x^2-2x-2)} \Big| f$$

$$\Rightarrow \boxed{f = (x^2+9)(x^2-2x-2)(x^2+2x+5) \in \mathbb{Q}[x]}$$

Raíces: $0 = x^2 + 2x + 5$

$$= (x+1)^2 - 1 + 5 \;\Rightarrow (x+1)^2 = -4 \;\Rightarrow x+1 = \pm 2i$$

$$\Rightarrow \boxed{x = -1-2i,\; x = -1+2i \text{ son raíces de } f}$$

Factorización

Raíces: f no tiene raíces en $\mathbb{Q}$

$$= \{x = \underbrace{1 \pm \sqrt{3}}_{\in\mathbb{R}},\ x = \underbrace{-1 \pm 2i}_{\in\mathbb{C}},\ x = \underbrace{\pm 3i}_{\in\mathbb{C}}\}$$

$$\Rightarrow \boxed{f = (x^2 - 2x - 2)(x^2 + 2x + 5)(x^2 + 9) \in \mathbb{Q}[x]}$$

$$\Rightarrow \boxed{f = (x-(1-\sqrt{3}))(x-(1+\sqrt{3}))(x^2 + 2x + 5)(x^2 + 9) \in \mathbb{R}[x]}$$

$$\Rightarrow \boxed{f = (x-(1-\sqrt{3}))(x-(1+\sqrt{3}))(x-3i)(x+3i)}$$

$$\boxed{(x-(-1+2i))(x-(-1-2i)) \in \mathbb{C}[x]} \checkmark$$

Ej. 3.

Hallar todos los polinomios mónicos $f \in \mathbb{Q}[x]$ de grado 6 tales que:

- $d = mcd(f : f') \neq 1$
- $(x^2 - (\sqrt{3} + i)x + \sqrt{3}i) \Big| f$ en $\mathbb{C}[x]$
- $f(2) = 5$

b) Factorizar f en $\mathbb{Q}[x], \mathbb{R}[x], \mathbb{C}[x]$ en cada caso.

Solución: • $q = (x^2 - (\sqrt{3} + i)x + \sqrt{3}i) | f$ en $\mathbb{C}[x]$

$\Longrightarrow$ las raíces de q son raíces de f

Raíces de q:

$$\alpha_{1,2} = \frac{\sqrt{3} + i \pm \sqrt{(\sqrt{3} + i)^2 - 4\sqrt{3}i}}{2} = \frac{\sqrt{3} + i + \Delta_{\pm}}{2}$$

$$(x+yi)^2 = (\Delta_\pm)^2 = (\sqrt{3}+i)^2 - 4\sqrt{3}i, \quad entonces$$

$$\bullet x^2 - y^2 = Re(\Delta) = 3 + i^2 = 2$$

$$\bullet 2xy = Im(\Delta) = 2\sqrt{3} - 4\sqrt{3}$$

$$\Longrightarrow x^2 - y^2 = 2, \quad xy = -\sqrt{3} \quad \Longrightarrow y = -\frac{\sqrt{3}}{x}$$

$$\Longrightarrow 2 = x^2 - \frac{3}{x^2} \Longrightarrow 0 = x^4 - 3 - 2x^2$$

que tiene raíces $x^2 = 3$ y ~~$x^2 = -1$~~, con $x \in \mathbb{R} \Longrightarrow x = \pm\sqrt{3}$

$$\Longrightarrow \Delta_\pm = \pm(\sqrt{3} - i)$$

Raíces de q:

$$\Longrightarrow \alpha_1 = \sqrt{3}, \quad \alpha_2 = i$$

- Raíces de q: $\Longrightarrow$ $\boxed{\alpha_1 = \sqrt{3}, \quad \alpha_2 = i}$ *pero* $q | f \in \mathbb{Q}[x]$

$\Longrightarrow \alpha_1 = \sqrt{3}, \quad \widetilde{\alpha_1} = -\sqrt{3}, \quad \alpha_2 = i, \quad \overline{\alpha_2} = -i$ son raíces de f

- Producto de irreducibles distintos que dividen a f, divide a f:

$$\Longrightarrow \boxed{\underbrace{(x-\sqrt{3})(x+\sqrt{3})}_{=(x^2-3)}\underbrace{(x-i)(x+i)}_{=(x^2+1)}\Big| f \in \mathbb{Q}[x]}$$

- $d = mcd(f : f') \neq 1 \Longrightarrow$ f tiene al menos una raíz múltiple:

$$gr(f) = 6 \Longrightarrow \begin{cases} f_1 = & (x^2-3)^2(x^2+1) \\ f_2 = & (x^2-3)(x^2+1)^2 \\ f_3 = & (x^2-3)(x^2+1)(x-\omega)^2 \end{cases}$$

- $f(2) = 5$

$$\begin{cases} f_1 = (x^2-3)^2(x^2+1) \implies f_1(2) = (2^2-3)^2(2^2+1) = 5 \checkmark \\ f_2 = (x^2-3)(x^2+1)^2 \implies f_2(2) = (2^2-3)(2^2+1)^2 = 25 \neq 5 \\ f_3 = (x^2-3)(x^2+1)(x-\omega)^2 \\ \implies f_3(2) = 5(2-\omega)^2 = 5 \Leftrightarrow 2-\omega = 1 \vee -1 \\ \Leftrightarrow \omega = 1 \vee \omega = 3 \end{cases}$$

$$\Longrightarrow \begin{cases} f_1 = & (x^2-3)^2(x^2+1) \\ f_2 = & \cancel{(x^2-3)(x^2+1)^2} \\ f_3 = & (x^2-3)(x^2+1)(x-1)^2 \\ f_4 = & (x^2-3)(x^2+1)(x-3)^2 \quad \checkmark \end{cases}$$

Factorización

- $f_1 = (x^2 - 3)^2(x^2 + 1) \in \mathbb{Q}[x]$
  $= (x - \sqrt{3})^2(x + \sqrt{3})^2(x^2 + 1) \in \mathbb{R}[x]$
  $= (x - \sqrt{3})^2(x + \sqrt{3})^2(x + i)(x - i) \in \mathbb{C}[x]$

- $f_3 = (x^2 - 3)(x^2 + 1)(x - 1)^2 \in \mathbb{Q}[x]$
  $= (x - \sqrt{3})(x + \sqrt{3})(x^2 + 1)(x - 1)^2 \in \mathbb{R}[x]$
  $= (x - \sqrt{3})(x + \sqrt{3})(x + i)(x - i)(x - 1)^2 \in \mathbb{C}[x]$

- $f_4 = (x^2 - 3)(x^2 + 1)(x - 3)^2 \in \mathbb{Q}[x]$
  $= (x - \sqrt{3})(x + \sqrt{3})(x^2 + 1)(x - 3)^2 \in \mathbb{R}[x]$
  $= (x - \sqrt{3})(x + \sqrt{3})(x + i)(x - i)(x - 3)^2 \in \mathbb{C}[x]$ ✓

Ej. 4.

Hallar todas las raíces de $f = x^6 - 3x^5 - 2x^4 + x^3 - 9x^2 + 4x - 6$ sabiendo que tiene raíces en común con $g = x^4 + x^3 + x + 1$ y con $h = x^3 + 2x^2 + x + 2$.

Ej. 14: α es raíz común de f y $g \Leftrightarrow \alpha$ es raíz de $(f : g)$

Calculemos $d = mcd(f : g)$

$$d = (f : g) = (f - x^2 g : g) = (-4x^5 - 2x^4 - 10x^2 + 4x - 6 : g)$$

$$= (\widetilde{f} + 4xg : g) = (2x^4 - 6x^2 + 8x - 6 : g) = (\widetilde{\widetilde{f}} - 2g : g)$$
$$= (-2x^3 - 6x^2 + 6x - 8 : g) = (x^3 + 3x^2 - 3x + 4 : g)$$
$$= (x^3 + 3x^2 - 3x + 4 : g) = (r : g - xr) = (r : -2x^3 + 3x^2 - 3x + 1)$$
$$(r : \widetilde{g} + 2r) = (r : 9(x^2 - x + 1)) = (r - x(x^2 - x + 1) : x^2 - x + 1)$$
$$= (4(x^2 - x + 1) : x^2 - x + 1) = x^2 - x + 1$$

Observación: sabiendo que f tiene una raíz en común con $g = x^4 + x^3 + x + 1$:

$$Gauss : \alpha = \frac{m}{n} \in \mathbb{Q} \text{ es raíz de } g \in \mathbb{Q}[x] \Rightarrow \alpha = \pm 1$$

Evaluemos $g(-1) = 0 \Longrightarrow (x+1)|g$

$$\Rightarrow g = (x+1)(x^3+1) = (x+1)^2(x^2 - x + 1)$$

Además, sus otras 2 raíces son

$$x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + 1 = 0 \Longrightarrow x = \frac{1 \pm \sqrt{3}i}{2} \in \mathbb{C}$$

Pero $f(-1) \neq 0$ y $f \in \mathbb{R}[x] \Longrightarrow x = \frac{1 \pm \sqrt{3}i}{2} \in \mathbb{C}$ son raíces de f

$$\Rightarrow d = mcd(f:g) = (f : x^2 - x + 1) = x^2 - x + 1$$

$$\Longrightarrow f = (x^2 - x + 1)(x^4 - 2x^3 - 5x^2 - 2x - 6)$$

$$f = (x^2 - x + 1)(x^4 - 2x^3 - 5x^2 - 2x - 6)$$

Estudiemos $h = x^3 + 2x^2 + x + 2 \Rightarrow h(-2) = 0$

$$\Rightarrow h = (x+2)(x^2+1) \quad \textit{pero } f(-2) \neq 0$$

$$x^2 + 1 = (x - i)(x + i)$$

$\Rightarrow \alpha_{1,2} = \pm i$ son raíces en común con f *pues* $f \in \mathbb{R}[x] \Rightarrow (x^2+1)|f$

Pero $\alpha_{1,2} = \pm i$ no son raíces de $x^2 - x + 1$

$$\Rightarrow (x^2 - x + 1 : x^2 + 1) = 1$$

$$\Rightarrow (x^2+1) \Big| (x^4 - 2x^3 - 5x^2 - 2x - 6)$$

$$\Rightarrow (x^4 - 2x^3 - 5x^2 - 2x - 6) = (x^2+1)(x^2 - 2x - 6)$$

$$\Rightarrow f = (x^2 - x + 1)(x^2 + 1)(x^2 - 2x - 6) \in \mathbb{Q}[x]$$

Raíces:

$$x^2 - 2x - 6 = (x+1)^2 - 7 \Rightarrow \beta_{1,2} = 1 \pm \sqrt{7} \text{ son raíces de } f$$

$$\Rightarrow f = (x^2 - x + 1)(x^4 - 2x^3 - 5x^2 - 2x - 6)$$
$$= (x^2 - x + 1)(x^2 + 1)(x^2 - 2x - 6) \in \mathbb{Q}[x]$$

y sus raíces son

$$\left\{ x = \frac{1 \pm \sqrt{3}i}{2},\ \alpha_{1,2} = \pm i,\ \beta_{1,2} = 1 \pm \sqrt{7} \right\} \checkmark$$

Factorización en $\mathbb{Q}[x]$

$$f = (x^2 - x + 1)(x^2 + 1)(x^2 - 2x - 6) \in \mathbb{Q}[x]$$

$$= (x^2 - x + 1)(x^2 + 1)(x - (1 + \sqrt{7}))(x - (1 - \sqrt{7})) \in \mathbb{R}[x]$$

Factorización en $\mathbb{C}[x]$

Además,

$$(x^2 - x + 1) = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} = 0 \Rightarrow \left(x - \frac{1}{2}\right)^2 = -\frac{3}{4}$$

$$\Rightarrow \omega_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \text{ son raíces de } f$$

$$\Longrightarrow f = \left(x - \left(\frac{1}{2} + \tfrac{\sqrt{3}}{2} i\right)\right)\left(x - \left(\frac{1}{2} - \tfrac{\sqrt{3}}{2} i\right)\right)$$

$$(x - i)(x + i)(x - (1 + \sqrt{7}))(x - (1 - \sqrt{7})) \in \mathbb{C}[x] \checkmark$$

Ej. 5.

Hallar todos los polinomios mónicos de grado 3 tales que

- el producto de sus raíces es 2
- la suma de las raíces de f' es $-\dfrac{2}{3}$
- $f(-1) = 1$

b) Factorizar f en $\mathbb{Q}[x], \mathbb{R}[x], \mathbb{C}[x]$,