Weak convergence implies strong convergence in $\ell^1(\mathbf{N})$

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Handout for 245B, Winter 2013

Perhaps my favorite homework problem from 245B was to establish the following:

Proposition 0.1. *If $f_n \rightharpoonup f$ in $\ell^1(\mathbf{N})$, then $f_n \to f$ strongly in $\ell^1(\mathbf{N})$.*

Proof. By subtracting f from f_n we may assume WLOG that $f_n \rightharpoonup 0$. For ease of notation we write

$$\langle f, g\rangle := \sum_{m\in\mathbf{N}} f(m)g(m)$$

for $f \in \ell^1(\mathbf{N})$ and $g \in \ell^\infty(\mathbf{N})$. By this we do not mean an ℓ^2 inner product (though on the intersection of ℓ^1 and ℓ^∞ it will agree with that inner product, except for a complex conjugate somewhere). Our assumption is that $\langle f_n, g\rangle \to 0$ for any $g \in \ell^\infty(\mathbf{N})$.

Taking g to be the kth standard basis vector δ_k we see in particular that

$$f_n(k) = \langle f_n, \delta_k\rangle \to 0 \tag{1}$$

for each $k \in \mathbf{N}$.

We prove the contrapositive. Assume $||f_n||_1 \not\to 0$. Then we have $\epsilon > 0$ and a subsequence f_{n_k} such that $||f_{n_k}||_1 > \epsilon$ for all $k \in \mathbf{N}$. We will use this bad subsequence to make a bad $g \in \ell^\infty(\mathbf{N})$.

Since $f_{n_1} \in \ell^1(\mathbf{N})$, there exists $M_1 > 0$ such that

$$\sum_{m\geq M_1} |f_{n_1}(m)| < \epsilon/100.$$

Note this means that $\sum_{m\leq M_1} |f_{n_1}(m)| > .99\epsilon$. Set $n_{k_1} = n_1$, the first element of a sub-subsequence n_{k_j}.

With this M_1 fixed, it follows from (1) that there is $n_{k_2} > n_{k_1}$ such that

$$\sum_{m<M_1} |f_{n_{k_2}}(m)| < \epsilon/100$$

(since M_1 is finite we can take n_{k_2} large enough that each of the $f_{n_{k_2}}(m)$ for $0 \leq m < M_1$ is sufficiently small). Now again since $f_{n_{k_2}} \in \ell^1(\mathbf{N})$, there exists $M_2 > M_1$ such that

$$\sum_{m \geq M_2} |f_{n_{k_2}}(m)| < \epsilon/100.$$

It follows that

$$\sum_{M_1 \leq m < M_2} |f_{n_{k_2}}(m)| > .98\epsilon.$$

We continue inductively, constructing a subsequence $f_{n_{k_j}}$ and a sequence $M_j \in \mathbf{R}_+$ such that for each $j \geq 2$,

$$\sum_{M_{j-1} \leq m < M_j} |f_{n_{k_j}}(m)| > .98\epsilon$$

(each time using the pointwise convergence of $f_{n_{k_{j-1}}}$ to choose n_{k_j} large enough that $f_{n_{k_j}}$ has at most $\epsilon/100$ of mass near 0, and using that $f_{n_{k_j}} \in \ell^1(\mathbf{N})$ to choose M_j sufficiently large that the tail has mass at most $\epsilon/100$).

We can use this subsequence with packets of mass in the ranges $\{M_j, \ldots, M_{j+1} - 1\}$ to construct a bad sequence $g \in \ell^\infty(\mathbf{N})$. Define

$$g(m) = \frac{\overline{f_{n_{k_j}}(m)}}{|f_{n_{k_j}}(m)|}$$

for $M_j \leq m < M_{j+1}$ for each $j \geq 1$ (and set it to zero on the remaining coordinates $m < M_1$). Then we have $||g||_\infty = 1$, and for each $j \geq 2$,

$$\begin{aligned}
|\langle f_{n_{k_j}} g\rangle| &\geq |\sum_{M_{j-1} \leq m < M_j} f_{n_{k_j}}(m) g(m)| - |\sum_{m < M_{j-1}} f_{n_{k_j}}(m) g(m)| - |\sum_{m \geq M_j} f_{n_{k_j}}(m) g(m)| \\
&\geq (\sum_{M_{j-1} \leq m < M_j} |f_{n_{k_j}}|) - ||g||_\infty (\sum_{m \notin \{M_{j-1}, \ldots, M_j - 1\}} |f_{n_{k_j}}(m)|) \\
&\geq .98\epsilon - .01\epsilon - .01\epsilon = .96\epsilon.
\end{aligned}$$

Hence f_n does not converge weakly to f, which concludes the proof by contrapositive. □

Remark 0.2. Jim Ralston told me this argument is a variant of the "traveling hump" method. We were able to use weak convergence and a lower bound on the ℓ^1 mass of the elements of the subsequence to track a traveling packet with mass at least $.98\epsilon$ on its journey out to infinity (viewing m as a spatial coordinate and n as a time coordinate).