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Estimaciones de dimensión para conjuntos de tipo Furstenberg y Teoremas de Restricción para medidas de Hausdorff

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

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Buenos Aires, diciembre de 2010

Estimaciones de dimensión para conjuntos de tipo Furstenberg y Teoremas de Restricción para medidas de Hausdorff

En esta tesis se estudian dos problemas del Análisis Armónico clásico desde el punto de vista de las medidas de Hausdorff. El primero es el problema de Furstenberg, que en su versión clásica se refiere a la determinación de la dimensión de Hausdorff ($\dim_H$) de los conjuntos de la clase F_α: dado $\alpha \in (0,1]$, un conjunto $E \subseteq \mathbb{R}^2$ está en la clase F_α si para cada $e \in \mathbb{S}$ existe un segmento unitario ℓ_e en la dirección de e tal que $\dim_H(\ell \cap E) \geq \alpha$. En el caso $\alpha = 1$, este problema resulta equivalente al problema de Kakeya. Si notamos $\gamma(\alpha) = \inf \{\dim_H(E) : E \in F_\alpha\}$, entonces vale que

$$\max\{1/2 + \alpha; 2\alpha\} \leq \gamma(\alpha) \leq (1 + 3\alpha)/2. \tag{1}$$

En este trabajo se estudia este problema desde una perspectiva más general, en términos de las medidas de Hausdorff h-dimensionales $\mathcal{H}^h$ asociadas a funciones de dimensión. Definimos los conjuntos de la clase de Furstenberg F_h asociados a una función h. La hipótesis natural para cada dirección es que $\mathcal{H}^h(\ell_e \cap E) > 0$. Generalizamos los resultados conocidos en términos de "saltos logarítmicos" y obtenemos resultados análogos a las cotas clásicas que permiten, además, extender la desigualdad (1) al caso extremo $\alpha = 0$. Precisamente, se prueba que la función de dimensión apropiada para los conjuntos de la clase F_h no puede ser mucho más chica que h^2 o que $\sqrt{\cdot}h$. Para las cotas superiores exhibimos explícitamente conjuntos en la clase F_h suficientemente chicos. Usamos para eso algunos resultados sobre Aproximación Diofántica, acerca de la dimensión de conjuntos de números "bien aproximables".

Obtenemos resultados acerca de la dimensión de conjuntos en la clase $F_{\alpha\beta}$, definida como F_α pero sólo para un conjunto $L \subset \mathbb{S}$ tal que $\dim_H(L) \geq \beta$. Probamos una versión de (1) que refleja la interacción entre los parámetros α y β. Este problema fue estudiado también en el conexto general.

En segundo lugar se estudió con el mismo enfoque el problema de la Restricción de la Trasformada de Fourier, que se refiere a la posibilidad de darle sentido a la restricción de $\widehat{f}$ a un subconjunto E de $\mathbb{R}^n$. La respuesta depende de la existencia de una medida μ en E con ciertas propiedades de dimensionalidad y de decaimiento para su transformada $\widehat{\mu}$. En este contexto se reformuló el teorema de restricción de Stein-Tomas en términos de medidas de Hausdorff.

Palabras clave: Conjuntos de Furstenberg, medidas de Hausdorff, funciones de dimensión, dimensión de Hausdorff, aproximación diofántica, restricción de la transformada de Fourier.

Dimension estimates for Furstenberg type sets and Restriction Theorems for Hausdorff measures

In this thesis we study two problems in classical Harmonic Analysis. The first is the Furstenberg problem, which in its classical form concerns the determination of the Hausdorff dimension ($\dim_H$) of the sets in the F_α-class: for a given $\alpha \in (0, 1]$, a set $E \subseteq \mathbb{R}^2$ is in the F_α-class if for each $e \in \mathbb{S}$ there exists a unit line segment ℓ_e in the direction of e such that $\dim_H(\ell \cap E) \geq \alpha$. For $\alpha = 1$, this problem is essentially equivalent to the "Kakeya needle problem". If we define $\gamma(\alpha) = \inf\{\dim_H(E) : E \in F_\alpha\}$, then

$$\max\{1/2 + \alpha; 2\alpha\} \leq \gamma(\alpha) \leq (1 + 3\alpha)/2. \tag{1}$$

In this work we approach this problem from a more general point of view, in terms of h-dimensional Hausdorff measures $\mathcal{H}^h$ associated to dimension functions. We define the class F_h of Furstenberg sets associated to a given dimension function h. The natural requirement for a set E to belong to F_h, is that $\mathcal{H}^h(\ell_e \cap E) > 0$ for each direction. We generalize the known results in terms of 'logarithmic gaps" and obtain analogues to the estimates given in (1). Moreover, these analogues allow us to extend our results to the endpoint $\alpha = 0$. Precisely, we prove that the correct dimension function for the class F_h can not be much smaller than h^2 or $\sqrt{\cdot}h$. For the upper bounds we exhibit an explicit construction of F_h-sets which are small enough. To that end we prove some results from Diophantine Approximation about the the dimension of the set of "well approximable numbers".

We obtain results about the dimension of Furstenberg sets in the class $F_{\alpha\beta}$, defined analogously to the class F_α but only for a fractal set $L \subset \mathbb{S}$ such that $\dim_H(L) \geq \beta$. We prove an inequality like (1) which reflects the interplay between α and β. This problem is also studied in the general scenario of Hausdorff measures.

The second problem studied in this work, in the same general scenario as before, is the Restriction Problem for the Fourier transform. Here the problem is to give a meaningful sense to the restriction of $\widehat{f}$ to a subset E of $\mathbb{R}^n$. The answer depends on the existence of a measure μ supported on E with precise conditions on the dimensionality of μ and decay properties of $\widehat{\mu}$. In this context we reformulate the Stein-Tomas restriction theorem for Hausdorff measures.

Agradecimientos

A Ursula, por su guía y su infatigable optimismo, que más de una vez me contagió la confianza necesaria para salir adelante.

A todo el grupo de Análisis Armónico y Geometría Fractal, por la oportunidad de participar y crecer junto con ellos.

A mis viejos, a Rami y a Lore.

A los amigos, de todas las etapas de esta larga carrera:
Los Perrucci, los Colo, los Muro, Vendrix, Morét, los DelPe, Magui, Vicky, los López, Santiago, Sandra, Chafaflip, Damián, Zúperman, Torreta, Santiago, Andrei, Ezequiel, Nino.

A los del Voleiból: Aro, Ari, Nico, Yani, Caro, Mati.

A Ele, que me iluminó la vida cuando la conocí, que me dio a Manu, el regalo más maravilloso que me pudo haber dado y que hace que juntos todo sea más fácil.

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Introduction

In many situations in geometric measure theory, one wants to determine the size of a given set or a given class of sets identified by some geometric properties. Throughout this thesis, size will mean Hausdorff dimension, denoted by $\dim_H$. The main purpose of the present work is the study of dimension estimates for *Furstenberg sets*. For a given α in $[0, 1]$, a bounded subset E of $\mathbb{R}^2$ is called *Furstenberg set* of type α or an F_α-set if for each direction e in the unit circle there is a line segment ℓ_e in the direction of e such that the Hausdorff dimension of the set $E \cap \ell_e$ is equal or greater than α. Since the defining property of these sets is stated in terms of containing sufficient linear subsets, the relevant question is about the minimal possible size for a generic member of the class. The schematic representations to keep in mind are the ones in Figure 1 and Figure 2.
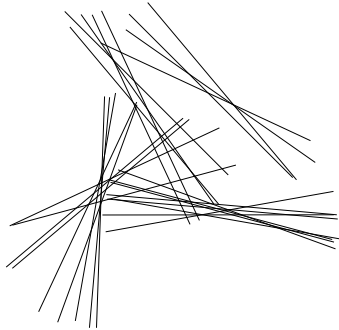
Figure 1: A Furstenberg set with overlap, that allows the set to be small.
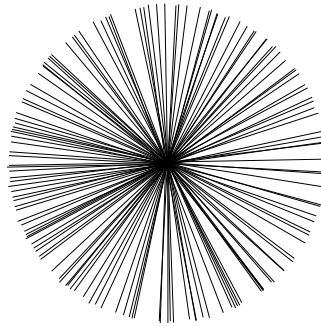
Figure 2: A "thick" Furstenberg set. The lack of overlapping makes the set bigger. This example is essentially the unit ball.

For $\alpha \in [0, 1]$ we call $\gamma(\alpha) = \inf\{\dim_H(E) : E$ is an F_α-set$\}$, then the Furstenberg problem is to determine $\gamma(\alpha)$. The best known bounds on $\gamma(\alpha)$ so far are pictured in Figure 3, which shows that for $\alpha \in (0, 1]$, *any* F_α-set E must have Hausdorff dimension not smaller than $\max\{2\alpha, \frac{1}{2} + \alpha\}$ and there exists at least one F_α-set F of Hausdorff dimension less or equal than $\frac{1}{2} + \frac{3}{2}\alpha$. Our purpose in this work is to approach this problem in the wider scenario of general Hausdorff measures (see below) to sharpen these known bounds and,

in addition, to be able to include the case $\alpha = 0$.

The Furstenberg problem appears for the first time in the work of Harry Furstenberg in [Fur70], regarding the problem of estimating the size of the intersection of fractal sets. Unavoidable references on this matter are [Wol99b], [Wol99a], [Wol02], [Wol03], [KT02]. See also [Tao01] for related topics and [KT01] for a discretized version of this problem. In this last article, the authors study some connections between the Furstenberg problem and two other very well known and unsolved problems: the Falconer distance problem and the Erdös ring problem (we will come back to these problems later, in Chapter 3).
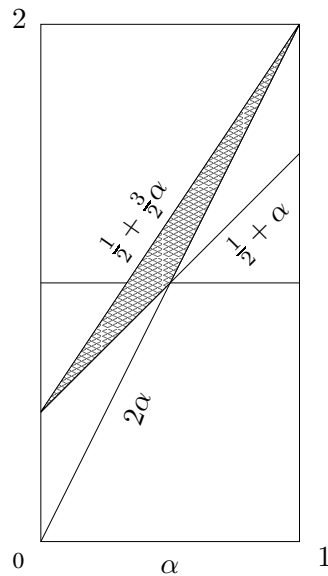
Figure 3: Possible values of $\gamma(\alpha)$.

For the particular case of $\alpha = 1$, when we require the set to contain a whole line segment in each direction, we actually are in the presence of the much more famous Kakeya problem. A Kakeya set (or Besicovitch set) is a compact set $E \subseteq \mathbb{R}^n$ that contains a unit line segment in every possible direction. The question here is about the minimal size for the class of Kakeya sets. Besicovitch [Bes19] proved that there exist Besicovitch sets of Lebesgue measure zero.

Originally, Kakeya [FK17] asks which is the possible minimal area that permits to *continuously* turn around a unit line segment in the plane and in [Bes28] Besicovitch actually shows that the continuous movement can be achieved using an arbitrary small area by the method known as *shifting triangles* or *Perron's trees*.

The next question, which is relevant for this thesis, is the *unsolved* "Kakeya conjecture" which asserts that these sets, although they can be small with respect to the Lebesgue measure, must have full Hausdorff dimension. This last problem is only solved in $\mathbb{R}^2$: all Kakeya sets in $\mathbb{R}^2$ have dimension 2. In

higher dimensions the Kakeya problem is still open, and one of the best known bounds for $\mathbb{R}^3$ appears in [Wol99b] and states that any Kakeya set $E \subseteq \mathbb{R}^3$ must satisfy the bound $\dim_H(E) \geq \frac{5}{2}$.
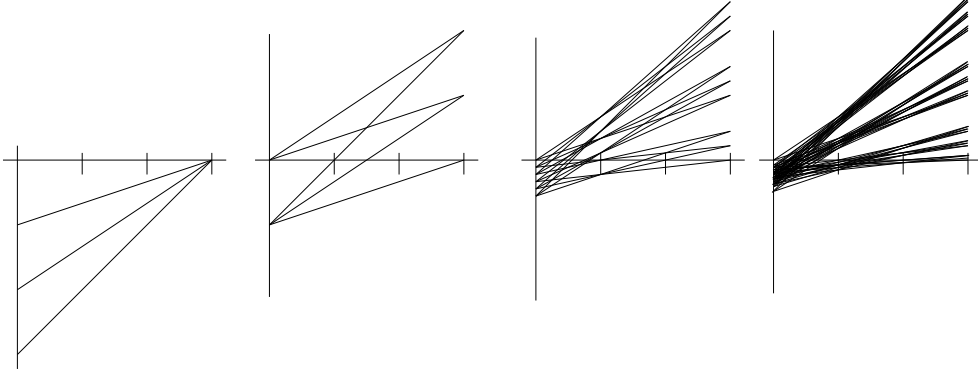
Figure 4: Schematic iterative procedure to construct a Kakeya set with zero area by *shifting triangles*. The key is to locate the shifted triangles to achieve an extreme overlapping.

These kind of geometric-combinatorial problems have deep implications in many different areas of general mathematics. Some of the connections to other subjects include Bochner-Riesz multipliers, restrictions estimates for the Fourier transform and also partial differential equations. For example, it has been shown that a positive answer to the Restriction Conjecture for the sphere $\mathbb{S}^{n-1}$ would imply that any Kakeya set in $\mathbb{R}^n$ must have full dimension, and therefore solve the Kakeya conjecture.

Our approach in this work, is to attack the Furstenberg problem using generalized Hausdorff measures. This approach is motivated by the well known fact that knowing the value of the dimension of a given set is not telling us yet anything about the corresponding measure at this critical dimension. In fact, if $\mathcal{H}^s$ is the Hausdorff s measure of an s-dimensional set E, $\mathcal{H}^s(E)$ can be 0, ∞ or finite. The case of a set E with $0 < \mathcal{H}^s(E) < +\infty$ is of special interest. We refer to it as an s-set, considering it as *truly* s-dimensional. For, if a set E with $\dim_H(E) = s$ has non σ-finite $\mathcal{H}^s$-measure, it is still too big to be correctly measured by $\mathcal{H}^s$. Analogously, the case of null measure reflects that the set is too thin to be measured by $\mathcal{H}^s$. To solve (partially) this problem, the appropriate tools are the "generalized Hausdorff Measures" introduced by Felix Hausdorff in his seminal paper [Hau18] in 1918. For any *dimension function*, i.e. a function belonging to the set

$$\mathbb{H} := \{h : [0, \infty) \to [0 : \infty), \text{non-decreasing, continuous, } h(0) = 0\}.$$

he defines

$$\mathcal{H}_\delta^h(E) = \inf \left\{ \sum_i h(\text{diam}(E_i)) : E \subset \bigcup_i^\infty E_i, \text{diam}(E_i) < \delta \right\}$$

and

$$\mathcal{H}^h(E) = \sup_{\delta > 0} \mathcal{H}^h_\delta(E).$$

Note that if $h_\alpha(x) := x^\alpha$, we actually recover the previous measure since $\mathcal{H}^{h_\alpha} = \mathcal{H}^\alpha$. We now have a finer criteria to classify sets by a notion of size. Precisely, consistently with the natural order of power functions, we introduce a notion of partial order in $\mathbb{H}$. We will say that g is dimensionally smaller than h and write $g \prec h$ if and only if

$$\lim_{x \to 0^+} \frac{h(x)}{g(x)} = 0. \tag{1}$$

This partial order allows us to distinguish between sets with the same Hausdorff dimension. For example:

$$x^{\alpha_1} \prec x^{\alpha_1} \log^{-\gamma}\left(\frac{1}{x}\right) \prec x^{\alpha_2} \log^{\beta}\left(\frac{1}{x}\right) \prec x^{\alpha_2}, \quad \beta, \gamma > 0, \quad \alpha_1 < \alpha_2.$$

Given a set E of Hausdorff dimension s that is not an s-set, one could try to find an adequate dimension function h having the property $0 < \mathcal{H}^h(E) < \infty$. In that case we will say that E is an h-set, which reflects that the proper notion of size may live outside the class of power laws. However, it can be proved that there are sets that are not h-sets for any $h \in \mathbb{H}$. One example of such a set is the set of Liouville numbers $\mathbb{L}$ presented later. On the other side, in [CMMS04] the authors consider Cantor type subsets of the real line defined in terms of a monotonic non-increasing positive sequence $\{a_n\}$ such that $\sum a_n = 1$ and provide an explicit construction of dimension functions h_a that make those sets h_a-sets, i.e., $0 < \mathcal{H}^{h_a}(C_a) < +\infty$.

   The contribution of this thesis is to provide sharp bounds on the size of generalized Furstenberg sets. A set E belongs to the Furstenberg-type class $F_\mathfrak{h}$ associated to a given $\mathfrak{h} \in \mathbb{H}$ if $\mathcal{H}^\mathfrak{h}(\ell_e \cap E) > 0$ for each $e \in \mathbb{S}$. Note that this definition is (as discussed before) somehow stronger that requesting it only to be α-dimensional, for the case that $\mathfrak{h}(x) = x^\alpha$.

   Let us remark here that there is one profound conceptual obstacle in the general setting. Whereas, to detect the Hausdorff dimension of a set E, it is enough to find a number s such that $\mathcal{H}^r(E) > 0$ for all $r < s$ and such that $\mathcal{H}^t(E) = 0$ for all $t > s$, in the general setting, if a given set E has null $\mathcal{H}^h$-measure for some $h \in \mathbb{H}$, then there is another dimension function g, $g \prec h$, such that E has also null $\mathcal{H}^g$-measure (Rogers [Rog70] Theorem 42). Hence there is an unavoidable need to study a notion of "gap" between dimension functions.

   We show that if E is a set in the class $F_\mathfrak{h}$, and $h(x)$ is a dimension function that is *much smaller* than $\mathfrak{h}^2(x)$ or $\sqrt{x}\,\mathfrak{h}(x)$, then $\mathcal{H}^h(E) = \infty$ (Theorem 3.4.1 and Theorem 3.5.4 respectively). We further exhibit a very *small* Furstenberg set F in $F_\mathfrak{h}$, for some particular choices of $\mathfrak{h}$ and show that for this set, if $\sqrt{x}\,\mathfrak{h}^{3/2}(x)$ is *much smaller* than h, then $\mathcal{H}^h(F) = 0$ (Theorem 6.2.2). This

generalizes the result of the classical setting, which says that the best known bounds on $\gamma(\alpha)$ are

$$\max\{2\alpha; \frac{1}{2} + \alpha\} \leq \gamma(\alpha) \leq \frac{1}{2} + \frac{3}{2}\alpha, \qquad \alpha \in (0, 1]. \tag{2}$$

To make the notion of *much smaller* precise, we use the notion of distance induced by the partial order in $\mathbb{H}$ (1). With this definition, the speed of convergence to zero of the quotient $\Delta(h, g)(x) = \frac{h(x)}{g(x)}$ can be seen as a distance between the dimension functions.

The $\mathfrak{h}(x) \to \mathfrak{h}^2(x)$ bound strongly depends on the known estimates for the Kakeya maximal operator: for an integrable function f on $\mathbb{R}^n$, the Kakeya maximal operator at scale δ applied to f, $\mathcal{K}_\delta(f) : \mathbb{S}^{n-1} \to \mathbb{R}$, is

$$\mathcal{K}_\delta(f)(e) = \sup_{x \in \mathbb{R}^n} \frac{1}{|T_e^\delta(x)|} \int_{T_e^\delta(x)} |f(x)| \, dx \qquad e \in \mathbb{S}^{n-1}, \tag{3}$$

where $T_e^\delta(x)$ is a $1 \times \delta$-tube (by this we mean a tube of length 1 and cross section of radius δ) centered at x in the direction of $e \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$. It is well known that in $\mathbb{R}^2$ the Kakeya maximal function satisfies the bound $\|\mathcal{K}_\delta(f)\|_2^2 \lesssim \log(\frac{1}{\delta})\|f\|_2^2$ (see [Wol99b]).

Our proof relies on an optimal use of this estimates for the Kakeya maximal function, exploiting the logarithmic factor in the above bound, which is necessary (see [Kei99]), because of the existence of Kakeya sets of zero measure.

The other lower bound, which is the relevant bound near the zero dimensional case, depends on some combinatorial arguments that we extended to this general setting. In addition, our techniques allow us to extend the bounds in (2) to "zero dimensional" classes. To manage these zero dimensional situations, we introduce the subclass $\mathbb{H}_0$ of $\mathbb{H}$ of all "zero dimensional dimension functions", i.e.,

$$\mathbb{H}_0 := \{h \in \mathbb{H} : h \prec h_\alpha \text{ for any } \alpha > 0\},$$

and replace the standard dyadic pigeonhole principle with an adapted hyperdyadic sequence of scales.

For the upper bounds the aim is to explicitly exhibit constructions of reasonably small Furstenberg sets. To achieve these optimal constructions, we needed the most general version of Jarník's sets $B_\mathfrak{g}$ defined for an appropriate $\mathfrak{g} \in \mathbb{H}$:

$$B_\mathfrak{g} = \left\{x \in [0, 1] \setminus \mathbb{Q} : \|xq\| < \frac{q}{\mathfrak{g}(q)} \text{ for infinitely many } q \in \mathbb{Z}\right\}. \tag{4}$$

Using dimension estimates for the set $B_\mathfrak{g}$ and appropriate related sets, we exhibit an $F_\mathfrak{h}$-set whose dimension function can not be much larger (in terms of logarithmic gaps) than $\sqrt{x} \, \mathfrak{h}^{3/2}(x)$ for the classical case of $\mathfrak{h}(x) = x^\alpha$.

We remark that with our results we are able to conclude that the value $\frac{1}{2}$ is sharp for the Hausdorff dimension of a family of Furstenberg sets. Note that for any $\alpha \in (0,1]$, there is a zone of uncertainty between the known lower and upper bounds. At the endpoint $\alpha = 0$ however, we can show that actually there is a family of Furstenberg sets associated to a zero dimensional dimension function $\mathfrak{h}$ such that if $\mathfrak{h} \in \mathbb{H}$ is defined by $\mathfrak{h}(x) = \frac{1}{\log(\frac{1}{x})}$ (note that $\mathfrak{h} \in \mathbb{H}_0$), then any $F_\mathfrak{h}$-set E must satisfy $\dim_H(E) \geq \frac{1}{2}$. On the other hand, in Chapter 6, we exhibit a particular set $E \in F_\mathfrak{h}$ for the same $\mathfrak{h}$ satisfying $\dim_H(E) \leq \frac{1}{2}$.

We also consider another related problem, both in the classical and generalized setting. We analyze the role of the dimension of the set of directions in the Furstenberg problem. We consider the class of $F_{\alpha\beta}$ sets, defined in the same way as the F_α class but with the directions taken in a subset L of the unit circle such that $\dim_H(L) \geq \beta$. We are able to prove that if E is any $F_{\alpha\beta}$-set, then

$$\dim_H(E) \geq \max\left\{ 2\alpha + \beta - 1; \frac{\beta}{2} + \alpha \right\}, \qquad \alpha, \beta > 0, \tag{5}$$

which are again the same kind of inequalities than (2). For the proof of one of the lower bounds we needed the estimates for the Kakeya maximal function but for more general measures. The other lower bound uses the δ-entropy of the set L of directions, which is the maximal possible cardinality of a δ-separated subset. Our results are proved in the context of the general Hausdorff measures and we obtain (5) as a corollary.

The only previously known bounds in this setting where for the particular case of $\alpha = 1$, $\beta \in (0,1]$ (see [Mit02]). The author there obtains that if E is an A-Kakeya set (that is, a planar set with a unit line segment in any direction $e \in A$ for a set $A \subseteq \mathbb{S}$), the $\dim_H(E) \geq 1 + \dim_H(A)$ (this is only one of the lower bounds).

Finally, our approach allows to sharpen the classic bounds in terms of "logarithmic gaps". Precisely, to estimate the Hausdorff dimension of a set in the class F_α, we prove that there is no need to take a "power like" step down from $\frac{1}{2} + \alpha$ or 2α. In fact, if for example h is any dimension function satisfying the relation $h(x) \geq Cx^{2\alpha}\log^{1+\theta}(\frac{1}{x})$ for $\theta > 2$ then $\mathcal{H}^h(E) > 0$ for any $E \in F_\alpha$. Analogous results are obtained for the other lower bound and also for the upper bounds.

In addition, we study the problem of estimating the Hausdorff dimension of "finite" Furstenberg sets. We look at the F^K-sets, which are those sets with at least K points lined up in the direction of e for each direction $e \in \mathbb{S}$. The intuition here is that those sets should be really small in the sense of Hausdorff dimension. We have found a non trivial configuration that yields a F^2-set of zero Hausdorff dimension, but it seems that there is no possible extension to $K \geq 3$. Clearly, from our results on the upper bounds we can deduce the existence of F^K-sets of Hausdorff dimension less or equal than $\frac{1}{2}$.

It is remarkable that if we measure the size with the Packing dimension, then any F^K-set must have Packing dimension at least $\frac{1}{2}$ for any $K \geq 2$.

Finally, in this thesis we also look at generalized Hausdorff measures applied to the Restriction Problem for the Fourier Transform. We reformulate the theorem of Stein-Tomas in terms of dimension functions, i.e., for a surface S which supports a measure μ satisfying

$$\mu(B(x,r)) \lesssim h(r) \tag{6}$$

for any ball $B(x,r)$ for some $h \in \mathbb{H}$ with an extra regularity condition and

$$|\widehat{\mu}(\xi)| \lesssim g(|\xi|)$$

for some bounded, positive function g such that $g(|\xi|) \to 0$ at infinity. The exponents of the underlying L^p, L^q spaces will depend on the decay ratio for g at infinity and the dimensionality of the function h (for a precise statement of the results we refer the reader to Chapter 7).

Our result generalizes the one obtained by Mockenhaupt [Moc00] and, as the original version, can be applied to subsets of $\mathbb{R}^n$ that are not necessarily "smooth".

The thesis is organized as follows. In Chapter 1 we include the measure theoretical background that we believe is needed. Perhaps it is not sufficient to make it self contained, but we kept the reference to the literature at the minimum possible. We present the definitions and properties of the most common notions of dimension, namely the Hausdorff, Minkowski, Packing and Fourier dimensions. In Chapter 2 we present the precise definitions and properties about dimension functions and the associated Hausdorff measures. We also present and study the *Dimension Partition* (see [CHM10]) of a given set, which is a way to classify the dimension functions by means of a zero-one-infinity law. We also included some classical constructions of Cantor type sets and technical lemmas rephrased in the language of dimension functions to be used in the sequel. Chapter 3 contains the results about the lower bounds announced here for the generalized Furstenberg sets $F_{\mathfrak{h}}$. In Chapter 4 we study the same bounds but in the case of the Furstenberg problem for a fractal set of directions. Chapter 5 and Chapter 6 are devoted to the upper bounds. First we introduce in Chapter 5 the needed results on diophantine approximation. The aim of this chapter is to collect the results about the Hausdorff measures estimates for generalized Jarník's sets. This will be the significant ingredient in the proofs of the upper bounds in Chapter 6, where we present explicit constructions of small generalized Furstenberg sets and obtain sharp estimates on the Hausdorff dimension of these classes in the spirit of Chapter 3. The key ingredient in the constructions is a variant of the Jarník's sets adapted to our needs. We also present here some constructions regarding the problem of finite Furstenberg sets. Finally, in Chapter 7, we focus on the Stein-Tomas restriction theorem and its subsequent generalization due to Mockenhaupt. We show here that if a measure μ has a Fourier transform

with some decay and satisfies condition (6) for a dimension function h with some extra regularity properties, we can obtain a restriction theorem for the support of μ.

Most of the results presented here have been included in research articles. The estimates of Chapter 3 have been published in [MR10]. The results in Chapter 6 have been presented in [MRa] and the results of Chapter 4 are contained in [MRb].

Chapter 1

Preliminaries

1.1   Measure Theoretical Background

We include in this section, for completeness, all the basic notions on measure theory that we will require on the sequel. References on this matter are [Mat95], [Rog98], [Fal86, Fal97, Fal03]. We start with the usual notion of measure on a set Ω. Although we will restrict ourselves to work on the euclidean space $\mathbb{R}^n$, we choose to introduce this preliminaries definitions and propositions in the abstract setting, since we believe that it highlights the most relevant notions.

Definition 1.1.1. A real valued set function μ defined on the subsets of a set Ω is called a measure if

1. $0 \leq \mu(A) \leq +\infty$ for all $A \subseteq \Omega$.

2. $\mu(\emptyset) = 0$.

3. $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$ (monotonicity).

4. (Subadditivity) For any countable sequence of sets $\{A_n\}_{n \in \mathbb{N}}$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i). \tag{1.1}$$

It is important to remark that a set function with the above properties is often referred to as an **outer** measure. We adopt a different terminology since it will more appropriate in our setting. In the sequel, we will omit the term "outer" and simply refer to any function defined as in Definition 1.1.1 as a measure. There is a class of subsets of Ω which are special to a given measure in terms of being well behaved under disjoint unions. More specifically, we have the following definition.

Definition 1.1.2. Let μ be a measure on Ω. We will say that a set $E \subseteq \Omega$ is μ-*measurable* if for every set $A \subseteq \Omega$,

$$\mu(A) = \mu(A \cap E) + \mu(A \setminus E),$$

or, equivalently

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for all $A \subseteq E$ and $B \subseteq \Omega \setminus E$.

We introduce the following definition to study the structure of the family of measurable sets.

Definition 1.1.3. Let $\mathcal{A}$ be a class of subsets of a set Ω. We will say that $\mathcal{A}$ is a σ-algebra if it is closed under countable union and complementation, i.e.,

1. $\emptyset \in \mathcal{A}$.

2. If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

3. If $A_1, A_2, \ldots$ are in $\mathcal{A}$, then $\bigcup_i A_i \in \mathcal{A}$.

It is fairly easy to see that the family M_μ of μ-measurable sets is a σ-algebra, and also that M_μ contains all the sets of μ-measure zero. Moreover, the most relevant property is that μ is countably additive on M_μ. Given $A_1, A_2, \ldots$ *disjoint* members of M_μ, then

$$\mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Obviously, if the underlying space is endowed with some structure, the notion of σ-algebra of measurable sets can be related to it. For example, if the measure is defined on a topological space, what can be said about the measurability of a set and its topological properties? Is every open set measurable? Can a given set be approximated in measure in some sense by set of a given topological structure (as G_δ or F_σ sets)? Wan can be said if, in addition, the topology arises from a metric in a metric space (X, d)? We want to briefly discuss these questions and introduce some notions to be used throughout the thesis.

Definition 1.1.4. Let (X, τ) be a topological space. The *Borel* σ-algebra $\mathcal{B} = \mathcal{B}(X)$ is the minimal σ-algebra containing the open sets. The elements of the σ-algebra are called *Borel* sets. A *Borel measure* μ on X is a measure on X where all the Borel sets are measurable ($\mathcal{B}(X) \subseteq M_\mu$).

One important notion to study about a measure is the regularity respect to a given family of sets.

Definition 1.1.5. Given a measure μ on Ω, and $\mathcal{C}$ a class of subsets of Ω, we will say that μ is $\mathcal{C}$-regular if for every set $A \subseteq \Omega$, there exists $D \in \mathcal{C}$, such that $A \subseteq D$ and $\mu(A) = \mu(D)$. When the class $\mathcal{C}$ is the σ-algebra of μ-measurable sets, we will simply say that μ is regular. The term Borel regular will be used when $\mathcal{C}$ is the Borel σ-algebra.

We now present two general methods to construct a measure on a set Ω, the latter in the presence of a metric defined on the underlying space.

1.1.1 Construction of a Measure

The intuition tells that in order to define a proper notion of size of an object, the reasonable procedure would be to compare it to a prefixed family of elementary known objects. Then, by doing this several times, one can have an idea of how many of elementary objects are needed. This is the intuition that leads to the methods presented in this section to construct a measure on a set Ω. We will follow the presentation made in [Rog98].

Definition 1.1.6. Let $\mathcal{C}$ be a collection of subsets of Ω, and τ a set function defined on $\mathcal{C}$ such that

1. $\emptyset \in \mathcal{C}$.

2. $0 \leq \tau(A) \leq +\infty$ for all $A \in \mathcal{C}$.

3. $\tau(\emptyset) = 0$.

Then τ is called a *pre-measure* with domain $\mathcal{C}$.

Method I

The following method to construct a measure from a pre-measure is known as *Method I*.

Definition 1.1.7. Let $\mathcal{C}$ be a collection of subsets of Ω. A covering of a set A from $\mathcal{C}$ is a countable family $\{A_i\}$ of elements of $\mathcal{C}$ such that $A \subset \bigcup_i A_i$.

Given a premeasure τ, we define a set function μ on each set $E \subseteq \Omega$ by

$$\mu(E) = \inf \sum_i \tau(A_i), \tag{1.2}$$

where the infimum is taken over all the coverings of E by elements of $\mathcal{C}$. We refer to the quantity (1.2) as the *size* of the covering. We immediately have the following proposition.

Proposition 1.1.8. *Every set function μ defined on Ω using Method I is a measure.*

The most immediate example of such a measure is the Lebesgue measure on $\mathbb{R}^n$. Here an elementary block would be a "coordinate parallelepiped". The pre-measure is defined by the usual notion of volume. Given a parallelepiped $A = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}$, his volume is

$$\text{vol}(A) = \prod_{i=1}^{n} (b_i - a_i).$$

Therefore, Method I applied to the family of parallelepipeds and taking the volume as the pre-measure gives the usual Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$:

$$\mathcal{L}^n(E) \inf \left\{ \sum_i \text{vol}(A_i) : E \subseteq \bigcup_i A_i \right\}.$$

It is important to remark that to apply this method there is almost no structure required. Therefore it can be performed in the abstract setting, only requiring a set Ω, a class $\mathcal{C}$ and a pre-measure τ.

Now, in a metric space (such as $\mathbb{R}^n$), the notion of diameter of a set allows us to control the scale at which one is covering a given set. That is, it is possible to distinguish a covering made by large sets from a covering made by small sets. Precisely, let (X, d) be a metric space. Given a bounded set $A \subseteq X$, let us denote the diameter of A with

$$\text{diam}(A) = \sup \left\{ |x - y| : x, y \in A \right\}.$$

We set $\text{diam}(A) = +\infty$ when A is unbounded and $\text{diam}(\emptyset) = 0$.

Definition 1.1.9. Given a class $\mathcal{C}$ of sets, we will say that a countable family $\{A_i\}$ is a δ-covering of a set $E \subseteq X$ from the class $\mathcal{C}$ if

1. $A_i \in \mathcal{C}$ for all $i \in \mathbb{N}$.

2. $E \subset \bigcup_{i \in \mathbb{N}} A_i$.

3. $\text{diam}(A_i) < \delta$ for all $i \in \mathbb{N}$.

Method II

Now we can introduce *Method II* to construct a measure (in a metric space). This method is also known as "Carathéodory construction". If τ is a pre-measure on X defined on a class $\mathcal{C}$, we define, as before, the *size* of the δ-covering $\{A_i\}$ with respect to τ to be $\sum_{i \in \mathbb{N}} \tau(A_i)$. If $\tau(A_i) = +\infty$ for some $i \in \mathbb{N}$ or if the series is not convergent, we say that the size is $+\infty$. So,

$$0 \leq \sum_{i \in \mathbb{N}} \tau(A_i) \leq +\infty.$$

Now for $\delta > 0$, let us consider

$$\mu_\delta(E) = \inf\left\{\sum_{i\in\mathbb{N}} \tau(A_i) : A_i \text{ is a } \delta\text{-covering of } E \text{ from } \mathcal{C}\right\}$$

(we adopt again the convention that $\inf\{\emptyset\} = +\infty$), and

$$\mu(E) = \sup_{\delta > 0} \mu_\delta(E). \tag{1.3}$$

Note that $0 \le \mu_\delta(E) \le +\infty$ and if $\delta_2 \le \delta_1$ then $\mu_{\delta_1}(E) \le \mu_{\delta_2}(E)$. Then expression (1.3) is equivalent to

$$\mu(E) = \lim_{\delta \to 0^+} \mu_\delta(E).$$

We have the following proposition. For the proof, we refer to the bibliography (see, for example, [Rog98]).

Proposition 1.1.10. *Given $\delta > 0$, μ_δ is a measure on X. In addition, μ is also a measure on X.*

An important property for a measure on a metric space should be that it behaves properly on separated sets. We make this more precise. Two sets A, B in a metric space (X, d) are said to be separated if $\inf\{d(x,y) : x \in A; y \in B\} > 0$.

Definition 1.1.11. A measure μ is called a *metric* measure if, for any pair of separated sets A, B, we have

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

The next lemma is about an important property of metric measures, which implies that Borel sets are measurable.

Lemma 1.1.12. *Let μ be a measure constructed by Method II. Then μ is a **metric** measure, i.e. is additive on separated sets. If E and F are sets with*

$$0 < \eta := \inf\{|x - y| : x \in E; y \in F\},$$

then

$$\mu(E \cup F) = \mu(E) + \mu(F).$$

Note on proof. The lemma follows from the fact that if E and F are as in the hypothesis, then for any $0 < \delta < \eta$, none of the sets of a δ-covering of $E \cup F$ can intersect both E and F at the same time.

From Lemma 1.1.12 we conclude that the σ-algebra of μ-measurable sets contains the Borel σ-algebra.

The following theorem has as a corollary the fact that every Borel set is μ-measurable for any measure μ constructed with the Method II. For the proof see, for example, [Rog98].

Theorem 1.1.13. *Let (X, d) be a metric space. Then every metric measure μ is a Borel measure.*

Corollary 1.1.14. *Let (X, d) be a metric space. Then every measure μ constructed using Method II is a Borel measure.*

Restriction of a Measure

In the previous sections, we introduced the notion of a measure μ on the whole space X or, in the metric case, (X, d). But it is often necessary to restrict the attention to a particular subset $A \subset X$. Specifically, what we want is to only measure the portion of a generic set E that is contained in A. We define the *restriction of* μ to A as

$$\mu|_A(E) := \mu(A \cap E).$$

It is clear that, with this definition, $\mu|_A$ is a measure. Moreover, every μ measurable set is also $\mu|_A$ measurable and if μ is Borel regular and A is μ measurable with $\mu(A) < \infty$, then $\mu|_A$ is Borel regular. It is also absolutely trivial to note that $\mu|_A(X \setminus A) = 0$. We will say in this case that the measure $\mu|_A$ is supported on A.

We introduce the precise notion of *support of a measure.*

Support of a Measure

Definition 1.1.15. If μ is a Borel measure on a separable metric space X, the support of μ, $\mathrm{supp}(\mu)$, is the smallest closed set F such that $\mu(X \setminus F) = 0$. i.e.,

$$\mathrm{supp}(\mu) := X \setminus \bigcup \{V : V \text{ open}, \mu(V) = 0\}.$$

An easy example: for a non-negative continuous function f, define the measure μ_f by

$$\mu_f(A) = \int_A f \, d\mathcal{L}^n.$$

In this case the support of μ coincides with the support of f, defined in the usual way:

$$\mathrm{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}.$$

In the next section we will focus on the particular case of Hausdorff measures, which are constructed by Method II and defined on the euclidean space $\mathbb{R}^n$.

1.2 Hausdorff Measures and Hausdorff dimension

We begin with the notion of Hausdorff measures. For a deep and comprehensive treatment of the subject, the reader is referred to [Rog98]. Let $s > 0$ be fixed. The s-dimensional Hausdorff measure at scale $\delta > 0$ for a set $E \subseteq \mathbb{R}^n$ is defined by

$$\mathcal{H}^s_\delta(E) = \inf \sum_{i=1}^{\infty} \left(\mathrm{diam}(U_i)\right)^s, \tag{1.4}$$

where the infimum is taken over all the δ-coverings of E. It should be noted that, since any set is contained in a *convex* set of the same diameter, the covering in (1.4) can be taken by convex sets. Clearly $\mathcal{H}^s_\delta$ is increasing with δ, so it makes sense to consider

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \mathcal{H}^s \delta(E) = \sup_{\delta} \mathcal{H}^s \delta(E) \tag{1.5}$$

to obtain the s-dimensional Hausdorff measure of E. The Hausdorff measure $\mathcal{H}^s$ is the corresponding measure obtained by Method II applied to the premeasure τ_s defined by $\tau_s(U) = \mathrm{diam}(U)^s$, and therefore it is a Borel measure. It seems natural to compare Hausdorff measures with integer exponents against the corresponding Lebesgue measure. Clearly, on $\mathbb{R}$ the measure $\mathcal{H}^1$ coincides with the usual Lebesgue measure. For $n > 1$ the relation between them is given by the formula

$$|E| = c_n \mathcal{H}^n(E), \tag{1.6}$$

where c_n denotes the volume of the unit ball $\mathbb{R}^n$. The proof of this inequality is nontrivial, and the proof requires the so called isodiametric inequality, which says that the set of maximal volume of a given diameter is the sphere. Proofs can be found in [EG92] or [Fed69].

We state without proof (see, for example, [Rog98], [Fal86]) the following lemma about the regularity of Hausdorff measures with respect to the following classes of sets.

Definition 1.2.1.

1. A set A will be called a G_δ-set if it is a countable intersection of open sets.

2. A set A will be called an F_σ-set if it is a countable union of closed sets.

Lemma 1.2.2.

1. *Given a set $E \subseteq \mathbb{R}^n$, there exists a G_δ set G such that $E \subseteq G$ and $\mathcal{H}^s(E) = \mathcal{H}^s(G)$.*

2. *Any $\mathcal{H}^s$-measurable sets of finite $\mathcal{H}^s$-measure contains an F_σ set of equal measure, and so contains a closed set differing from it by arbitrary small measure.*

1.2.1   Hausdorff dimension

We begin this section by illustrating the role of the parameter s in the definition of $\mathcal{H}^s$ with the following properties.

Lemma 1.2.3. *Let $E \subseteq \mathbb{R}^n$ be an arbitrary set. If $0 < s < t$, then*

1. $\mathcal{H}_\delta^t(E) \leq \delta^{t-s}\mathcal{H}_\delta^s(E)$.

2. *If $\mathcal{H}^t(E) > 0$ then $\mathcal{H}^s(E) = \infty$.*

3. *If $\mathcal{H}^s(E) < \infty$ then $\mathcal{H}^t(E) = 0$.*

4. *Let $C = [0,1]^n$ be the unit cube in $\mathbb{R}^n$. Then $0 < \mathcal{H}^n(C) < +\infty$.*

Proof. Item 1. follows immediately from the definitions of $\mathcal{H}_\delta^s$ and $\mathcal{H}_\delta^t$, since

$$
\begin{aligned}
\mathcal{H}_\delta^t(E) = \inf \sum_{i=1}^\infty \left(\mathrm{diam}(U_i)\right)^t \;\; &= \;\; \inf \sum_{i=1}^\infty \left(\mathrm{diam}(U_i)\right)^s \left(\mathrm{diam}(U_i)\right)^{t-s} \\
&\leq \;\; \delta^{t-s}\mathcal{H}_\delta^s(E).
\end{aligned}
$$

Since this is true for all δ, item 2 follows. Item 3 is immediate from 2. For 4, take $\delta > 0$ and divide the unit cube into k^n cubes of side length $\frac{1}{k}$. Therefore each one of them has diameter $\frac{1}{k}\sqrt{n}$. Consider k such that $\delta \geq \frac{1}{k}\sqrt{n}$. Then

$$
\mathcal{H}_\delta^n(C) \leq \sum_{i=1}^{k^n} \left(\frac{1}{k}\sqrt{n}\right)^n = k^n \left(\frac{1}{k}\sqrt{n}\right)^n = \sqrt{n}^n < \infty,
$$

and this bound is independent on δ. Hence we obtain that $\mathcal{H}^n(C) < \infty$. Now, let $\{U_i\}_{i\in\mathbb{N}}$ be a δ-covering of C. Since $\mathrm{diam}(U_i) \leq \delta$ we can find, for each U_i, a cube Q_i such that $U_i \subseteq Q_i$ and with a side length equal to the diameter of U_i. Then $C \subseteq \bigcup_{i=1}^\infty Q_i$ and

$$
\sum_{i=1}^\infty \left(\mathrm{diam}(U_i)\right)^n = \sum_{i=1}^\infty \left(\mathrm{vol}(Q_i)\right) \geq \mathrm{vol}\left(\bigcup_{i=1}^\infty Q_i\right) \geq \mathrm{vol}(C) = 1.
$$

Thus, $\mathcal{H}_\delta^n(C) \geq 1$ for all $\delta > 0$ and therefore $\mathcal{H}^n(C) > 0$. $\square$

Plotting, for a fixed $E \subset \mathbb{R}^n$, the graph of $\mathcal{H}^s(E)$ against s we obtain the following schematic representation:
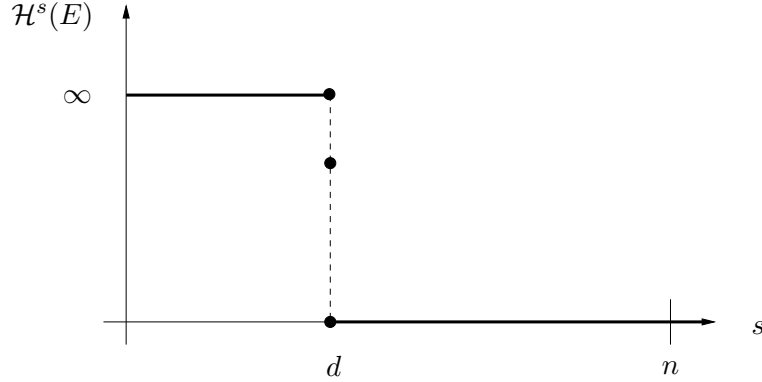
Figure 1.1: Hausdorff dimension

The properties stated above are essential to the definition of the notion of Hausdorff dimension, which can be understood in the following way. Considering the one parameter family of measures $\mathcal{H}^s$ with $s \in \mathbb{R}_{>0}$, one could fix a set $E \subseteq \mathbb{R}^n$ and look at the graph of $\mathcal{H}^s(E)$ against s. It could be at most one critical value of s, say $s = s_0$, where the measure of E drops from $+\infty$ to 0. Therefore, we are locating the cut point which separates the measures for which the set E is "too big" (on the left side of s_0), all of them taking the value $+\infty$, from those measures for which the set E is "too small" (on the right side of s_0), where all the measures take the value zero. This suggests that, if there is any candidate to measure properly the size of the set E, it should be $\mathcal{H}^{s_0}$. Hence, the notion of Hausdorff dimension is absolutely natural:

Definition 1.2.4. Let $E \subseteq \mathbb{R}^n$. The Hausdorff dimension of E, noted as $\dim_H(E)$ is defined by

$$\dim_H(E) = \sup\{s \in \mathbb{R}_{>0} : \mathcal{H}^s(E) = +\infty\} = \inf\{s \in \mathbb{R}_{>0} : \mathcal{H}^s(E) = 0\},$$

taking the supremum of the empty set to be 0. It is worth to note that we will see that for any set $E \subseteq \mathbb{R}^n$, we always have that $\mathcal{H}^s(E) = 0$ for all $s > n$.

About the value of $\mathcal{H}^{s_0}(E)$ for a set E with $\dim_H(E) = s_0$, nothing can be asserted in general. There are examples with zero, finite positive or infinite measure. Those sets E with $0 < \mathcal{H}^s(E) < \infty$ are called *s-sets*, and can be considered as *truly s*-dimensional. We will return to this in a much more general setting on Chapter 2.

Some properties of $\dim_H(E)$ are discussed in the next proposition.

Proposition 1.2.5. *For $\dim_H(E)$ we have the following properties.*

1. $\dim_H(E) \leq \dim_H(F)$ *whenever* $E \subseteq F$.

2. *Let $\{E_i\}_{i=1}^\infty$ be a countable family with $\dim_H(E_i) = d_i$ for all $i \in \mathbb{N}$. Then*

$$\dim_H\left(\bigcup_{i=1}^\infty E_i\right) = d = \sup_i\{d_i\} \quad \textit{(countable stability)}$$

3. *If $E \subseteq \mathbb{R}^n$ then $\dim_H(E) \leq n$.*

4. $\dim_H(\mathbb{R}^n) = n$.

5. *Given a set $E \subseteq \mathbb{R}^n$ with $\dim_H(E) = s$, there exists a G_δ set G with $E \subseteq G$ and $\dim_H(G) = s$.*

Proof. Item 1 follows from the validity of the same inequality for measures $\mathcal{H}^s$. For item 2, we calculate $\mathcal{H}^s(E)$ with $E = \bigcup_{i=1}^{\infty} E_i$ in separate cases:

- if $s > d$ then $\mathcal{H}^s(E_i) = 0$ for all i and then

$$\mathcal{H}^s(E) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(E_i) = 0,$$

  and therefore $\dim_H(E) \leq s$ for all $s > d$, which implies $\dim_H(E) \leq d$.

- if $s < d$ then there exists $i_0 \in \mathbb{N}$ with $d_{i_0} > s$. For that i_0 we have that $\mathcal{H}^s(E_{i_0}) = \infty$ and, since $E_i \subseteq E$,

$$\infty = \mathcal{H}^s(E_i) \leq \mathcal{H}^s(E).$$

  It follows that $\dim_H(E) \geq s$ for all $s < d$, which implies $\dim_H(E) \geq d$.

For item 3, it suffices to show that $\dim_H(\mathbb{R}^n) \leq n$. Using 2 we can do this by showing that $\dim_H(C) \leq n$ for $C = [0,1]^n$, but we already know this from item 4 of Lemma 1.2.3. Item 4 also follows from item 4 of Lemma 1.2.3 and for item 5 we use Lemma 1.2.2. □

An elementary tool in the problem of estimating dimensions is the following lemma about Lipschitz functions.

Lemma 1.2.6. *Let E be an $\mathcal{H}^s$-measurable subset of $\mathbb{R}^n$ and let $\psi : E \to \mathbb{R}^n$ be a Lipschitz map with constant $C > 0$, i.e.*

$$|\psi(x) - \psi(y)| \leq C|x - y| \quad \text{for all } x, y \in E. \tag{1.7}$$

Then

$$\mathcal{H}^s(\psi(E)) \leq C^s \mathcal{H}^s(E). \tag{1.8}$$

*This inequality is also known as a **scaling property** for the Hausdorff measures.*

Corollary 1.2.7. *Let $\psi : E \to \psi(E)$ be a Lipschitz map with constant $C > 0$. Then*

$$\dim_H(\psi(E)) \leq \dim_H(E).$$

Since orthogonal projections are Lipschitz transformations with constant 1, we have the following corollary.

Corollary 1.2.8. *Let $E \subseteq \mathbb{R}^n$ and S be a subspace. If $\Pi_S : \mathbb{R}^n \to S$ is the orthogonal projection onto S, then $\dim_H(\Pi_S(E)) \leq \dim_H(E)$.*

Another useful corollary is about similar sets. We say that A and B are similar sets if there exists an invertible affine map G between them. Note that any affine map results bi-Lipschitz.

Corollary 1.2.9. *Let A and B be two similar subsets of $\mathbb{R}^n$. Then their Hausdorff dimensions coincides $\dim_H(A) = \dim_H(B)$.*

As an immediate consequence of the preceding propositions, we can derive some other features of the Hausdorff dimension.

Open sets. It is clear that any open subset of $\mathbb{R}^n$ has full dimension, since it contains a cube of positive volume. This reflects the intuition that a "fat" set must behave dimensionally like the whole space. In fact, we can go further, since (1.6) implies the same for any set with positive Lebesgue measure.

Countable sets. If E is countable, then $\dim_H(E) = 0$. The counting measure $\mathcal{H}^0$ assigns mass 1 to every singleton, and therefore it has dimension zero. Then item 2 from Proposition 1.2.5 implies that the countable union also has dimension zero.

Smooth sets. Smooth curves have dimension 1 and smooth surfaces have dimension 2. More in general, if E is a continuously differentiable m-dimensional manifold, then $\dim_H(E) = m$.

Bi-Lipschitz stability. If $\varphi : E \to F$ is a bijective bi-Lipschitz map, then $\dim_H(E) = \dim_H(F)$.

Some non-trivial examples

Let us present here some examples which, although commonly included in the literature, are always appropriate to illustrate the nature of the objects with which we are dealing. We will present more examples and more general constructions in the sequel.

Cantor ternary set. Consider the following family of subsets of $[0, 1]$:

$$
\begin{aligned}
F_0 &= [0, 1] \\
F_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\
F_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}; \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]
\end{aligned}
$$

Given a set F_j, we construct F_{j+1} by removing from each interval of F_j their central middle third. Therefore each set F_j will be a union of 2^j intervals of length 3^{-j}. The Cantor ternary set is defined as

$$
C = \bigcap_{j=0}^{\infty} F_j. \tag{1.9}
$$

This is a perfect and totally disconnected set. It can be proved that the Hausdorff dimension of C is $\dim_H(C) = \frac{\log 2}{\log 3} = \log_3 2$. This follows essentially because C satisfies the self similarity equation

$$C = \frac{1}{3}C \cup \left(\frac{1}{3}C + \frac{2}{3}\right).$$

Therefore, assuming for a moment that $0 < \mathcal{H}^d(C) < \infty$ with some d, we can conclude that

$$\mathcal{H}^d(C) = \frac{2}{3^d}\mathcal{H}^d(C),$$

by the scaling property (1.8). Therefore d must be equal to $\frac{\log(2)}{\log(3)}$. The non-trivial part would be to prove the positivity of the $\mathcal{H}^d$ measure of C (the finiteness is much more easier, since the k-th step of the construction gives a natural and optimal covering). There is a natural generalization of the preceding construction. Consider a parameter $0 < \lambda < 1$ which will indicate the ratio of the length of an interval of one step and the length of its parent interval. Then, if E is a the middle-λ Cantor set, then $\dim_H(E) = \frac{\log(2)}{\log\left(\frac{2}{1-\lambda}\right)}$.

The following example is a particular case of the class of Cantor sets, which can be defined as the family of all the *perfect, totally disconnected* subsets of $\mathbb{R}$. Clearly, those sets are identified by their complement, which is a disjoint countable union of open intervals. Hence, a Cantor set is determined by the choice of this open intervals.

Cut out sets. Consider a positive, non-increasing and positive sequence $a = \{a_k\}$. Let I_a be a closed interval of length $\sum_{k=1}^\infty a_k$. The class $\mathcal{C}_a$ will be the family of all closed sets E contained in I_a that are of the form

$$E = I_a \setminus \bigcup_{k \geq 1} U_k,$$

where $\{U_k\}$ is a disjoint family of open intervals contained in I_a such that $|U_k| = a_k$ for all k. With this definition, each Cantor set in $\mathcal{C}_a$ has Lebesgue measure zero. The objective will be, roughly speaking, to determine the dimension of a Cantor set in terms of the defining sequence. But this does not make any sense unless we give a precise construction algorithm for a given sequence. We proceed as follows. Given the sequence $\{a_k\}$, remove from I_a an open interval I^0 of length a_1. This creates two closed subintervals I_1^1 and I_2^1 to the left and the right of I^0 respectively. Then we remove from the left interval an open interval of length a_2 and from the right interval we remove an interval of length a_3. Iterating this procedure, we obtain the perfect set $C \in \mathcal{C}_a$. Note that there is no ambiguity on the location of the intervals. To locate the first interval, for example, we could look at the subsequence corresponding to all the intervals to the left of I_0. The sum of this subsequence is then the position of the left endpoint of I^0. To study the Hausdorff measure and dimension of these sets, Besicovitch and Taylor in [BT54] study the decay of the sequence

$b_n = r_n/n$, where $r_n = \sum_{j \geq n} a_j$. They introduced the number

$$\alpha(a) = \varliminf_{n \to \infty} \alpha_n,$$

where $n b_n^{\alpha_n} = 1$ for all $n \in \mathbb{N}$. With this notation, the result in [BT54] is that $\dim_H(E) \leq \alpha(a)$ for all E in $\mathcal{C}_a$. It also can be proved that $\dim_H(C_a) = \alpha(a)$ (see [CMMS04] and [GMS07]).

1.3   Mass distribution and Frostman Lemma

Now we state two results that can be considered to be reciprocals in some sense, which characterize the Hausdorff dimension in terms of probability measures with some controlled local behavior. The first is a very elemental but useful criterion to estimate dimensions.

Lemma 1.3.1. Mass Distribution Principle. *Let $E \subseteq \mathbb{R}^n$ be a set and let μ be a probability measure on E. Let s, ε and C be positive constants such that for any $U \subseteq \mathbb{R}^n$ with $\mathrm{diam}(U) < \varepsilon$, μ satisfies the inequality*

$$\mu(U) \leq C \mathrm{diam}(U)^s.$$

Then $\mathcal{H}^s(E) > 0$.

The next lemma is called *Frostman's Lemma* and can be understood as a converse of Lemma 1.3.1

Lemma 1.3.2. Frostman's lemma *Let E be a Borel subset of $\mathbb{R}^n$ with $0 < \mathcal{H}^s(E) \leq \infty$. Then there is a compact set $F \subseteq E$ such that $0 < \mathcal{H}^s(F) < \infty$ and a constant $b > 0$ such that*

$$\mathcal{H}^s(F \cap B(x,r)) \leq b r^s \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and} \quad r > 0. \tag{1.10}$$

1.4   Energies

In this section we introduce a powerful tool to study dimension properties of sets. It is closely related to the preceding Frostman's Lemma. The purpose is to relate the local behavior of a measure μ , i.e. an inequality like

$$\mu(B(x,r)) \leq C r^s, \text{ for some suitable } C \text{ and all } x \in \mathbb{R}^n, s > 0 \tag{1.11}$$

with the finiteness of two quantities defined by integrals. For a measure μ and $t > 0$, the *t-Energy* $I_t(\mu)$ will be defined as

$$I_t(\mu) = \iint \frac{1}{|x-y|^t} \, d\mu(x) d\mu(y). \tag{1.12}$$

If we define the *t-Potential* V_μ^t as

$$V_\mu^t(y) = \int \frac{1}{|x-y|^t} d\mu(x) = \left(\mu * \frac{1}{|\cdot|^t} \right)(y),$$

we can express the energy as

$$I_t(\mu) = \int V_\mu^t(y) \; d\mu(y).$$

Now we have the following characterization of measures in terms of energies (see [Wol03]).

Lemma 1.4.1. *Consider a probability measure μ with compact support.*

1. *If μ satisfies (1.11) for some $s > 0$, then $I_t(\mu) < \infty$ for all $t < s$.*

2. *Conversely, if μ is a probability measure with compact support and finite energy $I_s(\mu)$ for $s > 0$, then there is another probability measure ν such that $\nu(X) \leq 2\mu(X)$ for all sets X and such that ν satisfies (1.11) for s.*

From Lemma 1.4.1 and Lemma 1.3.2 can be deduced the following characterization of Hausdorff dimension.

Proposition 1.4.2. *If E is compact then*

$$\dim_H(E) = \sup\left\{t : \exists \mu \in \mathcal{P}(E) \; with \; I_t(\mu) < \infty\right\}.$$

1.5   Box Counting, Packing and Fourier dimensions

We will discuss in this section the definition and some of the most relevant properties of two notions of dimension. The Box Counting dimension (also referred to as the *Minkowski dimension*) and the Packing dimension. For a comprehensive and extended development on this matter we refer the reader to [Fal86, Fal97, Fal03].

1.5.1   Box Counting Dimension

For a given set $E \subseteq \mathbb{R}^n$, denote with $\mathcal{M}_\delta(E)$ the smallest number of sets of diameter at most δ which can cover E. The *lower* and *upper box dimensions* of E respectively are defined as

$$\underline{\dim}_B(E) = \varliminf_{\delta \to 0} \frac{\log \mathcal{M}_\delta(E)}{-\log \delta} \qquad \text{and} \qquad \overline{\dim}_B(E) = \varlimsup_{\delta \to 0} \frac{\log \mathcal{M}_\delta(E)}{-\log \delta}.$$

If these quantities are equal, we refer to the common value as the *box dimension* of E

$$\dim_B(E) = \lim_{\delta \to 0} \frac{\log \mathcal{M}_\delta(E)}{-\log \delta}. \tag{1.13}$$

The intuition here is that, for $s = \dim_B(E)$, the number of balls of diameter δ needed to cover E is of the order of δ^{-s}. More precisely, (1.13) says that

$$\mathcal{M}_\delta(E)\delta^s \to \infty \qquad \text{if } s < \dim_B(E)$$

and

$$\mathcal{M}_\delta(E)\delta^s \to 0 \qquad \text{if } s > \dim_B(E).$$

It is very helpful to use some alternative definitions of the box dimension, which all follow simply by comparison.

Proposition 1.5.1. *The same notion of dimension is achieved if the quantity $\mathcal{M}_\delta$ above is taken to be any of the following:*

(i) *the smallest number of sets of diameter δ that cover E,*

(ii) *the smallest number of closed balls of radius δ that can cover E,*

(iii) *the smallest number of cubes of side δ that cover E,*

(iv) *the largest number of disjoint balls of radius δ with centers in E,*

(v) *the number of δ-mesh cubes that intersect E, hence the name "box counting".*

(An δ-mesh cube is a cube of the form $\prod_i [m_i\delta; (m_i + 1)\delta)$ where $m_1, \ldots, m_n$ are integers)

It is worth noting that, if δ_k is any sequence converging to zero, then trivially

$$\underline{\dim}_B(E) \leq \varliminf_{k \to 0} \frac{\log \mathcal{M}_{\delta_k}(E)}{-\log \delta_k} \qquad \text{and} \qquad \varlimsup_{k \to 0} \frac{\log \mathcal{M}_{\delta_k}(E)}{-\log \delta_k} \leq \overline{\dim}_B(E).$$

If, in addition, the sequence δ_k satisfies $\delta_{k+1} \geq c\delta_k$ for some $0 < c < 1$, then we obtain the reverse inequalities

$$\underline{\dim}_B(E) \geq \varliminf_{k \to 0} \frac{\log \mathcal{M}_{\delta_k}(E)}{-\log \delta_k} \qquad \text{and} \qquad \varlimsup_{k \to 0} \frac{\log \mathcal{M}_{\delta_k}(E)}{-\log \delta_k} \geq \overline{\dim}_B(E).$$

Therefore, for the upper bounds, we can choose appropriate coverings with no restrictions. For the lower bounds, we can choose particular coverings (suited to the set we are analyzing) but with the restriction above.

At this point it is very natural to ask which is the relation between the Box dimension and the Hausdorff dimension. For that, suppose that $\dim_H(E) = d$ and note that if a set E can be covered by $\mathcal{M}_\delta(E)$ sets of diameter δ, then for $s < d$,

$$1 < \mathcal{H}^s_\delta(E) \leq \mathcal{M}_\delta(E)\delta^s$$

Therefore, $0 < \log \mathcal{M}_\delta(E) + s \log(\delta)$ and from there, for small δ,

$$s \leq \varliminf_{\delta \to 0} \frac{\log \mathcal{M}_\delta(E)}{-\log \delta}.$$

Since $s < d$ is arbitrary, this implies that

$$\dim_H(E) \leq \underline{\dim}_B(E) \leq \overline{\dim}_B(E). \qquad (1.14)$$

We actually can obtain strict inequality here, as we shall see in forthcoming examples, but we want to remark that in any case, this inequality is a useful tool to obtain upper bounds on the Hausdorff dimension of a set by finding clever coverings of it.

The Box dimension shares some nice properties with the Hausdorff dimension, i.e.:

Monotonicity: If $E \subseteq F$, $\underline{\dim}_B(E) \leq \underline{\dim}_B(F)$ and $\overline{\dim}_B(E) \leq \overline{\dim}_B(F)$.

Smooth sets: If M is a smooth m-dimensional manifold, then $\dim_B(M) = m$.

Bi-Lipschitz invariance: If $\phi : E \to F$ is bi-Lipschitz, then $\dim_B(E) = \dim_B(F)$ (upper and lower).

We now discuss a remarkable difference between Box and Hausdorff dimensions. Consider a set E and its closure $\overline{E}$. Then

$$\underline{\dim}_B(E) = \underline{\dim}_B(\overline{E})$$

and

$$\overline{\dim}_B(E) = \overline{\dim}_B(\overline{E}).$$

To see why, take a finite collection $B_1, \ldots, B_k$ of *closed* balls of radii δ. If the closed set $\bigcup_{i=1}^{k} B_i$ contains E then also contains $\overline{E}$. Thus the smallest number of closed balls of radius δ that covers E coincides with the smallest number required to cover the larger set $\overline{E}$. Since this quantity is the only input with information of the set in the formula of the box dimension, the claim follows.

An immediate consequence of this is that a countable set E can have positive box dimension. For example, if it is dense on an open subset of $\mathbb{R}^n$, then $\dim_B(E) = n$. In particular, the set of rational numbers in an interval I has box dimension 1. Moreover, since the box dimension of a single point is clearly zero, we conclude that, in general,

$$\dim_B \left(\bigcup_i E_i \right) \neq \sup_i \{\dim_B(E_i)\} .$$

Example 1.5.2. Let $E := \{\frac{1}{n}\}_{n \in \mathbb{N}}$. Then $\dim_B E = \frac{1}{2}$.

We omit the calculation, since it can be found in [Fal97], or easily performed. We just comment that the result follows from the following fact. For any fixed $n \in \mathbb{N}$, the distance between $\frac{1}{n}$ and its two neighbors is of order $\frac{1}{n^2}$. Hence, for $\delta \sim \frac{1}{n^2}$, it will be clear that $n \leq \mathcal{M}_\delta(E) \leq 2n$. In the same spirit, it can also be proved that $\dim_B \left(\{\frac{1}{n^\alpha}\}_{n \in \mathbb{N}} \right) = \frac{1}{\alpha+1}$. These examples also illustrate that the first inequality in (1.14) can be strict.

As a final remark, we note that the Box Dimension, in opposite to Hausdorff dimension, is not defined in terms of a measure. The notion of Packing dimension defined in the next section is closely related to the Box dimension, and solves in some way the difficulties arising by the lack of an underlying measure in the definition of the latter.

1.5.2 Packing Dimension

Recall that Hausdorff dimension may be defined using economical coverings by small balls, whilst $\underline{\dim}_B$ may be defined using economical coverings by small balls *of equal radius*. The equivalent formulation (iv) in Proposition 1.5.1 suggest that it could be natural to look for a dimension that is defined in terms of dense packings of disjoint balls of different small radii. We follow the pattern of definition of Hausdorff measure and dimension. For $\delta > 0$ and a set $E \subseteq \mathbb{R}^n$, a δ-**packing** of E is a collection of disjoint balls or radius at most δ with centers in E. For $s \geq 0$ and $\delta > 0$, let

$$\mathcal{P}_\delta^s(E) = \sup \left\{ \sum_i \operatorname{diam}(U_i)^s : \{U_i\} \text{ is a } \delta\text{-packing of } E \right\}.$$

Since $\mathcal{P}_\delta^s(E)$ decreases with $\delta \to 0$, the limit $\mathcal{P}_0^s(E) = \lim_{\delta \to 0} \mathcal{P}_\delta^s(E)$ exists. Here we meet the same problems of the box dimension. By considering dense subsets, it is easy to see that $\mathcal{P}_0^s$ is not a measure, since it fails to be countable subadditive. But $\mathcal{P}_0^s$ does satisfy the properties of a premeasure, and therefore by a standard argument we can define the measure $\mathcal{P}^s$ of the set E as

$$\mathcal{P}^s(E) := \inf \left\{ \sum_i \mathcal{P}_0^s(E_i) : E \subseteq \bigcup_i E_i \right\},$$

which *is* a Borel measure on $\mathbb{R}^n$ called the s-dimensional packing measure of E. Clearly, for smooth sets, $\mathcal{P}^1$, $\mathcal{P}^2$, yields (essentially) length, area, analogously than $\mathcal{H}^1$, $\mathcal{H}^2$, etc. But for non integer exponents, $\mathcal{H}^s$ and $\mathcal{P}^s$ can be very different measures.

As for the Hausdorff dimension, for any $E \subseteq \mathbb{R}^n$, there is a number $\dim_P(E)$ called the *packing dimension* of E, such that $\mathcal{P}^s(E) = \infty$ for $s < \dim_P(E)$ and $\mathcal{P}^s(E) = 0$ for $s > \dim_P(E)$. Thus

$$\dim_P(E) = \inf\{s : \mathcal{P}^s(E) = 0\} = \sup\{s : \mathcal{P}^s = \infty\}.$$

It is sometimes convenient to express packing dimension in terms of upper box dimension. For $E \subseteq \mathbb{R}^n$ it is the case that

$$\dim_P(E) = \inf \left\{ \sup_i \overline{\dim}_B(E_i) : E \subset \bigcup_i E_i \right\}$$

(the infimum is taken over all countable coverings $\{E_i\}$ of E). The underlying measure structure implies monotonicity. Moreover, $\dim_P$ is also bi-Lipschitz invariant, is zero for any singleton and any open set has full dimension. Further, it coincides with the topological dimension on smooths manifolds, is countably stable (the dimension of a countable union is the supremum of the dimensions) and if $f : E \to f(E)$ is Lipschitz, then $\dim_P(f(E)) \leq \dim_P(E)$.

There are some basic inequalities between all the notions of dimensions that we have presented. They can be found in [Fal97, Fal03]. For any bounded

non-empty set $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) \leq \dim_P(E) \leq \overline{\dim}_B(E)$$

and

$$\dim_H(E) \leq \underline{\dim}_B(E) \leq \overline{\dim}_B(E),$$

and suitable examples shows that these inequalities may be strict.

1.5.3 Fourier Dimension

In many problems of harmonic analysis, an important ingredient is the rate of decay of the Fourier Transform $\widehat{\mu}$ of a measure μ supported on a set $E \subseteq \mathbb{R}^n$. In this section we want to relate this decay to a new notion of dimension for a subset of $\mathbb{R}^n$ and compare it to the Hausdorff dimension. This will be a consequence of the characterization of Hausdorff dimension in terms of energies presented in Section 1.4. We will introduce some elemental definitions in order to make a precise statement of the problem. For the proofs of the results on this section, we refer to [SW71], [Rud73], [Ste93], [Duo01], [Wol03] and [Gra04].

The Fourier Transform

Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz class on $\mathbb{R}^n$. Roughly speaking, a function f is Schwartz if it is smooth and all of its derivatives decay faster than the reciprocal of any polynomial at infinity. More precisely, we have the following definition.

Definition 1.5.3. A $C^\infty(\mathbb{R}^n)$ complex valued function f on $\mathbb{R}^n$ is called a Schwartz function if for all multiindices α and β there exists positive constants $C_{\alpha,\beta}$ such that

$$\rho_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| = C_{\alpha,\beta} < \infty.$$

The class of smooth functions with compact support $C_0^\infty(\mathbb{R}^n)$, is clearly contained in $\mathcal{S}(\mathbb{R}^n)$. Let $L^p(\mathbb{R}^n)$ denote the Lebesgue space of measurable p-integrable functions defined on $\mathbb{R}^n$ endowed with the norm

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and $\|f\|_\infty = \inf \{B > 0 : |\{x : |f(x)| > B\}| = 0\}$ (the essential supremum of f). It is also clear that the inclusion $\mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ holds. The following alternative characterization of Schwartz functions will be very useful. A C^∞ function f is in $\mathcal{S}(\mathbb{R}^n)$ if and only if for all positive integers N and all multiindices α there exists a positive constant $C_{\alpha,N}$ such that

$$|(\partial^\alpha f)(x)| \leq \frac{C_{\alpha,N}}{(1 + |x|)^N}.$$

If $f \in \mathcal{S}(\mathbb{R}^n)$, then its Fourier Transform is $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \xi} f(x) dx. \tag{1.15}$$

It is clear that (1.15) makes sense for any f in $L^1(\mathbb{R}^n)$, and, more generally, for finite measures. Let $\mathcal{M}(\mathbb{R}^n)$ be the space of finite complex-valued measures on $\mathbb{R}^n$ with the norm

$$\|\mu\| = |\mu|(\mathbb{R}^n),$$

where $|\mu|$ is the total variation. Then $L^1(\mathbb{R}^n)$ is contained in $\mathcal{M}(\mathbb{R}^n)$ via the identification $f \to \mu$, where the measure μ is defined on a set A by the formula

$$\mu(A) = \int_A f \ d\mu$$

We can generalize the definition of Fourier transform via

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \xi} d\mu(x).$$

A trivial bound on the size of the Fourier transform is the following:

$$\|\widehat{\mu}\|_\infty \leq \|\mu\|_1.$$

With respect to the smoothness of $\widehat{\mu}$, it is easy to see that $\mu \in \mathcal{M}(\mathbb{R}^n)$ implies the uniform continuity of $\widehat{\mu}$. Indeed, there is a close relation between localization and smoothness properties of μ and $\widehat{\mu}$, but at this point, we want to define the Fourier dimension of a set in $\mathbb{R}^n$, so we will postpone some other important properties and remarks on this subject to Chapter 7. We focus our attention on the decay properties at infinity for the Fourier transform. The first step toward this objective is the well known *Riemann-Lebesgue lemma*.

Proposition 1.5.4. *For a measure μ with an $L^1(\mathbb{R}^n)$ density f, we have that*

$$|\widehat{\mu}(\xi)| \to 0 \qquad as \qquad |\xi| \to \infty.$$

Outline of proof. This easily follows from the explicit computation of the Fourier transform of the characteristic of an interval. Then a typical L^1-approximation argument by step functions concludes the proof.

Is is clear that this result cannot hold for a general measure μ since it can be, for example, a probability supported on a null set. In this case, the problem of finding a measure with some prescribed decay is extremely difficult and depends, partially, on dimension properties of the support. Precisely, for a null set $E \subset \mathbb{R}^n$, the Hausdorff dimension of E can be determined by looking the behavior of the Fourier transform of measures supported by E. We will need the following lemma (see [Wol03],[Mat95]) that relates the t-energy of a measure μ defined in (1.12) to its L^2-size.

Lemma 1.5.5. *Let μ be a positive measure with compact support in $\mathbb{R}^n$ and $0 < t < n$. Then*

$$I_t(\mu) = \iint \frac{1}{|x-y|^t} \, d\mu(x)d\mu(y) = c_t \int |\widehat{\mu}(\xi)|^2 |\xi|^{t-n} \, d\xi, \qquad (1.16)$$

where $c_t = \frac{\Gamma(\frac{n-t}{2})\pi^{t-\frac{n}{2}}}{\Gamma(\frac{t}{2})}$. Here Γ is the gamma function: $\Gamma(s) = \int_0^\infty \frac{r^{s-1}}{e^r} \, dr$.

Now the connection between the decay of $\widehat{\mu}$ and the Hausdorff dimension becomes evident. More precisely, we have the following corollary.

Corollary 1.5.6. *Suppose μ is a compactly supported probability measure on $\mathbb{R}^n$ with*

$$|\widehat{\mu}(\xi)| \leq C|\xi|^{-\beta} \quad \text{for some} \quad 0 < \beta < \frac{n}{2}.$$

Then $\dim_H(\mathrm{supp}(\mu)) \geq 2\beta$.

Therefore, for a measure $\mu \in \mathcal{P}(\mathbb{R}^n)$, there is a maximum possible decay for its Fourier transform related to the size of its support, which leads to the following definition.

Definition 1.5.7. Let $E \subseteq \mathbb{R}^n$ be a Borel set. The *Fourier dimension* of E, $\dim_F(E)$, is defined as

$$\dim_F(E) = \sup\left\{ s : \exists \mu \in \mathcal{P}(E) \text{ with } |\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2} \right\}.$$

We have for any Borel set $E \subseteq \mathbb{R}^n$,

$$\dim_F(E) \leq \dim_H(E), \qquad (1.17)$$

and the inequality is often strict. Indeed, it is not easy to construct a set E for which the equality holds. The question about the reverse inequality is equivalent to ask whether a compact set E with $\dim_H(E) = \alpha$ must support a measure μ with

$$|\widehat{\mu}(\xi)| \leq C_\varepsilon (1 + |\xi|)^{-\frac{\alpha}{2}+\varepsilon} \qquad (1.18)$$

for all $\varepsilon > 0$. It is absolutely trivial to note that the answer to this question is no. The easiest way to see this is to consider the set $E = [0,1] \times \{0\} \subset \mathbb{R}^2$. E has dimension 1, but if μ is a measure supported on E then $\widehat{\mu}$ depends on ξ_1 only, so it cannot go to zero at infinity, hence $0 = \dim_F(E) < \dim_H(E) = 1$. Another less trivial example is given by the Cantor ternary set C. It is well known that C does not support any non-zero measure whose Fourier transform would tend to zero at infinity (see [KS63]).

Those sets E for which $\dim_F(E) = \dim_H(E)$ are called *Salem sets.*. Some non-trivial examples were constructed by Kahane [Kah70] and also by Kaufman [Kau81]. On the other hand, Kahane [Kah85] has shown that, in some probabilistic sense, many significant random sets are Salem sets. We introduce here a classic example due to Kaufman [Kau81].

Theorem 1.5.8. *For a fixed $\beta \geq 2$, define the following set:*

$$B_\beta = \left\{ x \in [0,1] \setminus \mathbb{Q} : \exists \text{ infinite many } \frac{p}{q} \text{ such that } \left| x - \frac{p}{q} \right| < \frac{1}{q^\beta} \right\}.$$

Then $\dim_F(B_\beta) = \dim_H(B_\beta) = \frac{2}{\beta}$. More precisely, for any $\varepsilon > 0$, there is a constant C_ε and a positive measure supported on a subset of B_β such that

$$|\widehat{\mu}(\xi)| \leq C_\varepsilon |\xi|^{-\frac{1}{\beta} + \varepsilon}.$$

The value for the Hausdorff dimension is known since the work of Jarník [Jar31], and is a classical theorem on Diophantine Approximation. There is a further improvement on Theorem 1.5.8 due to Bluhm [Blu98], which provides a deterministic construction of a Salem set E with $\dim_H(E) = \alpha$ for a prescribed $\alpha \in (0,1)$ supporting a measure μ_α whose Fourier transform obeys the following assymptotics

$$|\widehat{\mu_\alpha}(\xi)| \lesssim (1 + |\xi|)^{-\frac{1}{2+\alpha}} \log(e + |\xi|) \log\log(e + |\xi|).$$

We will return to this subject in Chapter 5.

1.6 Projections, Products and Intersections

There is a natural interest on the study of how the dimension properties of sets change with some elemental operations, such as projections, products and intersections. In this section we include only a few of results on this subject. We will focus our interest mainly on the properties of the Hausdorff dimension, including only some examples regarding Box, Packing and Fourier dimension.

1.6.1 Projections

The first result is a classical result of Marstrand on orthogonal projections of s-sets in $\mathbb{R}^2$. For a given $\theta \in [0, \pi)$, we will denote with L_θ the line in $\mathbb{R}^2$ through the origin that makes an angle of θ with the x axis, and proj_θ will denote the orthogonal projection onto that line. In the forthcoming theorem, "for almost all θ" refers to the arclegnth measure on the unit circle. The proofs can be found in [Fal86] or [Mar54a, Mar54b].

Theorem 1.6.1. *Let E be a Borel set in $\mathbb{R}^2$ such that $\dim_H(E) = s$.*

1. *If $s \leq 1$ then $\dim_H(\text{proj}_\theta(E)) = s$ for almost all $\theta \in [0, \pi)$.*

2. *If $s > 1$ then $|\text{proj}_\theta(E)| > 0$ for almost all $\theta \in [0, \pi)$.*

There is an easy consequence of this result that we include for future reference.

Corollary 1.6.2. *Given two subsets E and F of $\mathbb{R}$ and $\lambda \in \mathbb{R}$, consider the set*

$$E + \lambda F = \{x + \lambda y : x \in E, y \in F\}.$$

Then, for almost all $\lambda \in \mathbb{R}$, $\dim_H(E + \lambda F) = \min\{1, \dim_H(E \times F)\}$.

Outline of proof. Define Π_λ as the orthogonal projection onto the line $\ell(x) = \lambda x$. It is easy to define a bi-Lipschitz transformation between $E + \lambda F$ and $\Pi_\lambda(E \times F)$: take

$$\begin{aligned} \varphi : \Pi_\lambda(E \times F) &\to & E + \lambda F \\ (a, \lambda a) &\mapsto& a + \lambda \cdot \lambda a = (\lambda^2 + 1)a \end{aligned}$$

Then, for all $\lambda \in \mathbb{R}$, $\dim_H(\Pi_\lambda(E \times F)) = \dim_H(E + \lambda F)$ and the corollary follows from Theorem 1.6.1 (but only for a.e. λ).

1.6.2 Products

The aim of this section is to collect some of the known results about a problem, which is absolutely easy to state but by no means easy to completely solve. Given two sets (they can be taken to be compact to keep it simple) E and F, for which we know their dimension, what can we say about the dimension of the set $G = E \times F$?

The first attempt to an answer is to consider the most trivial example of two line segments. Therefore the product is a square and it in that case it is clear that

$$\dim(E \times F) = \dim(E) + \dim(F)$$

for Hausdorff, Box or Packing dimensions. It is also known that for smooth manifolds E and F the same formula is still valid. However, this is not the general rule, and the best that can be achieved, for Hausdorff dimension, is

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F).$$

It is worthwhile to note that even for sets of non integer dimensions, in some cases the equality does hold, and we will include some examples of that fact. The first easy but nontrivial example is the product $C \times C$ where C is the Cantor ternary set. In this case, it can be proved that $\dim_H(F \times F) = 2\dim_H(F) = 2\frac{\log 2}{\log 3}$.

Now we list several "product formulas" involving Hausdorff, Box and Packing dimensions (see [Fal03], [BP96], [Tri82]).

Hausdorff Dimension. For Borel sets $E, F \subseteq \mathbb{R}^n$,

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F). \tag{1.19}$$

Hausdorff and Box Dimensions. For any sets $E, F \subseteq \mathbb{R}^n$,

$$\dim_H(E \times F) \leq \dim_H(E) + \overline{\dim}_B(F). \tag{1.20}$$

Box Dimension. For any sets $E, F \subseteq \mathbb{R}^n$,

$$\overline{\dim}_B(E \times F) \leq \overline{\dim}_B(E) + \overline{\dim}_B(F). \qquad (1.21)$$

Packing and Hausdorff dimensions. For any sets $E, F \subseteq \mathbb{R}^n$,

$$\dim_H(E \times F) \leq \dim_H(E) + \dim_P(F) \leq \dim_P(E \times F). \qquad (1.22)$$

If we reformulate the left hand side of (1.22) as

$$\dim_H(E \times F) - \dim_H(E) \leq \dim_P(F),$$

one can prove some *approximation property* for the Packing Dimension in the following sense [BP96]: For any analytic set A in $\mathbb{R}^n$,

$$\sup_B \left\{ \dim_H(A \times B) - \dim_H(B) \right\} = \dim_P(A)$$

where the supremum is over all compact sets $B \subset \mathbb{R}^n$.

There are plenty of suitable examples that can be found in the literature to illustrate that all of the above inequalities can be strict. We only include the following example, since it will be crucial for some construction in Chapter 6. The complete details are in [Fal03], Example 7.8 (p. 104).

Example 1.6.3. There exists sets $E, F \subset \mathbb{R}$ with $\dim_H(E) = \dim_H(F) = 0$ and $\dim_H(E \times F) = 1$.

We only comment on the proof. The basic idea is to consider two very small sets E, F with the property $[0, 1] \subseteq E + F$. To achieve this, consider the real numbers in the unit interval $[0, 1]$ represented in their binary expansion $x = \sum_j r_j 2^{-j}; r_j = 0, 1$. Let $\{m_k; m_0 = 0\}_k$ be an increasing sequence tending to $+\infty$. Consider the following subsets of $[0, 1]$:

$$E := \{x \in [0, 1] : r_j = 0 \text{ if } m_k + 1 \leq j \leq m_{k+1}; k \text{ even}\}$$

$$F := \{x \in [0, 1] : r_j = 0 \text{ if } m_k + 1 \leq j \leq m_{k+1}; k \text{ odd }\}$$

Easy computations show that, if the sequence m_k is chosen to increase sufficiently fast, then $\dim_H(E) = \dim_H(F) \leq \underline{\dim}_B(E) = \underline{\dim}_B(F) = 0$ (we refer to Chapter 6 for a detailed construction). Now, to show why $\dim_H(E \times F) = 1$, note that we can define the map $f : E \times F \to \mathbb{R}$, $f(x, y) = x + y$ and, since it is clearly Lipschitz and the image of f contains the unit interval, it follows from Corollary 1.2.7 that $1 = \dim_H([0, 1]) \leq \dim_H(f(E \times F)) \leq \dim_H(E \times F)$.

1.6.3   Intersections

The study of intersection properties of fractals can be stated as follows. Consider a group G or transformations on $\mathbb{R}^n$, such as congruences, similarities or translations. This group can be endowed with a measure, the Lebesgue measure on the space of the parameters that describe the group. The question is then what can be said, in general, of the dimension of the set $E \cap \sigma(F)$ for $\sigma \in G$ in terms of the dimensions of E and F. Here the expression "in general" will be used when an intersection formula holds "for almost all $\sigma \in G$". It also will be used the term "often" when the formula holds for a set of positive measure of motions. Of course the underlying measure will always be the Lebesgue measure on G.

We include the following proposition about upper bounds for the particular case of translations. It is clear that the same result holds for larger groups of congruences and similarities.

Proposition 1.6.4. *If E and F are Borel subsets of $\mathbb{R}^n$ then*

$$\dim_H(E \cap (F + x)) \leq \max\{0, \dim_H(E \times F) - n\}$$

for almost all $x \in \mathbb{R}^n$.

There are general results for several particular choices of groups that can be found in [Fal03] (Theorem 8.2 and the discussion on the proof). We include the following theorem regarding the problem of finding lower bounds for $E \cap \sigma(F)$.

Theorem 1.6.5. *Let E and F Borel subsets of $\mathbb{R}^n$ and let G be a group of transformations on $\mathbb{R}^n$. Then*

$$\dim_H(E \cap \sigma(F)) \geq \dim_H(E) + \dim_H(E) - n$$

for a set of motions $\sigma \in G$ of positive measure in the following cases:

1. *G is the group of similarities and E and F are arbitrary sets.*

2. *G is the group of rigid motions, E is arbitrary an F is a rectifiable curve, surface or manifold.*

3. *G is the group of rigid motions and E and F are arbitrary, with either $\dim_H(E) > \frac{1}{2}(n+1)$ or $\dim_H(F) > \frac{1}{2}(n+1)$.*

Chapter 2

Dimension Functions

In this section we present a very well known construction of measures following Method II (Proposition 1.1.10) using as function on the diameters general *dimension functions* (see [Hau18]). We summarize some important definitions and results on dimension functions. We also illustrate with classical examples and introduce the necessary lemmas to be used in the following chapters.

2.1 The POSet of dimension functions

The family of dimension functions will be endowed with a notion of partial order, yielding a Partially Ordered Set (POSet) of functions. Let us begin with the basic definitions.

Definition 2.1.1. The following class of functions will be called *dimension functions*.

$$\mathbb{H} := \{h : [0, \infty) \to [0 : \infty), \text{non-decreasing, right continuous, } h(0) = 0\}.$$

The important subclass of those $h \in \mathbb{H}$ that satisfy a doubling condition will be denoted by $\mathbb{H}_d$:

$$\mathbb{H}_d := \{h \in \mathbb{H} : h(2x) \leq Ch(x) \text{ for some } C > 0\}.$$

Remark 2.1.2. Clearly, if $h \in \mathbb{H}_d$, the same inequality will hold (with some other constant) if 2 is replaced by any other $\lambda > 1$. We also remark that any concave function trivially belongs to $\mathbb{H}_d$. Also note that the monotonicity of h implies that $C \geq 1$.

If one only looks at the power functions, there is a natural total order given by the exponents. In $\mathbb{H}$ we also have a natural notion of order, but we can only obtain a *partial* order.

Definition 2.1.3. Let g, h be two dimension functions. We will say that g is dimensionally smaller than h and write $g \prec h$ if and only if

$$\lim_{x \to 0^+} \frac{h(x)}{g(x)} = 0.$$

2.1.1   Some examples

This partial order is compatible with the *total* order among the power laws, i.e.,

$$x^{\alpha_1} \prec x^{\alpha_2} \iff \alpha_1 < \alpha_2,$$

but allows us to refine that family:

$$x^{\alpha_1} \prec x^{\alpha_1} \log^{-\gamma} \left(\frac{1}{x} \right) \prec x^{\alpha_2} \log^{\beta} \left(\frac{1}{x} \right) \prec x^{\alpha_2}, \quad \beta, \gamma > 0, \quad \alpha_1 < \alpha_2.$$

Another example: if $h_{\alpha\beta}(x) := x^{\alpha} \log^{-\beta}(\frac{1}{x})$, then the order between them is given by the lexicographical order on the parameters:

$$h_{\alpha_1\beta_1} \prec h_{\alpha_2\beta_2} \iff (\alpha_1, \beta_1) <_{\ell} (\alpha_2, \beta_2)$$
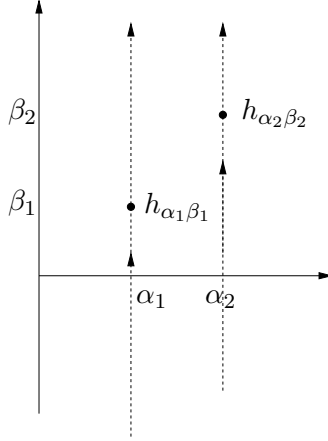

Figure 2.1: Lexicographical order in $\mathbb{H}$

To measure the "distance" between to dimension functions, we introduce the following notion:

Definition 2.1.4. Let $g, h \in \mathbb{H}$ with $g \prec h$. Define the "gap" between g and h as $\Delta(g, h)$ defined by

$$\Delta(g, h)(x) = \frac{h(x)}{g(x)}.$$

From this definition and the definition of partial order introduced above, we always have that $\lim_{x \to 0} \Delta(g, h)(x) = 0$, and therefore the speed of convergence to zero can be seen as a notion of distance between g and h.

We will be interested in the special subclass of dimension functions that allow us to classify zero dimensional sets.

Definition 2.1.5. A function $h \in \mathbb{H}$ will be called "zero dimensional dimension function" if $h \prec x^\alpha$ for any $\alpha > 0$. We denote by $\mathbb{H}_0$ the subclass of those functions. As a model to keep in mind, consider the family $h_\theta(x) = \frac{1}{\log^\theta(\frac{1}{x})}$.

A classical non trivial example of a zero dimensional set is the set of Liouville numbers. It is closely related to the set of β-approximable numbers B_β. For $\beta \geq 2$, define B_β as

$$B_\beta = \left\{ x \in [0, 1] \setminus \mathbb{Q} : \|xq\| < \frac{1}{q^{\beta-1}} \text{ for infinitely many } q \in \mathbb{Z} \right\}.$$

The set of Liouville numbers $\mathbb{L}$ can be defined as

$$\mathbb{L} = \bigcap_{n \in \mathbb{N}} B_n$$

and it is clearly smaller than any of the above sets. A classical result is that $\dim_H(B_\beta) = \frac{2}{\beta}$ (we will come back to this subject in Chapter 5). In particular, this implies that $\dim_H(\mathbb{L}) = 0$, since the monotony of the Hausdorff dimension yields $\dim_H(\mathbb{L}) \leq \frac{2}{\beta}$ for all $\beta \geq 2$.

2.2 Hausdorff measures

In this section we introduce the general Hausdorff measures. These measures are all constructed following Method II presented in Subsection 1.1.1. For a given $h \in \mathbb{H}$, the h-dimensional (outer) Hausdorff measure $\mathcal{H}^h$ will be defined as follows. For a set $E \subseteq \mathbb{R}^2$ and $\delta > 0$, write

$$\mathcal{H}^h_\delta(E) = \inf \left\{ \sum_i h(\text{diam}(E_i)) : E \subset \bigcup_i^\infty E_i, \text{diam}(E_i) < \delta \right\}.$$

Then the h-dimensional Hausdorff measure $\mathcal{H}^h$ of E is defined by

$$\mathcal{H}^h(E) = \sup_{\delta > 0} \mathcal{H}^h_\delta(E).$$

This notion generalizes the classical α-Hausdorff measure to functions h that are different to x^α. It is well known that a set of Hausdorff dimension α can have zero, positive or infinite α-dimensional measure. The desirable situation, in general, is to work with a set which is *truly* α-dimensional, that is, it has positive and finite α-dimensional measure. In this case we refer to this set as an α-set.

Now, given an α-dimensional set E without this last property, one could expect to find in the class $\mathbb{H}$ an appropriate function h to detect the precise "size" of it. By that we mean that $0 < \mathcal{H}^h(E) < \infty$, and in this case E is referred to as an h-set.

In order to illustrate the focus of this chapter, we start with a simple observation. The Hausdorff dimension of a set $E \subseteq \mathbb{R}^n$ is the unique real number s characterized by the following properties:

- $\mathcal{H}^r(E) = +\infty$ for all $r < s$.

- $\mathcal{H}^t(E) = 0$ for all $s < t$.

Therefore, to prove that some set has dimension s, it suffices to prove the preceding two properties, and this is independent of the possibles values of $\mathcal{H}^s(E)$. It is always true, no matter if $\mathcal{H}^s(E)$ is zero, finite and positive, or infinite.

The above observation could lead to the conjecture that in the wider scenario of dimension functions the same kind of reasoning can be made. In fact, Eggleston claims in [Egg52] that for any $A \subseteq \mathbb{R}^n$, one of the following three possibilities holds.

1. for all $h \in \mathbb{H}$, $\mathcal{H}^h(A) = 0$.

2. there is a function $h_0 \in \mathbb{H}$, such that if $h \succ h_0$ then $\mathcal{H}^h(A) = 0$, whilst if $h \prec h_0$, then $\mathcal{H}^h(A) = +\infty$.

3. for all $h \in \mathbb{H}$, $\mathcal{H}^h(A) = +\infty$.

Note that the most interesting situation is the one on item 2., since it is saying that the correct notion of size for the set A is represented by the function h_0. Clearly, this is the case when we are dealing with an h-set. However, this claim is false, in the sense that there are situations where none of the above three cases is met, and that is the reason why in the general setting the problem of determining an appropriate notion of size for a set is highly non trivial. Therefore the situation in the general setting does not behaves in the same way as in the classical setting.

In this wider scenario, we must overcome some difficulties that arise from two results due to C.A. Rogers. The first says that if a set E has null $\mathcal{H}^h$-measure for some $h \in \mathbb{H}$, then there exists a function $g \prec h$ such that $\mathcal{H}^g(E) = 0$ (see [Rog98]). Symmetrically, the second says that if a *compact* set E has non-σ-finite $\mathcal{H}^h$ measure, then there exists a function $g \succ h$ such that E has also non-σ-finite $\mathcal{H}^g$ measure (see [Rog62]). These results imply that if a *compact* set E satisfies that there exists a function h_0 such that $\mathcal{H}^h(E) > 0$ for any $h \prec h_0$ and $\mathcal{H}^h(E) = 0$ for any $h \succ h_0$, then it must be the case that $0 < \mathcal{H}^{h_0}(E) < +\infty$.

Consider now the set $\mathbb{L}$ of Liouville numbers. It is known that this set is dimensionless (see [EK06]), which means that it is not an h-set for any $h \in \mathbb{H}$. In addition, it is shown in [OR06] that there are two proper nonempty subsets

$\mathbb{L}_0, \mathbb{L}_\infty \subseteq \mathbb{H}$ of dimension functions such that $\mathcal{H}^h(\mathbb{L}) = 0$ for all $h \in \mathbb{L}_0$ and $\mathcal{H}^h(\mathbb{L}) = \infty$ for all $h \in \mathbb{L}_\infty$. From the results in [OR06] it follows that the Liouville numbers $\mathbb{L}$ must satisfy condition 2. in the classification of Eggleston. But suppose that h_0 is the claimed dimension function in that case. The discussion in the above paragraph implies that the set $\mathbb{L}$ is an h_0-set, which is a contradiction. Note that the results of Rogers appear ten years later that the paper by Eggleston.

Since there are many sets E that are not h-sets for any $h \in \mathbb{H}$ we need to look at a different approach for finding appropriate dimension functions for an arbitrary compact set E. Since in the present thesis we are interested in estimates for the size of general Furstenberg sets, we have to consider dimension functions that are a true *step down* or *step up* from the critical one. In the next section we introduce some notation and terminology that allows us to work in this general scenario.

2.3 Dimension Partition

For a given set $E \subseteq \mathbb{R}^n$, we introduce the notion of dimension partition (see [CHM10]).

Definition 2.3.1. By the *Dimension Partition* of a set E we mean a partition of $\mathbb{H}$ into (three) sets: $\mathcal{P}(E) = E_0 \cup E_1 \cup E_\infty$ with

- $E_0 = \{h \in \mathbb{H} : \mathcal{H}^h(E) = 0\}$.

- $E_1 = \{h \in \mathbb{H} : 0 < \mathcal{H}^h(E) < \infty\}$.

- $E_\infty = \{h \in \mathbb{H} : \mathcal{H}^h(E) = \infty\}$.

It is very well known that E_1 could be empty, reflecting the dimensionless nature of E. A classical example of this phenomenon is the set $\mathbb{L}$ of Liouville numbers. On the other hand, E_1 is never empty for an h-set, but it is not easy to determine this partition in the general case. We also remark that it is possible to find non-comparable dimension functions g, h and a set E with the property of being a g-set and an h-set simultaneously. Consider the following example:

Example 2.3.2. There exists a set E and two dimension functions $g, h \in \mathbb{H}$ which are not comparable and such that E is a g-set and also an h-set.

Proof. We will use the results of [CMMS04]. The set E will be the Cantor set C_a associated to a nonnegative decreasing sequence $a = \{a_i\}$ such that $\sum a_i = 1$. If we define $b_n = \frac{1}{n} \sum_{i \geq n} a_i$, then the main result of the cited work is that

$$\varliminf_{n \to \infty} nh(b_n) \sim \mathcal{H}^h(C_a) \tag{2.1}$$

for all $h \in \mathbb{H}$. Therefore, if $h = h_a$ is the dimension function that makes C_a an h-set, we have that

$$0 < \varliminf_{n \to \infty} nh(b_n) < \infty.$$

Now we want to define g. Consider the sequence $x_n = b_{n!}$ and take g satisfying the following properties:

1. $g(x) \geq h(x)$ for all $x > 0$.

2. $g(x_n) = h(x_n)$ for all $n \in \mathbb{N}$,

3. g is a polygonal spline (same as h), but it is constant in each interval $[b_{n!-1}, b_{(n-1)!}]$ and drops abruptly on $[b_{n!}, b_{n!-1}]$ (we are building up g from the right approaching the origin). More precisely, for each $n \in \mathbb{N}$,

$$g(x) = \begin{cases} \frac{1}{(n-1)!} & \text{if} \quad x \in [b_{n!-1}, b_{(n-1)!}] \\ \frac{1}{n!} & \text{if} \qquad x = b_{n!} \end{cases}$$

   and it is linear on $[b_{n!}, b_{n!-1}]$.

Conditions 1 and 2 imply that $\overline{\lim}_{x \to 0} \frac{h(x)}{g(x)} = 1 < \infty$. Note that we also have that $\underline{\lim}_{x \to 0} \frac{h(x)}{g(x)} = 0$, since

$$\frac{h(b_{n!-1})}{g(b_{n!-1})} = \frac{(n-1)!}{n!-1} \sim \frac{1}{n} \to 0.$$

It follows that h and g are not comparable. To see that C_a is also a g-set, we use again the characterization (2.1). Since

$$\varliminf_{n \to \infty} n g(b_n) \leq \varliminf_{n \to \infty} n! g(b_{n!}) = \varliminf_{n \to \infty} n! h(b_{n!}) < \infty,$$

we obtain that $\mathcal{H}^g(C_a) < \infty$. In addition, $g(x) \geq h(x)$ for all x, hence $g(b_n) \geq h(b_n)$ for all $n \in \mathbb{N}$ and it follows that

$$\varliminf_{n \to \infty} n g(b_n) \geq \varliminf_{n \to \infty} n h(b_n) > 0.$$

and therefore $\mathcal{H}^g(C_a) > 0$. $\square$

We refer the reader to [GMS07] for a detailed study of the problem of equivalence between dimension functions and Cantor sets associated to sequences. The authors also study the Packing measures and premeasures of those sets. For the construction of h-sets associated to certain sequences see the work of Cabrelli *et al* [CMMS04].

We refer to the work of Olsen and Renfro [OR06], [Ols05], [Ols03] for a detailed study of the exact Hausdorff dimension of the Liouville numbers $\mathbb{L}$, which is a known example of a zero dimensional set. Moreover, the authors prove that it is also a dimensionless set, i.e. there is no $h \in \mathbb{H}$ such that $0 < \mathcal{H}^h(\mathbb{L}) < \infty$ (equivalently, for any dimension function h, one has $\mathcal{H}^h(\mathbb{L}) \in \{0, \infty\}$). In that direction, further improvements are due to Elekes and Keleti [EK06]. There the authors prove much more than that there is no exact Hausdorff-dimension function for the set $\mathbb{L}$ of Liouville numbers: they prove that for any translation invariant Borel measure $\mathbb{L}$ is either of measure

zero or has non-sigma-finite measure. So in particular they answer the more interesting question that there is no exact Hausdorff-dimension function for $\mathbb{L}$ even in the stronger sense when requiring only sigma-finiteness instead of finiteness.

It follows from Example 2.3.2 that even for h-sets the dimension partition, and in particular E_1, is not completely determined. Note that the results of Rogers cited above imply that, for compact sets, E_0 and E_∞ can be thought of as open components of the partition, and E_1 as the "border" of these open components. An interesting problem is then to determine some criteria to classify the functions in $\mathbb{H}$ into those classes (see Theorem 5.1.1 below for a precise example).

To detect where this "border" is, we will introduce the notion of *chains* in $\mathbb{H}$. This notion allows to refine the notion of Hausdorff dimension by using an ordered family of dimension functions. More precisely, we have the following definition.

Definition 2.3.3. A family $\mathcal{C} \subset \mathbb{H}$ of dimension functions will be called a *chain* if it is of the form

$$\mathcal{C} = \{h_t \in \mathbb{H} : t \in \mathbb{R}, h_s \prec h_t \iff s < t\}$$

That is, a totally ordered one-parameter family of dimension functions.

Suppose that $h \in \mathbb{H}$ belongs to some chain $\mathcal{C}$ and satisfies that, for any $g \in \mathcal{C}$, $\mathcal{H}^g(E) > 0$ if $g \prec h$ and $\mathcal{H}^g(E) = 0$ if $g \succ h$. Then, even if $h \notin E_1$, in this chain, h does measure the size of E. It can be thought of as being "near the frontier" of both E_0 and E_∞. For example, if a set E has Hausdorff dimension α but $\mathcal{H}^\alpha(E) = 0$ or $\mathcal{H}^\alpha(E) = \infty$, take $h(x) = x^\alpha$ and $\mathcal{C}_H = \{x^t : t \geq 0\}$. In this chain, x^α is the function that best measures the size of E.
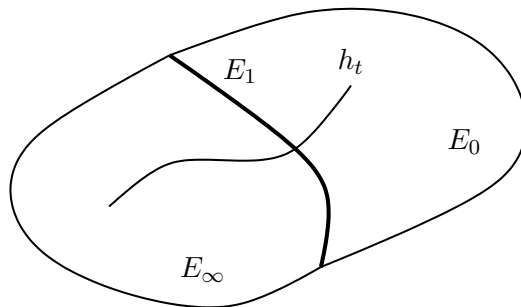

Figure 2.2: Dimension Partition $\mathbb{H} = E_\infty \cup E_1 \cup E_0$

We look for finer estimates, considering chains of dimension functions that yield "the same Hausdorff dimension". Further, for zero dimensional sets, this approach allows us to classify them by some notion of dimensionality. In some particular cases, it is possible to have some intuition or evidence to conjecture some explicit expression of an appropriate dimension function for

a given set. We will refer to such a function as one of the *Expected Dimension Functions* for a set E and will denote that class of them with $\mathcal{E}(E)$. The goal will be to find very fine chains to give precise bounds on how far from the expected dimension function the Hausdorff measure drops to zero or remains positive. In this setting, the sharpness of the results are associated to the "fineness" of the chains. In the above example, the chain $\mathcal{C}_H$ detects only the Hausdorff dimension of a given set. Suppose that we know that a given set E has Hausdorff dimension α_0. If we use the chain $\mathcal{C}_{\log} = \{x^{\alpha_0} \log(\frac{1}{x})^{-\beta}, \beta \in \mathbb{R}\}$ we can sharpen the dimension estimate for this set in terms of logarithmic gaps.

Let us note that we have not presented a precise definition of "fineness" of a chain. Nevertheless, in most of the particular cases, we will use chains $\mathcal{C}$ that actually enjoy a "group property": given two members of the chain, $h_r \prec h_s$, we have that $\frac{h_s(x)}{h_r(x)} = h_{r-s}(x)$. In that case, we can normalize the notion of gap and associate the fineness of the chain to the speed of convergence to zero of h_1. For example, the size of the gaps in $\mathcal{C}$ above are "power like" and those in $\mathcal{C}_{\log}$ are "log like". It is clear that we can add more parameters and obtain finer chains, for example we can fix a power α_0, a logarithmic power β_0, and consider the chain $\mathcal{C}_H = \{x^{\alpha_0} \log^{\beta_0}(\frac{1}{x}) \log \log^{\gamma}(\frac{1}{x})\}$. We refer to Example 5.1.4 for a precise example of this subject.

2.4 The exact dimension function for a class of sets

In the previous section we dealt with the problem of detecting an appropriate dimension function for a given set or, more generally, the problem of determining the dimension partition of that set. Now we introduce another related problem, which concerns the analogous problem but for a whole class of sets defined, in general, by geometric properties. We mention one example: A Kakeya set is a compact set containing a unit segment in every possible direction. It is known that there are Kakeya sets of zero measure and it is conjectured that they must have full Hausdorff dimension. The conjecture was proven by Davies [Dav71] in $\mathbb{R}^2$ and remains open for higher dimensions. Since in the class of planar Kakeya sets there are several distinct types of two dimensional sets (i.e. with positive or null Lebesgue measure), one would like to associate a dimension function to the whole class. A dimension function $h \in \mathbb{H}$ will be called the exact Hausdorff dimension function of the class of sets $\mathcal{A}$ if

- For every set E in the class $\mathcal{A}$, $\mathcal{H}^h(E) > 0$.

- There are sets $E \in \mathcal{A}$ with $\mathcal{H}^h(E) < \infty$.

Analogously, h is the exact Minkowski dimension for the class $\mathcal{A}$ if

- For every set E in the class $\mathcal{A}$, $\mathfrak{M}^h(E) > 0$.

- There are sets $E \in \mathcal{A}$ with $\mathfrak{M}^h(E) < \infty$.

where $\mathfrak{M}^h(E) := \sup_\delta \mathfrak{M}^h_\delta(E)$ and

$$\mathfrak{M}^h_\delta(E) = \inf\left\{ Nh(r) : E \subset \bigcup_{i=1}^N B(x_i, r); r < \delta \right\}.$$

In the direction of finding the exact dimension of the class of Kakeya sets in $\mathbb{R}^2$, Keich [Kei99] has proven that in the case of the Minkowski dimension the exact dimension function is $h(x) = x^2 \log(\frac{1}{x})$. For the case of the Hausdorff dimension, he provided some partial results. Specifically, he shows that in this case the exact dimension function h must decrease to zero at the origin faster than $x^2 \log(\frac{1}{x}) \log \log(\frac{1}{x})^{2+\varepsilon}$ for any given $\varepsilon > 0$, but slower than $x^2 \log(\frac{1}{x})$. This notion of speed of convergence tells us precisely that h is between those two dimension functions (see Definition 2.1.3). More precisely, the author explicitly construct a small Kakeya set, which is small enough to have finite g measure for $g(x) = x^2 \log(\frac{1}{x})$. Therefore, for h to be an exact dimension function for the class of Kakeya sets, it cannot be dimensionally greater than g. But this last condition is not sufficient to ensure that *any* Kakeya set has positive h-measure. The partial result from this work is that for any $\varepsilon > 0$ and any Kakeya set E, we have that $\mathcal{H}^{h_\varepsilon}(E) > 0$, where $h_\varepsilon = x^2 \log(\frac{1}{x}) \log \log(\frac{1}{x})^{2+\varepsilon}$.

2.5 Some classical examples

There are some classical examples of constructions that we will need in the general scenario of dimension functions. We include the elementary proofs since we believe that they also contribute to the understanding of the subject and the techniques involved.

In this section we introduce the construction of sets of Cantor type in the spirit of [Fal03]. By studying two quantities, the number of children of a typical interval and some separation property, we give a criterion to determine if the constructed set has positive h dimensional mass for a test function $h \in \mathbb{H}$.

We will need a preliminary elemental lemma about concave functions. The proof is straightforward.

Lemma 2.5.1. *Let $h \in \mathbb{H}$ be a concave dimension function. Then*

$$\min\{a, b\} \le \frac{a}{h(a)} h(b) \quad \text{for any } a, b \in \mathbb{R}_+.$$

Proof. We consider two separate cases:

- If $b \ge a$ then

$$\frac{a}{h(a)} h(b) \ge \frac{a}{h(a)} h(a) = a = \min\{a, b\}.$$

- If $a > b$, then the concavity of h implies that

$$\frac{h(b)}{b} \ge \frac{h(a)}{a} \iff \frac{a}{h(a)} h(b) \ge \frac{b}{h(b)} h(b) = b = \min\{a, b\}.$$

□

The following lemma is a natural extension of the "Mass Distribution Principle" to the dimension function setting.

Lemma 2.5.2 (*h*-dimensional mass distribution principle). *Let $E \subseteq \mathbb{R}^n$ be a set, $h \in \mathbb{H}$ and μ a probability measure on E. Let $\varepsilon > 0$ and $c > 0$ be positive constants such that for any $U \subseteq \mathbb{R}^n$ with $\mathrm{diam}(U) < \varepsilon$ we have*

$$\mu(U) \leq ch(\mathrm{diam}(U)).$$

Then $\mathcal{H}^h(E) > 0$.

Proof. For any δ-covering we have

$$0 < \mu(E) \leq \mu\left(\sum_i \mu(U)\right) \leq c\sum_i h(\mathrm{diam}(U)).$$

Then $\mathcal{H}_\delta^h > \frac{\mu(E)}{c}$ and therefore $\mathcal{H}^h(E) > 0$.   □

Now we present the construction of a Cantor type set (see of Example 4.6 in [Fal03]).

Lemma 2.5.3. *Let $\{E_k\}$ be a decreasing sequence of closed subsets of the unit interval. Set $E_0 = [0,1]$ and suppose that the following conditions are satisfied:*

1. *Each E_k is a finite union of closed intervals I_j^k.*

2. *Each level $k - 1$ interval contains at least m_k intervals of level k. We will refer to this as the "children" of an interval.*

3. *The gaps between the intervals of level k are at least of size ε_k, with $0 < \varepsilon_{k+1} < \varepsilon_k$.*

*Let $E = \bigcap_k E_k$. Define, for a **concave** dimension function $h \in \mathbb{H}$, the quantity*

$$D_k^h := m_1 \cdot m_2 \cdots m_{k-1} h(\varepsilon_k m_k).$$

If $\varliminf_k D_k^h > 0$, then $\mathcal{H}^h(E) > 0$.

Proof. The idea is to use the version of the mass distribution principle from Lemma 2.5.2. Clearly we can assume that the property (2) of Lemma 2.5.3 holds for *exactly* m_k intervals. So we can define a mass distribution on E assigning a mass of $\frac{1}{m_1 \cdots m_k}$ to each of the $m_1 \cdots m_k$ intervals of level k.

Now, for any interval U with $0 < |U| < \varepsilon_1$, take k such $\varepsilon_k < |U| < \varepsilon_{k-1}$. We will estimate the number of intervals of level k that could have non-empty intersection with U. For that, we note the following:

- U intersects at most one I_j^{k-1}, since $|U| < \varepsilon_{k-1}$. Therefore it could intersect at most m_k children of I_j^{k-1}.

- Suppose now that U intersects L intervals of level k. Then it must contain $(L-1)$ gaps at least of size ε_k. Therefore, $L-1 \leq \frac{|U|}{\varepsilon_k}$. Consequently $|U|$ intersects at most $\frac{|U|}{\varepsilon_k} + 1 \leq 2\frac{|U|}{\varepsilon_k}$ intervals of level k.

From these two observations, we conclude that

$$\mu(U) \leq \frac{1}{m_1 \cdots m_k} \min\left\{ m_k, \frac{2|U|}{\varepsilon_k} \right\} = \frac{1}{m_1 \cdots m_k \varepsilon_k} \min\{\varepsilon_k m_k, 2|U|\}.$$

Now, by the concavity of h, we obtain

$$\min\{\varepsilon_k m_k, 2|U|\} \leq \frac{\varepsilon_k m_k}{h(\varepsilon_k m_k)} h(2|U|).$$

In addition (also by concavity), h is doubling, so $h(2|U|) \lesssim h(|U|)$ and then

$$\mu(U) \lesssim \frac{\varepsilon_k m_k h(|U|)}{m_1 \cdots m_k \varepsilon_k h(\varepsilon_k m_k)} = \frac{h(|U|)}{m_1 \cdots m_{k-1} h(\varepsilon_k m_k)} = \frac{h(|U|)}{D_k^h}.$$

Finally, if $\underline{\lim}_k D_k^h > 0$, there exists k_0 such $\frac{1}{D_k^h} \leq C$ for $k \geq k_0$ and we can use the mass distribution principle with C and $\varepsilon = \varepsilon_{k_0}$. $\square$

Remark 2.5.4. In the particular case of $h(x) = x^s$, $s \in (0,1)$ we recover the result of [Fal03], where the parameter s can be expressed in terms of the sequences m_k and ε_k. For the set constructed in Lemma 2.5.3, we have

$$\dim(E) \geq \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k)}. \tag{2.2}$$

Suppose that the right hand side is positive, otherwise the inequality is obvious, and call

$$D_k^s := m_1 \cdots m_{k-1} (\varepsilon_k m_k)^s.$$

If $0 < s < \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k)}$ then, for all k,

$$\log(D_k^s) = \log(m_1 \cdots m_{k-1}) + s \log(\varepsilon_k m_k) > 0.$$

From there we conclude that $\underline{\lim}_k D_k^s \geq 1$. This implies that $\mathcal{H}^s(E) > 0$ for all $0 < s < \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k)}$ and therefore we obtain (2.2).

Chapter 3

Lower bounds for Furstenberg sets

3.1 Introduction

In this chapter we deal with the problem of finding sharp lower bounds for the generalized dimension of Furstenberg type sets. Let us begin with the basic definitions and some remarks about this problem.

Definition 3.1.1. For α in $(0, 1]$, a subset E of $\mathbb{R}^2$ is called *Furstenberg set* or F_α-set if for each direction e in the unit circle there is a line segment ℓ_e in the direction of e such that the Hausdorff dimension of the set $E \cap \ell_e$ is equal or greater than α.

We will also say that such set E belongs to the class F_α. It is known ([Wol99b], see also [Wol99a], [Wol02], [Wol03], [KT02], [Tao01] for related topics and [KT01] for a discretized version of this problem) that for any F_α-set $E \subseteq \mathbb{R}^2$ the Hausdorff dimension must satisfy the inequality $\dim_H(E) \geq \max\{2\alpha, \alpha + \frac{1}{2}\}$. On the other hand, there are examples of F_α-sets E with $\dim_H(E) \leq \frac{1}{2} + \frac{3}{2}\alpha$.

If we denote by

$$\gamma(\alpha) = \inf\{\dim_H(E) : E \in F_\alpha\},$$

then

$$\max\{2\alpha; \frac{1}{2} + \alpha\} \leq \gamma(\alpha) \leq \frac{1}{2} + \frac{3}{2}\alpha, \qquad \alpha \in (0, 1]. \tag{3.1}$$

Originally, in [Fur70] Furstenberg dealt with the problem of transversality of sets. Briefly, two closed subsets $A, B \subset \mathbb{R}$ are called *transverse* if

$$\dim_H(A \cap B) \leq \max\{\dim_H A + \dim_H B - 1, 0\}.$$

In addition, they will be called *strongly transverse* if every translate $A + t$ of A is transverse to B. More generally, the problem of the transversality between the dilations uA of A and B was considered. In this case the relevant

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quantity is $\dim_H(uA + t \cap B)$. This is where the connection pops in, since the dimension of this intersection can be seen as the dimension of the set $(A \times B) \cap \ell_{ut}$, where the line ℓ_{ut} in $\mathbb{R}^2$ is defined by the equation $y = ux + t$. In addition, Furstenberg proves, with some *invariance* hypothesis on A and B, the following: if the product $A \times B$ intersect one (and it suffices with only one) line in some direction on a set of dimension at least α, then for almost all directions the set $A \times B$ intersects a line in that direction also in a set of dimension at least α. Therefore, in that case the product is an F_α-set. Hence, any non trivial lower bound on the class F_α implies a lower bound for the dimension of the product $A \times B$ in this particular case.

We mentioned earlier in the introduction that there is a connection between the Furstenberg problem and the Falconer and Erdös problems. We now make this more precise.

We begin with the formulation of the Falconer distance problem. For a compact set $K \subseteq \mathbb{R}^2$, define the *distance set* $\text{dist}(K)$ by

$$\text{dist}(K) := \{|x - y| : x, y \in K\}.$$

The conjecture here is that $\dim_H(\text{dist}(K)) = 1$ whenever $\dim_H(K) \geq 1$. In the direction of proving this conjecture, it was shown by Bourgain in [Bou94] that the conclusion holds for any K of $\dim_H(K) \geq \frac{13}{9}$, improved later by Wolff in [Wol99a] to $\dim_H(K) \geq \frac{4}{3}$. On the other hand, Mattila shows in [Mat87] that if we assume that $\dim_H(K) \geq 1$, then $\dim_H(\text{dist}(K)) \geq \frac{1}{2}$. One may ask if there is an absolute constant $c > 0$ such that $\dim_H(\text{dist}(K)) \geq \frac{1}{2} + c_0$ whenever K is compact and satisfies $\dim_H(K) \geq 1$.

The Erdös ring problem, roughly speaking, asks about the existence of a Borel subring R of $\mathbb{R}$ such that $0 < \dim_H(R) < 1$.

The connection has been established only for some discretized version of the above three problems (see [KT01]). Consider the special case of Furstenberg sets belonging to the $F_{\frac{1}{2}}$ class. Note that for this family the two lower bounds for the Hausdorff dimension of Furstenberg sets coincide to become $\gamma(\frac{1}{2}) \geq \frac{1}{2}$ (see Figure 3 in the introduction). Essentially, the existence of the constant c_0 in the Falconer distance problem mentioned above is equivalent to the existence of another constant c_1 such that any $F_{\frac{1}{2}}$-set E must have $\dim_H(E) \geq \frac{1}{2} + c_1$. In addition, any of this two conditions would imply the non existence of a Borel subring of R of Hausdorff dimension exactly $\frac{1}{2}$.

3.2   Furstenberg type sets - Statement of results

The natural generalization of the class of Furstenberg sets to the wider scenario of dimension functions is the following.

Definition 3.2.1. Let $\mathfrak{h}$ be a dimension function. A set $E \subseteq \mathbb{R}^2$ is a Furstenberg set of type $\mathfrak{h}$, or an $F_\mathfrak{h}$-set, if for each direction $e \in \mathbb{S}$ there is a line segment ℓ_e in the direction of e such that $\mathcal{H}^\mathfrak{h}(\ell_e \cap E) > 0$.

Note that this hypothesis is stronger than the one used to define the original Furstenberg F_α-sets. However, the hypothesis $\dim_H(E \cap \ell_e) \geq \alpha$ is equivalent to $\mathcal{H}^\beta(E \cap \ell_e) > 0$ for any β smaller than α. If we use the wider class of dimension functions introduced above, the natural way to define $F_\mathfrak{h}$-sets would be to replace the parameters $\beta < \alpha$ with two dimension functions satisfying the relation $h \prec \mathfrak{h}$. But requiring $E \cap \ell_e$ to have positive $\mathcal{H}^h$ measure for any $h \prec \mathfrak{h}$ implies that it has also positive $\mathcal{H}^\mathfrak{h}$ measure (Theorem 42, [Rog70]).

Due to the existence of F_α-sets with $\mathcal{H}^\alpha(E \cap \ell_e) = 0$ for each e, it will be useful to introduce the following subclass of F_α:

Definition 3.2.2. A set $E \subseteq \mathbb{R}^2$ is an F_α^+-set if for each $e \in \mathbb{S}$ there is a line segment ℓ_e such that $\mathcal{H}^\alpha(\ell_e \cap E) > 0$.

By analogy to the classical estimate (3.1), we first note that if $\mathfrak{h}$ is a general dimension function (not x^α), $\alpha + \frac{1}{2}$ translates to $\sqrt{\cdot}\mathfrak{h}$ and 2α to $\mathfrak{h}^2$. Hence, when aiming to obtain an estimate of the Hausdorff measure of our set E, the naive approach would be to prove that if a dimension function h satisfies

$$h \prec \mathfrak{h}^2 \qquad \text{or} \qquad h \prec \sqrt{\cdot}\mathfrak{h}, \tag{3.2}$$

then $\mathcal{H}^h(E) > 0$. However, there is no hope to obtain such a general result, since for the special case of the identity function $h(x) = x$, this requirement would contradict (again by Theorem 42, [Rog70]) the existence of zero measure planar Kakeya sets. Therefore, it is clear that one needs to take a step down from the conjectured dimension function. The results of this chapter allow to generalize the inequalities (3.1) to the previous analogues in the dimension functions setting. Moreover, we show that if E is a set in the class $F_\mathfrak{h}$, and $h(x)$ is a dimension function that is sufficiently smaller than $\mathfrak{h}^2(x)$, then $\mathcal{H}^h(E) > 0$ (Theorem 3.4.1). The same conclusion holds when h is sufficiently smaller than $\sqrt{x}\,\mathfrak{h}(x)$ (Theorem 3.5.4). The precise meaning of *sufficiency* used in this paragraph will be stated in terms of dimensional gaps between dimension functions. Our results show that this step does not need to be as big as a power. It can be, for example, just the power of a log. Precisely, we find conditions on the step that guarantee lower bounds on the dimension of $F_\mathfrak{h}$-sets. The size of these gaps are associated to the techniques involved and are sharp in the sense that those techniques cannot give further improvements.

Our results, applied to the classical setting, improve the bounds (3.1) for the F_α^+ classes sharpening the logarithmic gaps. In the language of chains presented in Chapter 2, we find "log like" chains to detect the precise dimension of the class of Furstenberg sets.

Further, our techniques allow us to analyze Furstenberg-type sets of Hausdorff dimension zero. This can be done considering dimension functions h that are smaller than x^α for any $\alpha > 0$.

3.3 Techniques

In this section we introduce some reductions of the problem and classical techniques. The following remark can be understood as a *uniformization* of

the hypothesis on the fibers.

Remark 3.3.1. Given an $F_{\mathfrak{h}}$-set E for some $\mathfrak{h} \in \mathbb{H}$, it is always possible to find two constants $m_E, \delta_E > 0$ and a set $\Omega_E \subseteq \mathbb{S}$ of positive σ-measure such that

$$\mathcal{H}_{\delta}^{\mathfrak{h}}(\ell_e \cap E) > m_E > 0 \qquad \forall \delta < \delta_E \quad , \quad \forall e \in \Omega_E.$$

For each $e \in \mathbb{S}$, there is a positive constant m_e such that $\mathcal{H}^{\mathfrak{h}}(\ell_e \cap E) > m_e$. Now consider the following pigeonholing argument. Let $\Lambda_n = \{e \in \mathbb{S} : \frac{1}{n+1} \leq m_e < \frac{1}{n}\}$. At least one of the sets must have positive measure, since $\mathbb{S} = \cup_n \Lambda_n$. Let Λ_{n_0} be such set and take $0 < 2m_E < \frac{1}{n_0+1}$. Hence $\mathcal{H}^{\mathfrak{h}}(\ell_e \cap E) > 2m_E > 0$ for all $e \in \Lambda_{n_0}$. Finally, again by pigeonholing, we can find $\Omega_E \subseteq \Lambda_{n_0}$ of positive measure and $\delta_E > 0$ such that

$$\mathcal{H}_{\delta}^{\mathfrak{h}}(\ell_e \cap E) > m_E > 0 \qquad \forall e \in \Omega_E \quad \forall \delta < \delta_E. \tag{3.3}$$

To simplify notation throughout the remainder of the chapter, since inequality (3.3) holds for any Furstenberg set and we will only use the fact that m_E, δ_E and $\sigma(\Omega_E)$ are positive, it will be enough to consider the following definition of $F_{\mathfrak{h}}$-sets:

Definition 3.3.2. Let $\mathfrak{h}$ be a dimension function. A set $E \subseteq \mathbb{R}^2$ is Furstenberg set of type $\mathfrak{h}$, or an $F_{\mathfrak{h}}$-set, if for each $e \in \mathbb{S}$ there is a line segment ℓ_e in the direction of e such that $\mathcal{H}_{\delta}^{\mathfrak{h}}(\ell_e \cap E) > 1$ for all $\delta < \delta_E$ for some $\delta_E > 0$.

The following technique is a standard procedure in this area. The lower bounds for the Hausdorff dimension of a given set E, both in the classical and general setting, are achieved by bounding uniformly from below the size of the coverings of E. More precisely, the h-size of a covering $\mathcal{B} = \{B_j\}$ is $\sum_j h(r_j)$. Our aim will be then to prove essentially that $\sum_j h(r_j) \gtrsim 1$, provided that h is a small enough dimension function.

We introduce the following notation:

Definition 3.3.3. Let $\mathfrak{b} = \{b_k\}_{k \in \mathbb{N}}$ be a decreasing sequence with $\lim b_k = 0$. For any family of balls $\mathcal{B} = \{B_j\}$ with $B_j = B(x_j; r_j)$, $r_j \leq 1$, and for any set E, we define

$$J_k^{\mathfrak{b}} := \{j \in \mathbb{N} : b_k < r_j \leq b_{k-1}\}, \tag{3.4}$$

and

$$E_k := E \cap \bigcup_{j \in J_k^{\mathfrak{b}}} B_j.$$

In the particular case of the dyadic scale $\mathfrak{b} = \{2^{-k}\}$, we will omit the superscript and denote

$$J_k := \{j \in \mathbb{N} : 2^{-k} < r_j \leq 2^{-k+1}\}. \tag{3.5}$$

The idea will be to use the dyadic partition of the covering to obtain that

$$\sum_j h(r_j) \gtrsim \sum_{k=0}^{\infty} h(2^{-k}) \# J_k.$$

The lower bounds we need will be obtained if we can prove lower bounds on the quantity J_k in terms of the function h but independent of the covering.

The next lemma introduces a technique we borrow from [Wol99b] to decompose the set of all directions.

Lemma 3.3.4. *Let E be an $F_{\mathfrak{h}}$-set for some $\mathfrak{h} \in \mathbb{H}$ and $\mathfrak{a} = \{a_k\}_{k \in \mathbb{N}} \in \ell^1$ a non-negative sequence. Let $\mathcal{B} = \{B_j\}$ be a δ-covering of E with $\delta < \delta_E$ and let E_k and J_k be as above. Define*

$$\Omega_k := \left\{ e \in \mathbb{S} : \mathcal{H}_\delta^{\mathfrak{h}}(\ell_e \cap E_k) \geq \frac{a_k}{2\|\mathfrak{a}\|_1} \right\}.$$

Then $\mathbb{S} = \cup_k \Omega_k$.

Proof. Clearly $\Omega_k \subset \mathbb{S}$. To see why $\mathbb{S} = \cup_k \Omega_k$, assume that there is a direction $e \in \mathbb{S}$ that is not in any of the Ω_k. Then for that direction we would have that

$$1 < \mathcal{H}_\delta^{\mathfrak{h}}(\ell_e \cap E) \leq \sum_k \mathcal{H}_\delta^{\mathfrak{h}}\left(\ell_e \cap E \cap \bigcup_{j \in J_k} B_j\right) \leq \sum_k \frac{1}{2\|\mathfrak{a}\|_1} = \frac{1}{2},$$

which is a contradiction. $\qquad\square$

3.3.1 Kakeya estimates

In this section we summarize some of the results needed regarding the Kakeya maximal function, which will be the main tool for the proof of Theorem 3.4.1. We also include a version of the Kakeya maximal inequality with respect to general measures on the circle that we will use in Chapter 4, Theorem 4.2.1.

For an integrable function f on $\mathbb{R}^n$, the Kakeya maximal function at scale δ will be $\mathcal{K}_\delta(f) : \mathbb{S}^{n-1} \to \mathbb{R}$,

$$\mathcal{K}_\delta(f)(e) = \sup_{x \in \mathbb{R}^n} \frac{1}{|T_e^\delta(x)|} \int_{T_e^\delta(x)} |f(x)| \, dx \qquad e \in \mathbb{S}^{n-1},$$

where $T_e^\delta(x)$ is a $1 \times \delta$-tube (by this we mean a tube of length 1 and cross section of radius δ) centered at x in the direction $e \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$. It is well known that in $\mathbb{R}^2$ the Kakeya maximal function satisfies the bound (see [Wol99b])

$$\left\|\mathcal{K}_\delta(f)\right\|_2^2 \lesssim \log(\frac{1}{\delta}) \|f\|_2^2. \tag{3.6}$$

It is also known that the log growth is necessary (see [Kei99]), because of the existence of Kakeya sets of zero measure in $\mathbb{R}^2$. We will also need in the next chapter the main result of [Mit02], which is the following proposition.

Proposition 3.3.5. *Let μ be a Borel probability measure on $\mathbb{S}$ such that $\mu(B(x,r)) \lesssim \varphi(r)$ for some non-negative function φ for all $r \ll 1$. Define the Kakeya maximal operator $\mathcal{K}_\delta$ as usual:*

$$\mathcal{K}_\delta(f)(e) = \sup_{x \in \mathbb{R}^n} \frac{1}{|T_e^\delta(x)|} \int_{T_e^\delta(x)} |f(x)| \ dx, \qquad e \in \mathbb{S}^{n-1}.$$

Then we have the estimate

$$\|\mathcal{K}_\delta\|^2_{L^2(\mathbb{R}^2) \to L^2(\mathbb{S}, d\mu)} \lesssim C(\delta) = \int_\delta^1 \frac{\varphi(u)}{u^2} du. \tag{3.7}$$

Remark 3.3.6. It should be noted that if we choose $\varphi(x) = x^s$, then we obtain as a corollary that

$$\|\mathcal{K}_\delta\|^2_{L^2(\mathbb{R}^2) \to L^2(\mathbb{S}, d\mu)} \lesssim \delta^{s-1}. \tag{3.8}$$

In the special case of $s = 1$, the bound has the known logarithmic growth:

$$\|\mathcal{K}_\delta\|^2_{L^2(\mathbb{R}^2) \to L^2(\mathbb{S}, d\mu)} \sim \log(\frac{1}{\delta}).$$

3.4   The $\mathfrak{h} \to \mathfrak{h}^2$ bound

In this section we generalize the first inequality of (3.1), that is, $\dim_H(E) \geq 2\alpha$ for any F_α-set. For this, given a dimension function $h \prec \mathfrak{h}^2$, we impose some sufficient growth conditions on the gap $\Delta(h, \mathfrak{h}^2)(x) := \frac{\mathfrak{h}^2(x)}{h(x)}$ to ensure that $\mathcal{H}^h(E) > 0$. We have the following theorem:

Theorem 3.4.1. *Let $\mathfrak{h} \in \mathbb{H}_d$ be a dimension function and let E be an $F_\mathfrak{h}$-set. Let $h \in \mathbb{H}$ such that $h \prec \mathfrak{h}^2$. If $\sum_k \sqrt{\Delta(h, \mathfrak{h}^2)(2^{-k})k} < \infty$, then $\mathcal{H}^h(E) > 0$.*

Proof. By Definition 3.3.2, since $E \in F_\mathfrak{h}$, we have $\mathcal{H}^\mathfrak{h}_\delta(\ell_e \cap E) > 1$ for all $e \in \mathbb{S}$ and for any $\delta < \delta_E$.

Let $\{B_j\}_{j \in \mathbb{N}}$ be a covering of E by balls with $B_j = B(x_j; r_j)$. We need to bound $\sum_j h(2r_j)$ from below. Since h is non-decreasing, it suffices to obtain the bound

$$\sum_j h(r_j) \gtrsim 1 \tag{3.9}$$

for any $h \in \mathbb{H}$ satisfying the hypothesis of the theorem. Clearly we can restrict ourselves to δ-coverings with $\delta < \frac{\delta_E}{5}$.

Define $\mathfrak{a} = \{a_k\}$ with $a_k = \sqrt{\Delta(h, \mathfrak{h}^2)(2^{-k})k}$. By hypothesis, $\mathfrak{a} \in \ell^1$. Also define, as in the previous section, for each $k \in \mathbb{N}$, $J_k = \{j \in \mathbb{N} : 2^{-k} < r_j \leq 2^{-k+1}\}$ and $E_k = E \cap \cup_{j \in J_k} B_j$. Since $\mathfrak{a} \in \ell^1$, we can apply Lemma 3.3.4 to obtain the decomposition $\mathbb{S} = \bigcup_k \Omega_k$ associated to this choice of $\mathfrak{a}$.

We will apply the maximal function inequality to a weighted union of indicator functions. For each k, let $F_k = \bigcup_{j \in J_k} B_j$ and define the function

$$f := \mathfrak{h}(2^{-k})2^k \chi_{F_k}.$$

We will use the L^2 norm estimates for the maximal function. The L^2 norm of f can be easily estimated as follows:

$$
\begin{aligned}
\|f\|_2^2 &= \mathfrak{h}^2(2^{-k})2^{2k} \int_{\cup_{J_k} B_j} dx \\
&\lesssim \mathfrak{h}^2(2^{-k})2^{2k} \sum_{j \in J_k} r_j^2 \\
&\lesssim \mathfrak{h}^2(2^{-k})\#J_k,
\end{aligned}
$$

since $r_j \leq 2^{-k+1}$ for $j \in J_k$. Hence,

$$\|f\|_2^2 \lesssim \#J_k \mathfrak{h}^2(2^{-k}). \tag{3.10}$$

Now fix k and consider the Kakeya maximal function $\mathcal{K}_\delta(f)$ of level $\delta = 2^{-k+1}$ associated to the function f defined for this value of k.

In Ω_k we have the following pointwise lower estimate for the maximal function. Let ℓ_e be the line segment such that $\mathcal{H}_\delta^{\mathfrak{h}}(\ell_e \cap E) > 1$, and let T_e be the rectangle of width 2^{-k+2} around this segment. Define, for each $e \in \Omega_k$,

$$J_k(e) := \{j \in J_k : \ell_e \cap E \cap B_j \neq \emptyset\}.$$

With the aid of the Vitali covering lemma, we can select a subset of disjoint balls $\widetilde{J}_k(e) \subseteq J_k(e)$ such that

$$\bigcup_{j \in J_k(e)} B_j \subseteq \bigcup_{j \in \widetilde{J}_k(e)} B(x_j; 5r_j).$$

Note that every ball B_j, $j \in J_k(e)$, intersects ℓ_e and therefore at least half of B_j is contained in the rectangle T_e, yielding $|T_e \cap B_j| \geq \frac{1}{2}\pi r_j^2$. Hence, by definition of the maximal function, using that $r_j \geq 2^{-k+1}$ for $j \in J_k(e)$,

$$
\begin{aligned}
|\mathcal{K}_{2^{-k+1}}(f)(e)| &\geq \frac{1}{|T_e|} \int_{T_e} f \, dx = \frac{\mathfrak{h}(2^{-k})2^k}{|T_e|} \left|T_e \cap \cup_{J_k(e)} B_j\right| \\
&\gtrsim \mathfrak{h}(2^{-k})2^{2k} \left|T_e \cap \cup_{\widetilde{J}_k(e)} B_j\right| \\
&\gtrsim \mathfrak{h}(2^{-k})2^{2k} \sum_{j \in \widetilde{J}_k(e)} r_j^2 \\
&\gtrsim \mathfrak{h}(2^{-k})\#\widetilde{J}_k(e) \\
&\gtrsim \sum_{\widetilde{J}_k(e)} \mathfrak{h}(r_j).
\end{aligned}
$$

Now, since

$$\ell_e \cap E_k \subseteq \bigcup_{j \in J_k(e)} B_j \subseteq \bigcup_{j \in \widetilde{J}_k(e)} B(x_j; 5r_j)$$

and for $e \in \Omega_k$ we have $\mathcal{H}_\delta^{\mathfrak{h}}(\ell_e \cap E_k) \gtrsim a_k$, we obtain

$$|\mathcal{K}_{2^{-k+1}}(f)(e)| \gtrsim \sum_{\widetilde{J}_k(e)} \mathfrak{h}(r_j) \gtrsim \sum_{j \in \widetilde{J}_k(e)} \mathfrak{h}(5r_j) \gtrsim a_k.$$

Therefore we have the estimate

$$\|\mathcal{K}_{2^{-k+1}}(f)\|_2^2 \gtrsim \int_{\Omega_k} |f_{2^{-k+1}}^*(e)|^2 \, d\sigma \gtrsim a_k^2 \, \sigma(\Omega_k) = \sigma(\Omega_k) k \Delta(h, \mathfrak{h}^2)(2^{-k}). \tag{3.11}$$

Combining (3.10), (3.11) and using the maximal inequality (3.6), we obtain

$$\sigma(\Omega_k) k \Delta(h, \mathfrak{h}^2)(2^{-k}) \lesssim \|f_{2^{-k+1}}^*\|_2^2 \lesssim \log(2^k) \|f\|_2^2 \lesssim k \# J_k \mathfrak{h}^2(2^{-k}),$$

and therefore

$$\frac{\sigma(\Omega_k)}{h(2^{-k})} \lesssim \# J_k.$$

Now we are able to estimate the sum in (3.9). Let h be a dimension function satisfying the hypothesis of Theorem 3.4.1. We have

$$\begin{aligned} \sum_j h(r_j) &\geq \sum_k h(2^{-k}) \# J_k \\ &\gtrsim \sum_k \sigma(\Omega_k) \geq \sigma(\mathbb{S}) > 0. \end{aligned}$$

$\square$

Applying this theorem to the class F_α^+, we obtain a sharper lower bound on the generalized Hausdorff dimension:

Corollary 3.4.2. *Let E an F_α^+-set. If h is any dimension function satisfying the relation $h(x) \geq C x^{2\alpha} \log^{1+\theta}(\frac{1}{x})$ for $\theta > 2$ then $\mathcal{H}^h(E) > 0$.*

Remark 3.4.3. At the endpoint $\alpha = 1$, this estimate is worse than the one due to Keich. He obtained, using strongly the full dimension of a ball in $\mathbb{R}^2$, that if E is an F_1^+-set and h is a dimension function satisfying the bound $h(x) \geq C x^2 \log(\frac{1}{x}) \left(\log\log(\frac{1}{x})\right)^\theta$ for $\theta > 2$, then $\mathcal{H}^h(E) > 0$.

Remark 3.4.4. Note that the proof above relies essentially on the L^1 and L^2 size of the ball in $\mathbb{R}^2$, not on the dimension function $\mathfrak{h}$. Moreover, we only use the "gap" between h and $\mathfrak{h}^2$ (measured by the function $\Delta(h, \mathfrak{h}^2)$). This last observation leads to conjecture that this proof can not be used to prove that an $F_{\mathfrak{h}}$-set has positive $\mathfrak{h}^2$ measure, since in the case of $\mathfrak{h}(x) = x$, as we remarked in the introduction, this would contradict the existence of Kakeya sets of zero measure in $\mathbb{R}^2$.

Also note that the absence of conditions on the function $\mathfrak{h}$ allows us to consider the "zero dimensional" Furstenberg problem. However, this bound does not provide any substantial improvement, since the zero dimensionality property of the function $\mathfrak{h}$ is shared by the function $\mathfrak{h}^2$. This is because the proof above, in the case of the F_α-sets, gives the worse bound $(\dim_H(E) \geq 2\alpha)$ when the parameter α is in $(0, \frac{1}{2})$.

3.5 The $\mathfrak{h} \to \mathfrak{h}\sqrt{\cdot}$ bound, positive dimension

In this section we will turn our attention to those functions h that satisfy the bound $h(x) \lesssim x^\alpha$ for $\alpha \leq \frac{1}{2}$. For these functions we are able to improve on the previously obtained bounds. We need to impose some growth conditions on the dimension function $\mathfrak{h}$. This conditions can be thought of as imposing a lower bound on the dimensionality of $\mathfrak{h}$ to keep it away from the zero dimensional case.

Remark 3.5.1. Throughout this section, the expected dimension function should be about $\mathfrak{h}\sqrt{\cdot}$. We therefore need a step down from this function. For this, we will look at the gap $\Delta(h, \mathfrak{h})(x) = \frac{\mathfrak{h}(x)}{h(x)}$ for a test function $h \prec \mathfrak{h}$. The reason for us to only consider $\mathfrak{h}$ and h for the definition of the gap is because we are interested in very small gaps, of logarithmic order or even smaller, and therefore the relevant information is contained in the dimensionality of $\mathfrak{h}$.

The next lemma says that we can split the $\mathfrak{h}$-dimensional mass of a set E contained in an interval I into two sets that are positively separated.

Lemma 3.5.2. *Let $\mathfrak{h} \in \mathbb{H}$, $\delta > 0$, I an interval and $E \subseteq I$. Let $\eta > 0$ be such that $\mathfrak{h}^{-1}(\frac{\eta}{8}) < \delta$ and $\mathcal{H}_\delta^{\mathfrak{h}}(E) \geq \eta > 0$. Then there exist two subintervals I^-, I^+ that are $\mathfrak{h}^{-1}(\frac{\eta}{8})$-separated and with $\mathcal{H}_\delta^{\mathfrak{h}}(I^\pm \cap E) \gtrsim \eta$.*

Proof. Let $t = \mathfrak{h}^{-1}(\frac{\eta}{8})$ and subdivide I in N ($N \geq 3$) consecutive (by that we mean that they intersect only at endpoints and leave no gaps between them) subintervals I_j such that $|I_j| = t$ for $1 \leq j \leq N-1$ and $|I_N| \leq t$. Since $|I_j| < \delta$ and $\mathfrak{h}(|I_j|) \leq \frac{\eta}{8}$, we have

$$\mathcal{H}_\delta^{\mathfrak{h}}(E \cap I_j) \leq \mathfrak{h}(|I_j|) \leq \frac{\eta}{8} \tag{3.12}$$

and

$$\eta \leq \mathcal{H}_\delta^{\mathfrak{h}}(E) = \mathcal{H}_\delta^{\mathfrak{h}}\left(\bigcup_j E \cap I_j\right) \leq \sum_j \mathcal{H}_\delta^{\mathfrak{h}}(E \cap I_j).$$

Now we can group the subintervals in the following way. Let n be the first index for which we have $\sum_{j=1}^n \mathcal{H}_\delta^{\mathfrak{h}}(E \cap I_j) > \frac{\eta}{4}$.

Since $\sum_{j=1}^{n-1} \mathcal{H}_\delta^{\mathfrak{h}}(E \cap I_j) \leq \frac{\eta}{4}$, and by (3.12) the mass of each interval is not too large, we have the bound

$$\frac{\eta}{4} < \sum_{j=1}^n \mathcal{H}_\delta^{\mathfrak{h}}(E \cap I_j) \leq (\frac{1}{4} + \frac{1}{8})\eta = \frac{3\eta}{8}.$$

Take $I^- = I_1 \cup \cdots \cup I_n$, skip the interval I_{n+1}, and consider I^+ to be the union of the remaining intervals. It is easy to see that

$$\sum_{j=1}^{n+1} \mathcal{H}_\delta^\mathfrak{h}(E \cap I_j) \leq \frac{\eta}{2},$$

and therefore

$$\sum_{j=n+2}^{N} \mathcal{H}_\delta^\mathfrak{h}(E \cap I_j) \geq \frac{\eta}{2}.$$

So, we obtain $\mathcal{H}_\delta^\mathfrak{h}(I^\pm \cap E) \geq \frac{\eta}{4}$ and the intervals I^- and I^+ are $|I_j|$-separated. But $|I_j| = \mathfrak{h}^{-1}(\frac{\eta}{8})$, so the lemma is proved. $\qquad\square$

The next lemma will provide estimates for the number of lines with certain separation property that intersect two balls of a given size.

Lemma 3.5.3. *Let $\mathfrak{b} = \{b_k\}_{k \in \mathbb{N}}$ be a decreasing sequence with $\lim b_k = 0$. Given a family of balls $\mathcal{B} = \{B(x_j; r_j)\}$, we define $J_k^\mathfrak{b}$ as in (3.4) and let $\{e_i\}_{i=1}^{M_k}$ be a b_k-separated set of directions. Assume that for each i there are two line segments $I_{e_i}^+$ and $I_{e_i}^-$ lying on a line in the direction e_i that are s_k-separated for some given s_k Define $\Pi_k = J_k^\mathfrak{b} \times J_k^\mathfrak{b} \times \{1, .., M_k\}$ and $\mathcal{L}_k^\mathfrak{b}$ by*

$$\mathcal{L}_k^\mathfrak{b} := \left\{ (j_+, j_-, i) \in \Pi_k : I_{e_i}^- \cap B_{j_-} \neq \emptyset \; I_{e_i}^+ \cap B_{j_+} \neq \emptyset \right\}.$$

If $\frac{1}{5}s_k > b_{k-1}$ for all k, then

$$\#\mathcal{L}_k^\mathfrak{b} \lesssim \frac{b_{k-1}}{b_k} \frac{1}{s_k} \left(\# J_k^\mathfrak{b} \right)^2.$$

Proof. Consider a fixed pair j_-, j_+ and its associated B_{j_-} and B_{j_+} We will use as distance between two balls the distance between the centers, and for simplicity we denote $d(j_-, j_+) = d(B_{j_-}, B_{j_+})$. If $d(j_-, j_+) < \frac{3}{5}s_k$ then there is no i such that (j_-, j_+, i) belongs to $\mathcal{L}_k^\mathfrak{b}$.

Now, for $d(j_-, j_+) \geq \frac{3}{5}s_k$, we will look at the special configuration given by Figure 3.1 when we have $r_{j_-} = r_{j_+} = b_{k-1}$ and the balls are tangent to the ends of I^- and I^+. This will give a bound for any possible configuration, since in any other situation the cone of allowable directions is narrower.
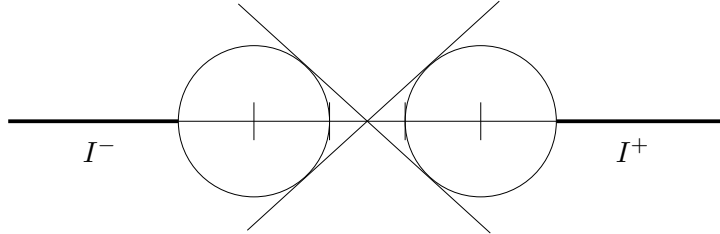

Figure 3.1: Cone of allowable directions I

Let us focus on one half of the cone (Figure 3.2). Let θ be the width of the cone. In this case, we have to look at $\frac{\theta}{b_k}$ directions that are b_k-separated. Further, we note that $\theta = \frac{2\theta_k}{s_k}$, where θ_k is the bold arc at distance $s_k/2$ from the center of the cone.
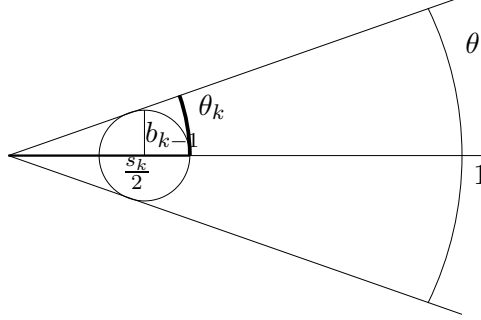

Figure 3.2: Cone of allowable directions II

Let us see that $\theta_k \sim b_{k-1}$. If we use the notation of Figure 3.3, we have to prove that $\theta_k \lesssim b_{k-1}$ for $a \in (0, +\infty)$. We have $\theta_k = \theta(a + 2b_{k-1})$. Also $\theta < \tan^{-1}(\frac{b_{k-1}}{a})$, so

$$\theta_k < \tan^{-1}(\frac{b_{k-1}}{a})(a + 2b_{k-1}) \sim b_{k-1}.$$
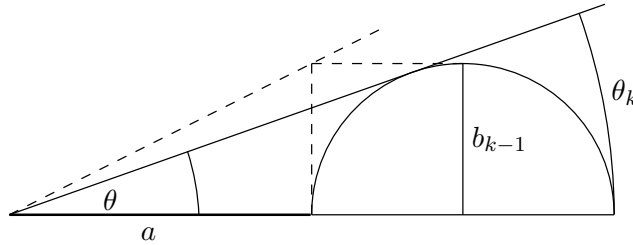
Figure 3.3: The arc θ_k is comparable to b_{k-1}

We conclude that $\theta_k \sim b_{k-1}$, and therefore the number D of lines in b_k-separated directions with non-empty intersection with B_{j_-} and B_{j_+} has to satisfy $D \le \frac{\theta}{b_k} = \frac{2\theta_k}{s_k b_k} \sim \frac{b_{k-1}}{b_k} \frac{1}{s_k}$.

The lemma follows by summing on all pairs (j_-, j_+). $\qquad\qquad\square$

Now we can prove the main result of this section. We have the following theorem:

Theorem 3.5.4. *Let $\mathfrak{h} \in \mathbb{H}_d$ be a dimension function such that $\mathfrak{h}(x) \lesssim x^\alpha$ for some $0 < \alpha < 1$ and E be an $F_{\mathfrak{h}}$-set. Let $h \in \mathbb{H}$ with $h \prec \mathfrak{h}$ and recall that $\Delta(h, \mathfrak{h}) = \frac{\mathfrak{h}}{h}$. If $\sum_k \Delta(h, \mathfrak{h})(2^{-k})^{\frac{2\alpha}{2\alpha+1}} < \infty$, then $\mathcal{H}^{h\sqrt{\cdot}}(E) > 0$.*

Proof. We begin in the same way as in the previous section. Again by Definition 3.3.2, since $E \in F_\mathfrak{h}$, we have $\mathcal{H}_\delta^\mathfrak{h}(\ell_e \cap E) > 1$ for all $e \in \mathbb{S}$ for any $\delta < \delta_E$.

Consider the sequence $\mathfrak{a} = \left\{ \Delta(h, \mathfrak{h})^{\frac{2\alpha}{2\alpha+1}}(2^{-k}) \right\}_k$. Let k_0 be such that

$$\mathfrak{h}^{-1}\left(\frac{a_k}{16\|\mathfrak{a}\|_1} \right) < \delta_E \qquad \text{for any } k \geq k_0. \tag{3.13}$$

Now take any δ-covering $\mathcal{B} = \{B_j\}$ of E by balls with $\delta < \min\{\delta_E, 2^{-k_0}\}$. Using Lemma 3.3.4 we obtain $\mathbb{S} = \bigcup_k \Omega_k$ with

$$\Omega_k = \left\{ e \in \Omega : \mathcal{H}_\delta^\mathfrak{h}(\ell_e \cap E_k) \geq \frac{a_k}{2\|\mathfrak{a}\|_1} \right\}. \tag{3.14}$$

Again we have $E_k = E \cap \bigcup_{j \in J_k} B_j$, but by our choice of δ, the sets E_k are empty for $k < k_0$. Therefore the same holds trivially for Ω_k and we have that $\mathbb{S} = \bigcup_{k \geq k_0} \Omega_k$.

The following argument is Remark 1.5 in [Wol99b]. Since for each $e \in \Omega_k$ we have (3.13), we can apply Lemma 3.5.2 with $\eta = \frac{a_k}{2\|\mathfrak{a}\|_1}$ to $\ell_e \cap E_k$. Therefore we obtain two intervals I_e^- and I_e^+, contained in ℓ_e with

$$\mathcal{H}_\delta^\mathfrak{h}(I_e^\pm \cap E_k) \gtrsim a_k$$

that are $\mathfrak{h}^{-1}(ra_k)$-separated for $r = \frac{1}{16\|\mathfrak{a}\|_1}$.

Let $\{e_j^k\}_{j=1}^{M_k}$ be a 2^{-k}-separated subset of Ω_k. Therefore $M_k \gtrsim 2^k \sigma(\Omega_k)$. Define $\Pi_k := J_k \times J_k \times \{1, .., M_k\}$ and

$$\mathcal{T}_k := \left\{ (j_-, j_+, i) \in \Pi_k : I_{e_i}^- \cap E_k \cap B_{j_-} \neq \emptyset \ I_{e_i}^+ \cap E_k \cap B_{j_+} \neq \emptyset \right\}. \tag{3.15}$$

We will count the elements of $\mathcal{T}_k$ in two different ways.

First, fix j_- and j_+ and count for how many values of i the triplet (j_-, j_+, i) belongs to $\mathcal{T}_k$.

For this, we will apply Lemma 3.5.3 for the choice $\mathfrak{b} = \{2^{-k}\}$. The estimate we obtain is the number of 2^{-k}-separated directions e_i, that intersect simultaneously the balls B_{j_-} and B_{j_+}, given that these balls are separated. We obtain

$$\#\mathcal{T}_k \lesssim \frac{1}{\mathfrak{h}^{-1}(ra_k)} (\#J_k)^2. \tag{3.16}$$

Second, fix i. In this case, we have by hypothesis that $\mathcal{H}_\delta^\mathfrak{h}(I_{e_i}^+ \cap E_k) \gtrsim a_k$, so $\sum_{j_+} \mathfrak{h}(r_{j_+}) \gtrsim a_k$. Therefore,

$$a_k \lesssim \sum_{(j_-, j_+, i) \in \mathcal{T}_k} \mathfrak{h}(r_{j_+}) \leq K\mathfrak{h}(2^{-k}),$$

where K is the number of elements of the sum. Therefore $K \gtrsim \frac{a_k}{\mathfrak{h}(2^{-k})}$. The same holds for j_-, so

$$\#\mathcal{T}_k \gtrsim M_k \left(\frac{a_k}{\mathfrak{h}(2^{-k})} \right)^2. \tag{3.17}$$

Combining the two bounds,

$$
\begin{aligned}
\#J_k &\gtrsim (\#\mathcal{T}_k)^{1/2}\mathfrak{h}^{-1}(ra_k)^{1/2} \\
&\gtrsim M_k^{1/2}\frac{a_k}{\mathfrak{h}(2^{-k})}\mathfrak{h}^{-1}(ra_k)^{1/2} \\
&\gtrsim 2^{\frac{k}{2}}\sigma(\Omega_k)^{1/2}\frac{a_k}{\mathfrak{h}(2^{-k})}\mathfrak{h}^{-1}(ra_k)^{1/2}.
\end{aligned}
$$

Consider now a dimension function $h \prec \mathfrak{h}$ as in the hypothesis of the theorem. Then again

$$
\begin{aligned}
\sum_j h(r_j)r_j^{1/2} &\geq \sum_k \mathfrak{h}(2^{-k})\frac{2^{-\frac{k}{2}}\#J_k}{\Delta(h,\mathfrak{h})(2^{-k})} \\
&\gtrsim \sum_{k\geq k_0}\sigma(\Omega_k)^{1/2}\frac{a_k\mathfrak{h}^{-1}(ra_k)^{1/2}}{\Delta(h,\mathfrak{h})(2^{-k})}.
\end{aligned}
\tag{3.18}
$$

To bound this last expression, we use first that there exists $\alpha \in (0,1)$ with $\mathfrak{h}(x) \lesssim x^\alpha$ and therefore $\mathfrak{h}^{-1}(x) \gtrsim x^{\frac{1}{\alpha}}$. We then recall the definition of the sequence $\mathfrak{a}$, $a_k = \Delta(h,\mathfrak{h})(2^{-k})^{\frac{2\alpha}{1+2\alpha}}$ to obtain

$$
\begin{aligned}
\sum_j h(r_j)r_j^{1/2} &\gtrsim \sum_{k\geq k_0}\sigma(\Omega_k)^{1/2}\frac{a_k^{\frac{1+2\alpha}{2\alpha}}}{\Delta(h,\mathfrak{h})(2^{-k})} \\
&= \sum_{k\geq k_0}\sigma(\Omega_k)^{1/2} \gtrsim 1.
\end{aligned}
\tag{3.19}
$$

$\square$

The next corollary follows from Theorem 3.5.4 in the same way as Corollary 3.4.2 follows from Theorem 3.4.1.

Corollary 3.5.5. *Let E be an F_α^+-set. If h is a dimension function satisfying the relation $h(x) \geq Cx^\alpha\sqrt{x}\log^\theta(\frac{1}{x})$ for $\theta > \frac{1+2\alpha}{2\alpha}$ then $\mathcal{H}^h(E) > 0$.*

Remark 3.5.6. Note that at the critical value $\alpha = \frac{1}{2}$, we can compare Corollary 3.4.2 and Corollary 3.5.5. The first says that in order to obtain $\mathcal{H}^h(E) > 0$ for an $F_{\frac{1}{2}}^+$-set E it is sufficient to require that the dimension function h satisfies the bound $h(x) \geq Cx\log^\theta(\frac{1}{x})$ for $\theta > 3$. On the other hand, the latter says that it is sufficient that h satisfies the bound $h(x) \geq x\log^\theta(\frac{1}{x})$ for $\theta > 2$. In both cases we prove that an $F_{\frac{1}{2}}^+$-set must have Hausdorff dimension at least 1, but Corollary 3.5.5 gives a better estimate on the logarithmic gap.

3.6   The $\mathfrak{h} \to \mathfrak{h}\sqrt{\cdot}$ bound, dimension zero

In this section we look at a class of very small Furstenberg sets. We will study, roughly speaking, the extremal case of F_0-sets and ask ourselves if inequality (3.1) can be extended to this class. According to the definition of F_α-sets, this class should be the one formed by sets having a zero dimensional linear set in every direction. Recall that we refer to a dimension function $\mathfrak{h}$ as "zero dimensional" if $\mathfrak{h} \prec x^\alpha$ for all $\alpha > 0$.

Our approach to the problem, using dimension functions, allows us to tackle the problem about the dimensionality of these sets in some cases. We study the case of $F_\mathfrak{h}$-sets associated to one particular choice of $\mathfrak{h}$. We will look at the function $\mathfrak{h}(x) = \dfrac{1}{\log(\frac{1}{x})}$ as a model of "zero dimensional" dimension function. Our next theorem will show that in this case inequality (3.1) can indeed be extended. The trick here will be to replace the dyadic scale on the radii in J_k with a faster decreasing sequence $\mathfrak{b} = \{b_k\}_{k \in \mathbb{N}}$.

The main difference will be in the estimate of the quantity of lines in b_k-separated directions that intersect two balls of level J_k with a fixed distance s_k between them. This estimate is given by Lemma 3.5.3.

In this case we will choose a particular sequence a_k. It will be sufficient to take $a_k = \frac{1}{k^2}$ so if we take a sequence $\mathfrak{a} = \{a_k\}_{k \in \mathbb{N}}$ going to zero slowly than $\frac{1}{k^2}$, then $a_k k^2 \geq 1$ for almost all k and then, for any $h \in \mathbb{H}$,

$$h^{-1}(a_k) = h^{-1}(\frac{1}{k^2} a_k k^2) \geq h^{-1}(\frac{1}{k^2}).$$

Note that the problem in the above bound is the rapid decay of $\mathfrak{h}^{-1}$, which is solved by the positivity assumption. In this case, since we are dealing with a zero dimensional function $\mathfrak{h}$, the inverse involved decays dramatically to zero. Therefore the strategy cannot be the same as before, where we choose optimally the sequence $\mathfrak{a}$. In this case, we will obtain a result by choosing an appropriate sequence of scales. It follows then that it will be enough if we can prove the theorem with $\mathfrak{a} = \{\frac{1}{k^2}\}$.

We can prove the next theorem, which provide a class of examples of zero dimensional $F_\mathfrak{h}$-sets.

Theorem 3.6.1. *Let* $\mathfrak{h}(x) = \frac{1}{\log(\frac{1}{x})}$ *and let* E *be an* $F_\mathfrak{h}$*-set. Then* $\dim_H(E) \geq \frac{1}{2}$.

Proof. Take a non-negative sequence $\mathfrak{b}$ which will be determined later. We will apply the splitting Lemma 3.5.2 as in the previous section. For this, take k_0 as in (3.13) associated to the sequence $\mathfrak{a} = \{k^{-2}\}_{k \in \mathbb{N}}$. Now, for a given generic δ-covering of E with $\delta < \min\{\delta_E, 2^{-k_0}\}$, we use Lemma 3.3.4 to obtain a decomposition $\mathbb{S} = \bigcup_{k \geq k_0} \Omega_k$ with

$$\Omega_k = \left\{ e \in \mathbb{S} : \mathcal{H}_\delta^\mathfrak{h}(\ell_e \cap E_k) \geq ck^{-2} \right\},$$

where $E_k = E \cap \bigcup_{J_k^{\mathfrak{b}}} B_j$, $J_k^{\mathfrak{b}}$ is the partition of the radii associated to $\mathfrak{b}$ and $c > 0$ is a suitable constant.

We apply the splitting Lemma 3.5.2 to $\ell_e \cap E_k$ to obtain two $\mathfrak{h}^{-1}(ck^{-2})$-separated intervals I_e^- and I_e^+ with $\mathcal{H}_\delta^{\mathfrak{h}}(I_e^\pm \cap E_k) \gtrsim k^{-2}$.

Now, let $\{e_j^k\}_{j=1}^{M_k}$ be a b_k-separated subset of Ω_k. Therefore $M_k \gtrsim \Omega_k / b_k$.

We also define, as in Theorem 3.5.4, $\Pi_k := J_k^{\mathfrak{b}} \times J_k^{\mathfrak{b}} \times \{1, .., M_k\}$ and

$$\mathcal{T}_k^{\mathfrak{b}} := \left\{ (j_-, j_+, i) \in \Pi_k : I_{e_i}^- \cap E_k \cap B_{j_-} \neq \emptyset \ I_{e_i}^+ \cap E_k \cap B_{j_+} \neq \emptyset \right\}.$$

By Lemma 3.5.3, we obtain

$$\#\mathcal{T}_k^{\mathfrak{b}} \lesssim \frac{b_{k-1}}{b_k} \frac{1}{\mathfrak{h}^{-1}(ck^{-2})} (\#J_k^{\mathfrak{b}})^2, \tag{3.20}$$

and the same calculations as in Theorem 3.5.4 (inequality (3.17)) yield

$$\#J_k^{\mathfrak{b}} \gtrsim \left(\frac{\sigma(\Omega_k)}{b_{k-1}} \right)^{1/2} \frac{\mathfrak{h}^{-1}(ck^{-2})^{1/2}}{k^2 \mathfrak{h}(b_{k-1})} \geq \left(\frac{\sigma(\Omega_k)}{b_{k-1}} \right)^{1/2} \frac{e^{-ck^2}}{k^2}.$$

Now we estimate a sum like (3.18). For $\beta < \frac{1}{2}$ we have

$$\begin{aligned}
\sum_j r_j^\beta &\geq \sum_k b_k^\beta \#J_k \\
&\geq \sum_k \sigma(\Omega_k)^{1/2} \frac{b_k^\beta}{b_{k-1}^{\frac{1}{2}}} \frac{e^{-ck^2}}{k^2} \\
&\gtrsim \sqrt{\sum_k \sigma(\Omega_k) \frac{b_k^{2\beta}}{b_{k-1}} \frac{1}{e^{ck^2} k^4}}. \tag{3.21}
\end{aligned}$$

In the last inequality we use that the terms are all non-negative. The goal now is to take some rapidly decreasing sequence such that the factor $\frac{b_k^{2\beta}}{b_{k-1}}$ beats the factor $k^{-4} e^{-ck^2}$.

Let us take $0 < \varepsilon < \frac{1-2\beta}{2\beta}$ and consider the hyperdyadic scale $b_k = 2^{-(1+\varepsilon)^k}$. With this choice, we have

$$\frac{b_k^{2\beta}}{b_{k-1}} = 2^{(1+\varepsilon)^{k-1} - (1+\varepsilon)^k 2\beta} = 2^{(1+\varepsilon)^k (\frac{1}{1+\varepsilon} - 2\beta)}.$$

Replacing this in inequality (3.21) we obtain

$$\begin{aligned}
\left(\sum_j r_j^\beta \right)^2 &\geq \sum_k \sigma(\Omega_k) 2^{(1+\varepsilon)^k (\frac{1}{1+\varepsilon} - 2\beta)} \frac{e^{-ck^2}}{k^4} \\
&\geq \sum_k \sigma(\Omega_k) \frac{2^{(1+\varepsilon)^k (\frac{1}{1+\varepsilon} - 2\beta)}}{e^{ck^2} k^4}.
\end{aligned}$$

Finally, since by the positivity of $\frac{1}{1+\varepsilon} - 2\beta$ the double exponential in the numerator grows much faster than the denominator, we obtain

$$\frac{2^{(1+\varepsilon)^k(\frac{1}{1+\varepsilon}-2\beta)}}{e^{ck^2}k^4} \gtrsim 1,$$

and therefore $\left(\sum_j r_j^\beta\right)^2 \gtrsim \sum_k \sigma(\Omega_k) \gtrsim 1$

$\square$

Corollary 3.6.2. *Let $\theta > 0$. If E is an $F_{\mathfrak{h}}$-set with $\mathfrak{h}(x) = \frac{1}{\log^\theta(\frac{1}{x})}$ then* $\dim_H(E) \geq \frac{1}{2}$.

Proof. This follows immediately, since in this case the only change will be $\mathfrak{h}^{-1}(ck^{-2}) = \frac{1}{e^{(ck^2)^{\frac{1}{\theta}}}}$, so the double exponential still grows faster and therefore

$$\frac{2^{(1+\varepsilon)^k(\frac{1}{1+\varepsilon}-2\beta)}}{e^{(ck^2)^{\frac{1}{\theta}}}k^4} \gtrsim 1.$$

$\square$

This shows that there is a whole class of F_0-sets that must be at least $\frac{1}{2}$-dimensional.

3.7   Examples & Further remarks

We include in this sections some examples and remarks concerning the discussion made in the previous section about the change of scale from a dyadic to an hyperdyadic regime.

3.7.1   The dyadic scale does not work for dimension zero.

Suppose that the dimension function $\mathfrak{h}$ is not dominated by any power. We often will take $\mathfrak{h}(x) = \frac{1}{\log(\frac{1}{x})}$ as a test with this kind of growth condition. In this case, the inverse is $\mathfrak{h}^{-1}(x) = e^{-\frac{1}{x}}$ and satisfies

$$\lim_{x\to 0} \frac{\mathfrak{h}^{-1}(x)}{x^t} = 0$$

for all $t > 0$. This means that $\mathfrak{h}$ goes very slowly to zero and therefore $\mathfrak{h}^{-1}$ does it very fast. In this setting, we cannot bound the factor $\mathfrak{h}^{-1}(Ca_k)$ in (3.19) by a power of a_k.

What we want is to find a gap $\Delta(h,\mathfrak{h}) = \frac{\mathfrak{h}}{h}$ that satisfies

$$\frac{a_k}{\Delta(h,\mathfrak{h})(2^{-k})}\mathfrak{h}^{-1}(Ca_k)^{\frac{1}{2}} \sim C$$

for some summable sequence a_k. For our choice for the test function $\mathfrak{h}$, we obtain $\frac{a_k}{\Delta(h,\mathfrak{h})(2^{-k})}e^{-\frac{C}{a_k}} \sim C$, so $h(2^{-k}) \sim \frac{e^{\frac{C}{a_k}}}{a_k k}$.

Now we note that if h is any dimension function, the sequence $h(2^{-k})$ should go to zero with k. But this would imply that

$$1 \geq h(2^{-k}) \sim \frac{e^{\frac{C}{a_k}}}{a_k k} \geq \frac{e^{\frac{C}{a_k}}}{k} \Rightarrow e^{\frac{C}{a_k}} \leq k \Rightarrow a_k \geq \frac{C}{\log(k)},$$

and therefore $\mathfrak{a} \notin \ell^1$. That means that any gap $\Delta(h,\mathfrak{h})$ will be useless.

As an alternative, we could try to prove that the Hausdorff dimension of E should be at least $\frac{1}{2}$. If we take $\beta < \frac{1}{2}$ and try to estimate

$$\sum_j r_j^\beta,$$

essentially we obtain

$$\sum_k \sigma(\Omega_k)\mathfrak{h}^{-1}(Ca_k)2^{k(\frac{1}{2}-\beta)}.$$

We would want the sequence a_k to satisfy

$$\mathfrak{h}^{-1}(Ca_k)2^{k(\frac{1}{2}-\beta)} \sim C,$$

but again in the case of the test function this implies that $\mathfrak{a}$ is not summable.

3.7.2 No more than a hyperdyadic scale

How fast could we take the scale b_k? The intuition says that we need a scale as fast as we can get. In this section we show that we cannot take a scale faster than the hyperdyadic. What we do is to choose, for each $0 < \eta < 1$, a scale $b_k \searrow 0$ with

$$\lim_{k\to\infty} \frac{b_k^\eta}{b_{k-1}} = +\infty.$$

Moreover, we want to compare this ratio with $\mathfrak{h}^{-1}(k^{-2})$. Suppose (a weaker assumption on b_k) that for all k we have $b_k^\eta \geq b_{k-1}$. Then $b_k \geq b_0^{(\frac{1}{\eta})^k}$ with $0 < b_0 < 1$ and $\frac{1}{\eta} > 1$ and therefore the sequence b_k must go to zero not faster than a double exponential.

3.7.3 An example about very small Furstenberg sets and Packing dimension

The conclusion of Theorem 3.6.1 shows that even with a zero dimensional amount of mass in each fiber, any Furstenberg set must have dimension not smaller that $\frac{1}{2}$. It seems that there is a condition of minimal size on the fibers to obtain such a lower bound.

This observation leads to conjecture that if we consider sets with only a fixed finite number of point in each direction (this is an extreme zero dimensional situation), we should be able to obtain smaller lower bounds. Take, for example, a set $E \subset \mathbb{R}^2$ with at least k points lined up in the direction of e for any $e \in \mathbb{S}$. We will refer to these sets as F^K-sets. The intuition here says that any such set must be really small, and perhaps it should be zero dimensional. In fact we present in Chapter 6, for the case $K = 2$, such a zero dimensional Furstenberg set. But the presented configuration of points seems not be extendable to $K \geq 3$.

We refer to Chapter 6, Remark 6.2.6 for a discussion on the problem of locating the minimum amount of mass on each fiber to obtain that lower bound.

We want to remark here that if we measure the size of the Furstenberg sets with the packing dimension, the situation is absolutely different. More precisely, for any $K \geq 2$, any F^K-set $E \subset \mathbb{R}^2$ must have $\dim_P(E) \geq \frac{1}{2}$. For, if E is an F^2 set, then the application φ defined by

$$\varphi(a, b) = \frac{a - b}{\|a - b\|}$$

is Lipschitz when restricted to $G_\varepsilon := E \times E \setminus \{(x, y) \in E \times E : \|(x, y) - (a, a)\| < \varepsilon ; a \in E\}$. Roughly, we are considering the application that recovers the set of directions but restricted "off the diagonal". It is clear that we can assume without loss of generality that all the pairs are the endpoints of unit line segments. Therefore, since E is an F^K-set, $\varphi(G_\varepsilon) = \mathbb{S}$ if ε is small enough. We obtain the inequality

$$1 = \dim_H(\mathbb{S}) \leq \dim_H(G_\varepsilon) \leq \dim_H(E \times E).$$

The key point is the product formulas for Hausdorff and Packing dimensions described in (1.22). We obtain that

$$1 \leq \dim_H(E \times E) \leq \dim_H(E) + \dim_P(E) \leq 2\dim_P(E) \qquad (3.22)$$

and then $\dim_P(E) \geq \frac{1}{2}$. It also follows that if we achieve small Hausdorff dimension then the Packing dimension is forced to increase. In particular, the F^2-set to be presented in Chapter 6 has Hausdorff dimension 0 and therefore it has Packing dimension 1.

Chapter 4

Fractal sets of directions

4.1 Introduction and preliminaries

In this chapter we are interested in the study of dimension properties of Furstenberg sets associated to fractal sets of directions. Let us introduce the definition of our object of study.

Definition 4.1.1. For α, β in $(0, 1]$, a subset E of $\mathbb{R}^2$ will be called an $F_{\alpha\beta}$-set if there is a subset L of the unit circle such that $\dim_H(L) \geq \beta$ and, for each direction e in L, there is a line segment ℓ_e in the direction of e such that the Hausdorff dimension of the set $E \cap \ell_e$ is equal or greater than α.

This generalizes the classical definition of Furstenberg sets, when the whole circle is considered as set of directions. The purpose here is to study how the parameter β affects the bounds above. From our results we will derive the following proposition.

Proposition 4.1.2. *For any set $E \in F_{\alpha\beta}$, we have that*

$$\dim_H(E) \geq \max \left\{ 2\alpha + \beta - 1; \frac{\beta}{2} + \alpha \right\}, \qquad \alpha, \beta > 0. \tag{4.1}$$

It is not hard to prove Proposition 4.1.2 directly, but we will study this problem in a wider scenario and derive it as a corollary. Moreover, by using general Hausdorff measures, we will extend the inequalities (4.1) to the zero dimensional case.

Let us note that the only known result in this direction, besides the results for F_α-sets, are those due to Mitsis in [Mit02]. There the author proves a sharp bound for the Kakeya maximal operator on $\mathbb{R}^2$ but for general measures on $\mathbb{S}$ instead of the arclength measure. As a consequence, Mitsis obtains an estimate on the lower bounds for, essentially, $F_{1\beta}$-sets. Precisely, using his terminology, the result is as follows: For a given $A \subset \mathbb{S}$, we will say that a compact set $E \subset \mathbb{R}^2$ is an A-Kakeya set if for each direction e in A, there is a line segment ℓ_e in the direction of e contained in E. Note that this is a particular case of $F_{1\beta}$-set. From the estimates on the Kakeya maximal

operator it can be deduced that $\dim_H(E) \geq 1 + \dim_H(A)$ for any A-Kakeya set. Note that this is another way to conclude that the Kakeya sets in $\mathbb{R}^2$ must have full dimension. It is interesting to note that we can compare the the three lower bounds for this particular case of Kakeya sets. Let E be a Kakeya set on $\mathbb{R}^2$. First, from the results on F_α-sets, we conclude that $\dim_H(E) = 2$ since any line segment ℓ_e has dimension 1 and then $\dim_H(E) \geq 2\alpha|_{\alpha=1} = 2 \cdot 1 = 2$. On the other hand, the results of Mitsis show that $\dim_H(E) \geq 1 + \dim_H(\mathbb{S}) = 1 + 1 = 2$. Finally, our results provide a unified description of the interplay between the Hausdorff dimension of the fibers and the set of directions. The Kakeya sets have full dimension because, for any of them, we have the bound $\dim_H(E) \geq 2\alpha + \beta - 1|_{\alpha=1,\beta=1} = 2 \cdot 1 + 1 - 1 = 2$

As in the previous chapter, we will also need to distinguish the subclass of $F_{\alpha\beta}$ of those sets with positive mass both in the set of directions and in the line segments ℓ_e. More precisely, we have the following definition.

Definition 4.1.3. For each pair α, β in $(0,1]$, a subset E of $\mathbb{R}^2$ will be called an $F_{\alpha\beta}^+$-set if there is a subset L of the unit circle such that $\mathcal{H}^\beta(L) > 0$ and, for each direction e in L, there is a line segment ℓ_e in the direction of e such that $\mathcal{H}^\alpha(\ell_e \cap E) > 0$.

Now we present the problem. Let us begin with the definition of $F_{\mathfrak{h}\mathfrak{g}}$-sets. Let $\mathfrak{h}$ and $\mathfrak{g}$ be two dimension functions. A set $E \subseteq \mathbb{R}^2$ is a Furstenberg set of type $\mathfrak{h}\mathfrak{g}$, or an $F_{\mathfrak{h}\mathfrak{g}}$-set, if there is a subset L of the unit circle such that $\mathcal{H}^{\mathfrak{g}}(L) > 0$ and, for each direction e in L, there is a line segment ℓ_e in the direction of e such that $\mathcal{H}^{\mathfrak{h}}(\ell_e \cap E) > 0$. Note that this is the natural generalization of the $F_{\alpha\beta}^+$ class.

Now, for the sake of clarity in the proof of our results, we will perform the same reduction made in Chapter 3. A standard pigeonhole argument allows us to work with the following definition.

Definition 4.1.4. Let $\mathfrak{h}$ and $\mathfrak{g}$ be two dimension functions. A set $E \subseteq \mathbb{R}^2$ is a Furstenberg set of type $\mathfrak{h}\mathfrak{g}$, or an $F_{\mathfrak{h}\mathfrak{g}}$-set, if there is a subset L of the unit circle such that $\mathcal{H}^{\mathfrak{g}}(L) > 0$ and, for each direction e in L, there is a line segment ℓ_e in the direction of e such that $\mathcal{H}^{\mathfrak{h}}_\delta(\ell_e \cap E) > 1$ for all $\delta < \delta_E$ for some $\delta_E > 0$.

Following the intuition suggested by Proposition 4.1.2, one could conjecture that if E belong to the class $F_{\mathfrak{h}\mathfrak{g}}$ then an appropriate dimension function for E should be dimensionally greater than $\frac{\mathfrak{h}^2\mathfrak{g}}{\mathbf{id}}$ and $\mathfrak{h}\sqrt{\mathfrak{g}}$. This will be the case, indeed, and we will provide some estimates on the gap between those conjectured dimension functions and a generic test function $h \in \mathbb{H}$ to ensure that $\mathcal{H}^h(E) > 0$, and also illustrate with some examples. We will consider the two results separately. Namely, for a given pair of dimension functions $\mathfrak{h}, \mathfrak{g} \in \mathbb{H}$, in Section 4.2 we obtain sufficient conditions on a test dimension function $h \in \mathbb{H}$, $h \prec \frac{\mathfrak{h}^2\mathfrak{g}}{\mathbf{id}}$ to ensure that $\mathcal{H}^h(E) > 0$ for any set $E \in F_{\mathfrak{h}\mathfrak{g}}$. In Section 4.3 we consider the analogous problem for $h \prec \mathfrak{h}\sqrt{\mathfrak{g}}$.

In Section 4.2 the main tool will be an L^2 estimate for the Kakeya maximal function for general measures (see Chapter 3, Section 3.3.1).

In Section 4.3 we perform a more combinatorial kind of proof. We introduce the notion of δ-entropy of a set E in the next definition:

Definition 4.1.5. Let $E \subset \mathbb{R}^n$ and $\delta \in \mathbb{R}_{>0}$. The δ-entropy of E is the maximal possible cardinality of a δ-separated subset of E. We will denote this quantity with $\mathcal{N}_\delta(E)$.

The main idea is to relate the δ-entropy to some notion of size of the set. Clearly, the entropy is essentially the Box dimension or the Packing dimension of a set (see [Mat95] or [Fal03] for the definitions) since both concepts are defined in terms of separated δ balls with centers in the set. However, for our proof we will need to relate the entropy of a set to some quantity that has the property of being (in some sense) stable under countable unions. One choice is therefore the notion of Hausdorff content, which enjoys the needed properties: it is an outer measure, it is finite, and it reflects the entropy of a set in the following manner. Recall that the $\mathfrak{g}$-dimensional Hausdorff content of a set E is defined as

$$\mathcal{H}^{\mathfrak{g}}_\infty(E) = \inf \left\{ \sum_i \mathfrak{g}(\mathrm{diam}(U_i)) : E \subset \bigcup_i U_i \right\}. \tag{4.2}$$

Note that the $\mathfrak{g}$-dimensional Hausdorff content $\mathcal{H}^{\mathfrak{g}}_\infty$ is clearly not the same than the Hausdorff measure $\mathcal{H}^{\mathfrak{g}}$. In fact, they are the measures obtained by applying Method I and Method II (see [Mat95]) respectively to the premeasure that assigns to a set A the value $\mathfrak{g}(\mathrm{diam}(A))$.

For future reference, we state the following estimate for the δ-entropy of a set with positive $\mathfrak{g}$-dimensional Hausdorff content as a lemma.

Lemma 4.1.6. *Let $\mathfrak{g} \in \mathbb{H}$ and let A be any set. Let $\mathcal{N}_\delta(A)$ be the δ-entropy of A. Then $\mathcal{N}_\delta(A) \geq \frac{\mathcal{H}^{\mathfrak{g}}_\infty(A)}{\mathfrak{g}(\delta)}$.*

Proof. Let $\{x_i\}_{i=1}^N$ be a maximal δ-separated subset. By maximality, we can cover A with balls $B(x_i, \delta)$. Therefore, for the $\mathfrak{g}$-dimensional Hausdorff content $\mathcal{H}^{\mathfrak{g}}_\infty$, we have the bound

$$\mathcal{H}^{\mathfrak{g}}_\infty(A) \leq \sum_i^N \mathcal{H}^{\mathfrak{g}}_\infty(B(x_i, \delta)) \leq N\mathfrak{g}(\delta) \tag{4.3}$$

and it follows that $\mathcal{N}_\delta(A) \geq N \geq \frac{\mathcal{H}^{\mathfrak{g}}_\infty(A)}{\mathfrak{g}(\delta)}$. $\qquad\square$

Of course, this result is meaningful when $\mathcal{H}^{\mathfrak{g}}_\infty(A) > 0$. We will use it in the case in which $\mathcal{H}^{\mathfrak{g}}(A) > 0$, which is equivalent to $\mathcal{H}^{\mathfrak{g}}_\infty(A) > 0$. For a detailed study of the properties of $\mathcal{H}^{\mathfrak{g}}$ and $\mathcal{H}^{\mathfrak{g}}_\infty$ see [Del02] and [Del03].

Note that the lemma above only requires the finiteness and the subadditivity of the Hausdorff content. The relevant feature that will be needed in

our proof is the σ-subadditivity, which is a property that the Box dimension does not share.

We will use again the notation from Definition 3.3.3 and an analogue of Lemma 3.3.4:

Definition 4.1.7. Let $\mathfrak{b} = \{b_k\}_{k \in \mathbb{N}}$ be a decreasing sequence with $\lim b_k = 0$. For any family of balls $\mathcal{B} = \{B_j\}$ with $B_j = B(x_j; r_j)$, $r_j \leq 1$, and for any set E, we define

$$J_k^{\mathfrak{b}} := \{j \in \mathbb{N} : b_k < r_j \leq b_{k-1}\}, \tag{4.4}$$

and

$$E_k := E \cap \bigcup_{j \in J_k^{\mathfrak{b}}} B_j. \tag{4.5}$$

In the particular case of the dyadic scale $\mathfrak{b} = \{2^{-k}\}$, we will omit the superscript and denote

$$J_k := \{j \in \mathbb{N} : 2^{-k} < r_j \leq 2^{-k+1}\}. \tag{4.6}$$

Lemma 4.1.8. *Let E be an $F_{\mathfrak{h}\mathfrak{g}}$-set for some $\mathfrak{h}, \mathfrak{g} \in \mathbb{H}$ with the directions in $L \subset \mathbb{S}$ and let $\mathfrak{a} = \{a_k\}_{k \in \mathbb{N}} \in \ell^1$ be a non-negative sequence. Let $\mathcal{B} = \{B_j\}$ be a δ-covering of E with $\delta < \delta_E$ and let E_k and J_k be as above. Define*

$$L_k := \left\{ e \in \mathbb{S} : \mathcal{H}_\delta^{\mathfrak{h}}(\ell_e \cap E_k) \geq \frac{a_k}{2\|\mathfrak{a}\|_1} \right\}.$$

Then $L = \cup_k L_k$.

4.2 The Kakeya type bound

In this section we will prove a generalized version of the announced bound $\dim_H(E) \geq 2\alpha + \beta - 1$ for $E \in F_{\alpha\beta}$. We have the following theorem.

Theorem 4.2.1 ($\mathfrak{h}\mathfrak{g} \to \frac{\mathfrak{h}^2\mathfrak{g}}{\mathbf{id}}$). *Let $\mathfrak{g} \in \mathbb{H}$ and $\mathfrak{h} \in \mathbb{H}_d$ be two dimension functions and let E be an $F_{\mathfrak{h}\mathfrak{g}}$-set. Let $h \in \mathbb{H}$ such that $h \prec \frac{\mathfrak{h}^2\mathfrak{g}}{\mathbf{id}}$. For $\delta > 0$, let $C(\delta)$ be as in (3.7). If $\sum_k \sqrt{\frac{\mathfrak{h}^2(2^{-k})C(2^{-k+1})}{h(2^{-k})}} < \infty$, then $\mathcal{H}^h(E) > 0$.*

Proof. Let $E \in F_{\mathfrak{h}\mathfrak{g}}$ and let $\{B_j\}_{j \in \mathbb{N}}$ be a covering of E by balls with $B_j = B(x_j; r_j)$. We need to bound $\sum_j h(2r_j)$ from below. Since h is non-decreasing, it suffices to obtain the bound

$$\sum_j h(r_j) \gtrsim 1 \tag{4.7}$$

for any $h \in \mathbb{H}$ satisfying the hypothesis of the theorem.

Define $\mathfrak{a} = \{a_k\}$ by $a_k^2 = \frac{\mathfrak{h}^2(2^{-k})C(2^{-k+1})}{h(2^{-k})}$. Therefore, by hypothesis $\mathfrak{a} \in \ell^1$. Also define, as in the previous section, for each $k \in \mathbb{N}$, $J_k = \{j \in \mathbb{N} : 2^{-k} <$

$r_j \leq 2^{-k+1}\}$ and $E_k = E \cap \cup_{j \in J_k} B_j$. Since $\mathfrak{a} \in \ell^1$, we can apply Lemma 3.3.4 to obtain the decomposition of the set of directions as $L = \bigcup_k L_k$ associated to this choice of $\mathfrak{a}$.

We will apply the maximal function inequality to a weighted union of indicator functions. For each k, let $F_k = \bigcup_{j \in J_k} B_j$ and define the function

$$f := \mathfrak{h}(2^{-k})2^k \chi_{F_k}.$$

We will use the L^2 norm estimates for the maximal function. We can compute directly the L^2 norm of f:

$$
\begin{aligned}
\|f\|_2^2 &= \mathfrak{h}^2(2^{-k})2^{2k} \int_{\cup_{J_k} B_j} dx \\
&\lesssim \mathfrak{h}^2(2^{-k})2^{2k} \sum_{j \in J_k} r_j^2 \\
&\lesssim \mathfrak{h}^2(2^{-k}) \# J_k,
\end{aligned}
$$

since $r_j \leq 2^{-k+1}$ for $j \in J_k$. Therefore

$$\|f\|_2^2 \lesssim \# J_k \mathfrak{h}^2(2^{-k}). \tag{4.8}$$

The same arguments used in the proof of Theorem 3.4.1 in Chapter 3 allows us to obtain a lower bound for the maximal function. Essentially, the maximal function is pointwise bounded from below by the average of f over the tube centered on the line segment ℓ_e for any $e \in L_k$. Therefore, we have the following bound for the (L^2, μ) norm. Here, μ is a measure supported on L that obeys the law $\mu(B(x,r) \leq \mathfrak{g}(r)$ for any ball $B(x,r)$ given by Frostman's lemma.

$$\|\mathcal{K}_{2^{-k+1}}(f)\|_{L^2(d\mu)}^2 \gtrsim a_k^2 \mu(L_k) = \frac{\mu(L_k)\mathfrak{h}^2(2^{-k})C(2^{-k})}{h(2^{-k})}. \tag{4.9}$$

Combining (4.9) with the maximal inequality (3.7), we obtain

$$\frac{\mu(L_k)\mathfrak{h}^2(2^{-k})C(2^{-k})}{h(2^{-k})} \lesssim \|\mathcal{K}_{2^{-k+1}}(f)\|_2^2 \lesssim C(2^{-k+1})\|f\|_2^2 \leq C(2^{-k})\|f\|_2^2.$$

We also have the bound (4.8), which implies that

$$\frac{\mu(L_k)}{h(2^{-k})} \lesssim \# J_k.$$

Now we are able to estimate the sum in (4.7). Let h be a dimension function satisfying the hypothesis of Theorem 4.2.1. We have

$$
\begin{aligned}
\sum_j h(r_j) &\geq \sum_k h(2^{-k}) \# J_k \\
&\gtrsim \sum_k \mu(L_k) \geq \mu(L) > 0.
\end{aligned}
$$

$\square$

Corollary 4.2.2. *Let E an $F_{\alpha\beta}^+$-set. If h is any dimension function satisfying*

$$h(x) \geq Cx^{2\alpha+\beta-1} \log^\theta(\frac{1}{x}) \tag{4.10}$$

for $\theta > 2$, then $\mathcal{H}^h(E) > 0$.

Proof. It follows directly, since in this case we have $C(\delta) \lesssim \delta^{\beta-1}$, and therefore the sum in Theorem 4.2.1 is

$$\sum_k \sqrt{\frac{\mathfrak{h}^2(2^{-k})C(2^{-k})}{h(2^{-k})}} \quad \lesssim \quad \sum_k \sqrt{\frac{2^{-k2\alpha}2^{-k(\beta-1)}}{h(2^{-k})}}$$

$$\leq \quad \sum_k \sqrt{\frac{2^{-k(2\alpha+\beta-1)}}{(2^{-k})^{2\alpha+\beta-1} \log^\theta(2^k)}}$$

$$= \quad \sum_k \frac{1}{k^{\frac{\theta}{2}}} < \infty.$$

$\square$

Remark 4.2.3. Note that the bound $\dim(E) \geq 2\alpha+\beta-1$ for $E \in F_{\alpha\beta}$ follows directly from this last corollary.

4.3   The combinatorial bound

In this section we deal with the bound $\mathfrak{h}\mathfrak{g} \to \mathfrak{h}\sqrt{\mathfrak{g}}$, which is the significant bound near the endpoint $\alpha = \beta = 0$ and generalizes the bound $\dim_H(E) \geq \frac{\beta}{2} + \alpha$ for $E \in F_{\alpha\beta}$. Note that the second bound in (4.1) is meaningless for small values of α and β. We will again consider separately the cases of $\mathfrak{h}$ being zero dimensional or positive dimensional. In the next theorem, the additional condition on $\mathfrak{h}$ reflects the positivity of the dimension function.

   We will use again the two relevant lemmas from Chapter 3. Lemma 3.5.2 is the "splitting lemma", which says that a linear set with positive $\mathfrak{h}$-dimensional mass can be splitted into two well separated linear subsets. Lemma 3.5.3 is the combinatorial ingredient in the proof of both Theorem 4.3.1 and Theorem 4.3.4. This lemma provides an estimate on the number of lines with certain separation that intersect two balls of a given size.

   With these two lemmas we are now ready to prove the main result of this section. We have the following theorem. Recall that $h_\alpha(x) = x^\alpha$.

Theorem 4.3.1 ($\mathfrak{h}\mathfrak{g} \to \mathfrak{h}\sqrt{\mathfrak{g}}$, $\mathfrak{h} \succ h_\alpha$). *Let $\mathfrak{g} \in \mathbb{H}$, $\mathfrak{h} \in \mathbb{H}_d$ be two dimension functions such that $\mathfrak{h}(x) \lesssim x^\alpha$ for some $0 < \alpha < 1$ and let E be an $F_{\mathfrak{h}\mathfrak{g}}$-set. Let $h \in \mathbb{H}$ with $h \prec \mathfrak{h}\sqrt{\mathfrak{g}}$. If $\sum_k \left(\frac{\mathfrak{h}(2^{-k})\sqrt{\mathfrak{g}}(2^{-k})}{h(2^{-k})}\right)^{\frac{2\alpha}{2\alpha+1}} < \infty$, then $\mathcal{H}^h(E) > 0$.*

Proof. Let $E \in F_{\mathfrak{h}\mathfrak{g}}$ and let $\{B_j\}_{j \in \mathbb{N}}$ be a covering of E by balls with $B_j = B(x_j; r_j)$. Define $\Delta(h, \mathfrak{h}\sqrt{\mathfrak{g}}) = \frac{\mathfrak{h}\sqrt{\mathfrak{g}}}{h}$ and consider the sequence $\mathfrak{a}$ defined as $\mathfrak{a} = \left\{ \Delta(h, \mathfrak{h}\sqrt{\mathfrak{g}})^{\frac{2\alpha}{2\alpha+1}} (2^{-k}) \right\}_k$. Also define, as in the previous section, for each $k \in \mathbb{N}$, $J_k = \{j \in \mathbb{N} : 2^{-k} < r_j \leq 2^{-k+1}\}$ and $E_k = E \cap \cup_{j \in J_k} B_j$. Since by hypothesis $\mathfrak{a} \in \ell^1$, we can apply Lemma 3.3.4 to obtain the decomposition of the set of directions as $L = \bigcup_k L_k$ associated to this choice of $\mathfrak{a}$, where L_k is defined as

$$L_k := \left\{ e \in \mathbb{S} : \mathcal{H}_\delta^{\mathfrak{h}}(\ell_e \cap E_k) \geq \frac{a_k}{2\|\mathfrak{a}\|_1} \right\}.$$

We can apply Lemma 3.5.2 with $\eta = \frac{a_k}{2\|\mathfrak{a}\|_1}$ to $\ell_e \cap E_k$. Therefore we obtain two intervals I_e^- and I_e^+, contained in ℓ_e with

$$\mathcal{H}_\delta^{\mathfrak{h}}(I_e^\pm \cap E_k) \gtrsim a_k$$

that are $\mathfrak{h}^{-1}(ra_k)$-separated for $r = \frac{1}{16\|\mathfrak{a}\|_1}$.

Now, let $\{e_j^k\}_{j=1}^{N_k}$ be a 2^{-k}-separated subset of L_k. Taking into account the estimate for the entropy given in Lemma 4.1.6. We obtain then that

$$N_k \gtrsim \frac{\mathcal{H}_\infty^{\mathfrak{g}}(L_k)}{\mathfrak{g}(2^{-k})}. \tag{4.11}$$

Define $\Pi_k := J_k \times J_k \times \{1, .., N_k\}$ and

$$\mathcal{T}_k := \left\{ (j_-, j_+, i) \in \Pi_k : I_{e_i}^- \cap E_k \cap B_{j_-} \neq \emptyset \ I_{e_i}^+ \cap E_k \cap B_{j_+} \neq \emptyset \right\}. \tag{4.12}$$

The idea is again to count the elements of $\mathcal{T}_k$ in two ways.

First, if we fix a pair j_- and j_+ and count for how many values of i the triplet (j_-, j_+, i) belongs to $\mathcal{T}_k$, we obtain, by using Lemma 3.5.3 for the choice $\mathfrak{b} = \{2^{-k}\}$, that

$$\#\mathcal{T}_k \lesssim \frac{1}{\mathfrak{h}^{-1}(ra_k)} (\#J_k)^2. \tag{4.13}$$

Second, fix i. In this case, we have by hypothesis that $\mathcal{H}_\delta^{\mathfrak{h}}(I_{e_i}^+ \cap E_k) \gtrsim a_k$, so $\sum_{j_+} \mathfrak{h}(r_{j_+}) \gtrsim a_k$. Therefore,

$$a_k \lesssim \sum_{(j_-, j_+, i) \in \mathcal{T}_k} \mathfrak{h}(r_{j_+}) \leq K\mathfrak{h}(2^{-k}),$$

where K is the number of elements of the sum. Therefore $K \gtrsim \frac{a_k}{\mathfrak{h}(2^{-k})}$. The same holds for j_-, so

$$\#\mathcal{T}_k \gtrsim N_k \left(\frac{a_k}{\mathfrak{h}(2^{-k})} \right)^2. \tag{4.14}$$

Combining the two bounds,

$$\begin{aligned} \#J_k &\gtrsim (\#\mathcal{T}_k)^{1/2} \mathfrak{h}^{-1}(ra_k)^{1/2} \\ &\gtrsim N_k^{1/2} \frac{a_k}{\mathfrak{h}(2^{-k})} \mathfrak{h}^{-1}(ra_k)^{1/2}. \end{aligned}$$

Therefore, for any $h \in \mathbb{H}$ as in the hypothesis of the theorem, we have the estimate

$$
\sum_j h(r_j) \quad \gtrsim \quad \sum_k \frac{(\mathfrak{h}\sqrt{\mathfrak{g}})(2^{-k})}{\Delta(h, \mathfrak{h}\sqrt{\mathfrak{g}})(2^{-k})} \# J_k \tag{4.15}
$$

$$
\gtrsim \quad \sum_k \frac{a_k \mathfrak{h}^{-1}(ra_k)^{\frac{1}{2}}\sqrt{\mathfrak{g}}(2^{-k})N_k^{\frac{1}{2}}}{\Delta(h, \mathfrak{h}\sqrt{\mathfrak{g}})(2^{-k})}. \tag{4.16}
$$

Recall now that from (4.11) we have $\sqrt{\mathfrak{g}}(2^{-k})N_k^{\frac{1}{2}} \gtrsim \mathcal{H}_\infty^{\mathfrak{g}}(L_k)^{\frac{1}{2}}$. In addition, $\mathfrak{h}(x) \lesssim x^\alpha$, which implies that $\mathfrak{h}^{-1}(x) \gtrsim x^{\frac{1}{\alpha}}$. Therefore we obtain the bound

$$
\sum_j h(r_j) \quad \gtrsim \quad \sum_k \frac{\mathcal{H}_\infty^{\mathfrak{g}}(L_k)^{1/2}a_k^{\frac{1+2\alpha}{2\alpha}}}{\Delta(h, \mathfrak{h}\sqrt{\mathfrak{g}})(2^{-k})}
$$

$$
= \quad \sum_k \mathcal{H}_\infty^{\mathfrak{g}}(L_k)^{1/2} \gtrsim 1.
$$

In the last inequality, we used the σ-subadditivity of $\mathcal{H}_\infty^{\mathfrak{g}}$. $\square$

Corollary 4.3.2. *Let E be an $F_{\alpha\beta}^+$-set for $\alpha, \beta > 0$. If h is a dimension function satisfying $h(x) \geq Cx^{\frac{\beta}{2}+\alpha} \log^\theta(\frac{1}{x})$ for $\theta > \frac{1+2\alpha}{2\alpha}$, then $\mathcal{H}^h(E) > 0$.*

Remark 4.3.3. Note that again the bound $\dim(E) \geq \alpha + \frac{\beta}{2}$ for $E \in F_{\alpha\beta}$ follows directly from this last corollary.

In the next theorem we consider the case of a family of very small Furstenberg sets. More precisely, we deal with a family that corresponds to the case $\alpha = 0$, $\beta \in (0,1]$ in the classical setting.

Theorem 4.3.4 ($\mathfrak{h}\mathfrak{g} \to \mathfrak{h}\sqrt{\mathfrak{g}}$, $\mathfrak{h}$ zero dimensional, $\mathfrak{g}$ positive)**.** *Let $\beta > 0$ and define $\mathfrak{g}(x) = x^\beta, \mathfrak{h}(x) = \frac{1}{\log(\frac{1}{x})}$. If E is an $F_{\mathfrak{h}\mathfrak{g}}$-set, then $\dim(E) \geq \frac{\beta}{2}$.*

Proof. Once again, we follow the same arguments of Chapter 3. Let $E \in F_{\mathfrak{h}\mathfrak{g}}$ and let $\{B_j\}_{j\in\mathbb{N}}$ be a covering of E by balls with $B_j = B(x_j; r_j)$. Now we consider a scaling sequence $\mathfrak{b}$ to be determined later and, by using Lemma 3.3.4, we obtain a decomposition $L = \bigcup_{k \geq k_0} L_k$ with

$$
L_k = \left\{ e \in L : \mathcal{H}_\delta^{\mathfrak{h}}(\ell_e \cap E_k) \geq ck^{-2} \right\},
$$

where $E_k = E \cap \bigcup_{J_k^{\mathfrak{b}}} B_j$, $J_k^{\mathfrak{b}}$ is the partition of the radii as in (3.4) associated to $\mathfrak{b}$ and $c > 0$ is a suitable constant. We apply Lemma 3.5.2 and also define, as in Theorem 4.3.1, $\Pi_k := J_k^{\mathfrak{b}} \times J_k^{\mathfrak{b}} \times \{1, .., N_k\}$ and

$$
\mathcal{T}_k^{\mathfrak{b}} := \left\{ (j_-, j_+, i) \in \Pi_k : I_{e_i}^- \cap E_k \cap B_{j_-} \neq \emptyset \ I_{e_i}^+ \cap E_k \cap B_{j_+} \neq \emptyset \right\},
$$

where $\{e_j^k\}_{j=1}^{N_k}$ is a b_k-separated subset of L_k. By Lemma 3.5.3, we obtain

$$\#\mathcal{T}_k^{\flat} \lesssim \frac{b_{k-1}}{b_k}\frac{1}{\mathfrak{h}^{-1}(ck^{-2})}(\#J_k^{\flat})^2. \tag{4.17}$$

For the lower bound on $\#\mathcal{T}_k^{\flat}$, we have the extra information about the entropy of L_k, i.e., $N_k \gtrsim \mathcal{H}_\infty^\beta(L_k)/b_k^\beta$. We therefore obtain the analogous of (3.17):

$$\#\mathcal{T}_k \gtrsim \frac{\mathcal{H}_\infty^\beta(L_k)}{b_k^\beta}\left(\frac{k^{-2}}{\mathfrak{h}(b_{k-1})}\right)^2.$$

The last two inequalities together imply that

$$\#J_k^{\flat} \gtrsim \mathcal{H}_\infty^\beta(L_k)^{\frac{1}{2}}\left(\frac{b_k^{1-\beta}}{b_{k-1}}\right)^{1/2}\frac{e^{-ck^2}}{k^2}.$$

It follows then that, for $s < \frac{\beta}{2}$,

$$\begin{aligned}
\sum_j r_j^s &\geq \sum_k b_k^s \#J_k \\
&= \sum_k \mathcal{H}_\infty^\beta(L_k)^{\frac{1}{2}}\frac{b_k^{\frac{1}{2}+s-\frac{\beta}{2}}}{b_{k-1}^{\frac{1}{2}}}\frac{1}{k^2 e^{ck^2}} \\
&\gtrsim \sqrt{\sum_k \mathcal{H}_\infty^\beta(L_k)\frac{b_k^{1+2s-\beta}}{b_{k-1}}\frac{1}{k^4 e^{ck^2}}}.
\end{aligned}$$

Consider the hyperdyadic scale $b_k = 2^{-(1+\varepsilon)^k}$ with some $\varepsilon > 0$ to be determined. With this choice, we have

$$\frac{b_k^{1+2s-\beta}}{b_{k-1}} = 2^{(1+\varepsilon)^{k-1}-(1+\varepsilon)^k(1+2s-\beta)} = 2^{(1+\varepsilon)^k\left(\frac{1}{1+\varepsilon}-(1+2s-\beta)\right)}.$$

Since $1 + 2s - \beta < 1$, we can choose $\varepsilon > 0$ such that $\frac{1}{1+\varepsilon} - (1 + 2s - \beta) > 0$. More precisely, take ε such that $0 < \varepsilon < \frac{\beta-2s}{1+2s-\beta}$.

Therefore,

$$\begin{aligned}
\left(\sum_j r_j^s\right)^2 &\gtrsim \sum_k \mathcal{H}_\infty^\beta(L_k)\frac{b_k^{1+2s-\beta}}{b_{k-1}}\frac{1}{k^4 e^{ck^2}} \\
&= \sum_k \mathcal{H}_\infty^\beta(L_k)\frac{2^{(1+\varepsilon)^k\left(\frac{1}{1+\varepsilon}-(1+2s-\beta)\right)}}{k^4 e^{ck^2}} \\
&\gtrsim \sum_k \mathcal{H}_\infty^\beta(L_k) \gtrsim 1.
\end{aligned}$$

$\square$

We have the following immediate corollary.

Corollary 4.3.5. *Let $\theta > 0$. If E is an $F_{\mathfrak{hg}}$-set with $\mathfrak{h}(x) = \frac{1}{\log^{\theta}(\frac{1}{x})}$ and $\mathfrak{g}(x) = x^{\beta}$, then $\dim(E) \geq \frac{\beta}{2}$.*

The next question would be: Which should it be the expected dimension function for an $F_{\mathfrak{hg}}$-set if $\mathfrak{h}(x) = \mathfrak{g}(x) = \frac{1}{\log(\frac{1}{x})}$? The preceding results lead us to the following conjecture:

Conjecture 4.3.6. *Let $\mathfrak{h}(x) = \mathfrak{g}(x) = \frac{1}{\log(\frac{1}{x})}$ and let E be an $F_{\mathfrak{hg}}$-set. Then $\frac{1}{\log^{\frac{3}{2}}(\frac{1}{x})}$ should be an appropriate dimension function for E, in the sense that a logarithmic gap can be estimated.*

We do not know, however, how to prove this.

4.4 A remark on the notion of size for the set of directions

In Section 4.1 we have emphasized that the relevant ingredient for the combinatorial proof in Section 4.3 is the notion of δ-entropy of a set. In addition, we have discussed the possibility of consider the Box dimension as an adequate notion of size to detect this quantity. In this section we present an example that shows that in fact the notion of Packing measure is also inappropriate. We want to remark here that none of them will give any further (useful) information to this problem and therefore there is no chance to obtain similar results in terms of those notions of dimensions. To make it clear, consider the classical problem of proving the bound $\dim(E) \geq \alpha + \frac{\beta}{2}$ for any $E \in F_{\alpha\beta}$ where β is the Box or Packing dimension of the set L of directions.

We illustrate this remark with the extreme case of $\beta = 1$. It is absolutely trivial that nothing meaningful can be said if we only know that the Box dimension of L is 1, since any countable dense subset L of $\mathbb{S}$ satisfies $\dim_B(L) = 1$ but in that case, since L is countable, we can only obtain that $\dim(E) \geq \alpha$.

For the Packing dimension, it is also easy to see that if we only know that $\dim_P(L) = 1$ we do not have any further information about the Hausdorff dimension of the set E. To see why, consider the following example. Let C_{α} be a regular Cantor set such that $\dim(C_{\alpha}) = \dim_B(C_{\alpha}) = \alpha$. Let L be a set of directions with $\dim_H(L) = 0$ and $\dim_P(L) = 1$.

Now, we build the Furstenberg set E in polar coordinates as

$$E := \{(r, \theta) : r \in C_{\alpha}, \theta \in L\}. \tag{4.18}$$

This can be seen as a "Cantor target", but with a fractal set of directions instead of the whole circle. By the Hausdorff dimension estimates, we know that $\dim(E) \geq \alpha$. We show that in this case we also have that $\dim(E) \leq \alpha$, which implies that in the general case this is the best that one could expect,

even with the additional information about the Packing dimension of L. For the upper bound, consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x \cos y, x \sin y)$. Clearly $E = f(C_\alpha \times L)$, and therefore

$$\dim(E) = \dim(f(C_\alpha \times L)) \leq \dim(C_\alpha \times L) = \dim_B(C_\alpha) + \dim(L) = \alpha$$

by the known product formulae that can be found, for example, in [Fal03].

Chapter 5

Diophantine approximation

In this chapter we describe some aspects of the problem of diophantine approximation, which will be the relevant ingredient in the constructions to be presented in Chapter 6.

5.1 Jarník's Theorem

The central problem in the theory of diophantine approximation is, at its simplest level, to approximate irrational numbers by rationals. More specifically, the study is based on a quantitative analysis of the property that rational numbers are dense in the real line. A celebrated theorem of Dirichlet, which can be understood as the starting point of this theory, states that for every real number x, there are infinitely many rationals $\frac{p}{q}$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{5.1}$$

If we denote with $\|x\|$ the distance from x to the nearest integer, i.e. $\|x\| := \min\{|x - m| : m \in \mathbb{Z}\}$, then Dirichlet's theorem states that every real number satisfies

$$\|xq\| < \frac{1}{q}, \tag{5.2}$$

for infinitely many integers q. If we now consider a parameter $\beta > 1$, the validity of the analogous inequality

$$\|xq\| < \frac{1}{q^\beta}, \tag{5.3}$$

imposes much more restrictive conditions on x and it is known that almost no real number x satisfies the above inequality for infinitely many integers. Moreover, if we allow β to grow and take arbitrary large values, the sets

$$B_\beta = \left\{ x \in [0,1] \setminus \mathbb{Q} : \|xq\| < \frac{1}{q^{\beta-1}} \text{ for infinitely many } q \in \mathbb{Z} \right\},$$

are all of zero Lebesgue measure.

More generally, we could consider the set of those real numbers that are ψ-approximable: for any decreasing function $\psi \geq 0$, a real number x is called ψ-approximable if there are infinitely many integers q that satisfy the inequality

$$\|xq\| < \psi(q). \tag{5.4}$$

In [Khi24] Khintchine established a criterion for the "size" of the set of ψ-approximable numbers expressed in terms of the Lebesgue measure: if the sum $\sum_q \psi(q)$ diverges, then (5.1) is satisfied for infinite many integers for almost all x, whilst the set of ψ approximable number is of null Lebesgue measure if $\sum_q \psi(q)$ converges. We are interested in this latter case. In the sequel, we refer to ψ as the approximation function.

In the case that $\sum \psi(q)$ converges, since the set has null Lebesgue measure, it is very natural to ask about the dimension of the set

$$\mathcal{A}(\psi) := \{x \in [0,1] \setminus \mathbb{Q} : \|xq\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}. \tag{5.5}$$

In [Jar31], Jarník provides a nice and simple criteria for the size of $\mathcal{A}(\psi)$ in terms of Hausdorff measures. We reproduce here the statement of the one dimensional case, although the original result is valid in higher dimensions.

Theorem 5.1.1 (Jarník). *Let $h \in \mathbb{H}$ such that $\frac{h(r)}{r} \to \infty$ as $r \to 0$ and $\frac{h(r)}{r}$ is decreasing. Then*

$$\mathcal{H}^h(\mathcal{A}(\psi)) = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} qh\left(\frac{\psi(q)}{q}\right) < \infty \\ \infty & \text{if} \quad \sum_{q=1}^{\infty} qh\left(\frac{\psi(q)}{q}\right) = \infty \quad \text{and } \psi \text{ is monotonic.} \end{cases}$$

Remark 5.1.2. Recall that $h_1(x) = x$ and note that the condition $\frac{h(r)}{r} \to \infty$ as r goes to zero is equivalent to $h \prec h_1$, which reflects that h is a dimension function of dimensionality at most 1. Since we will use this characterization, we introduce the class of dimension functions which satisfy the hypothesis of Theorem 5.1.1. Let

$$\mathbb{J} := \left\{ h \in \mathbb{H} : h \prec h_1; \frac{h(r)}{r} \text{ is decreasing} \right\}. \tag{5.6}$$

Let us note that from [Ols05] (p. 160) it follows that any function $h \prec h_1$ which is concave in a neighborhood of the origin belongs to $\mathbb{J}$.

We also remark that if we specialize on the power functions, we can recover the classic theorem of Jarník, i.e.:

Theorem 5.1.3. *For $\beta \geq 2$, define the following set:*

$$B_\beta = \left\{ x \in [0,1] \setminus \mathbb{Q} : \|xq\| < \frac{1}{q^{\beta-1}} \text{ for infinitely many } q \in \mathbb{Z} \right\}.$$

Then $\dim(B_\beta) = \frac{2}{\beta}$.

It is important to remark that Jarník's theorem not only completely determines the dimension partition for the set, but also implies that, at the critical dimension, the set has infinite measure, i.e., $\mathcal{H}^{\frac{2}{\beta}}(B_\beta) = \infty$. For related topics and further reading, we refer to [BBDV09] (and references therein), [BD99], [DK04], [DD01], [Kau81] and [Blu98].

Now, for our purposes, we will introduce a generalization of the set B_β which is slightly different from $\mathcal{A}(\psi)$ in Theorem 5.1.1.

Let $\mathfrak{g}$ be any positive increasing function such that

$$\mathfrak{g}(x) \gtrsim x^2 \tag{5.7}$$

and $x \to \frac{x}{\mathfrak{g}(x)}$ is monotonic. Note that the requirement on $\mathfrak{g}$ in (5.7) is the analogous version of the condition $\beta \geq 2$ on Theorem 5.1.3. The monotonicity is imposed since the approximation function will be $\frac{x}{\mathfrak{g}(x)}$. Define

$$B_\mathfrak{g} := \left\{ x \in [0,1] \setminus \mathbb{Q} : \|xq\| \leq \frac{q}{\mathfrak{g}(q)} \text{ for infinitely many } q \in \mathbb{N} \right\}. \tag{5.8}$$

We will show that $h_\mathfrak{g}(x) = \frac{1}{\left(\mathfrak{g}^{-1}(\frac{1}{x})\right)^2}$ is, in some sense, an appropriate dimension function for $B_\mathfrak{g}$. First note that

$$\sum_{q=1}^\infty q h_\mathfrak{g}\left(\frac{\psi(q)}{q}\right) = \sum_{q=1}^\infty q h_\mathfrak{g}\left(\frac{1}{\mathfrak{g}(q)}\right) = \sum_{q=1}^\infty \frac{1}{q} = \infty,$$

which implies that $\mathcal{H}^{h_\mathfrak{g}}(B_\mathfrak{g}) = \infty$. Nevertheless, the last sum is not far from being convergent. For, let h be another dimension function such that $h \succ h_\mathfrak{g}$ and such that h satisfies the hypothesis of Theorem 5.1.1. Recall that $\Delta(h_\mathfrak{g}, h)(x) = \frac{h(x)}{h_\mathfrak{g}(x)}$, then

$$\sum_{q=1}^\infty q h\left(\frac{\psi(q)}{q}\right) = \sum_{q=1}^\infty \frac{1}{q} \Delta(h_\mathfrak{g}, h)\left(\frac{1}{\mathfrak{g}(q)}\right)$$

and therefore the divergence or convergence of this sum will depend on the size of the gap $\Delta(h_\mathfrak{g}, h)$ between $h_\mathfrak{g}$ and h.

We illustrate this on the classical case with the following example.

Example 5.1.4. For a given $\beta \geq 2$ consider $\mathfrak{g}(x) = x^\beta$ and define the chain

$$\mathcal{C} := \left\{ h_\theta(x) := x^{\frac{2}{\beta}} \log^{-\theta}(\frac{1}{x}) : \theta \in \mathbb{R} \right\}. \tag{5.9}$$

Clearly $\mathcal{H}^{h_\theta}(B_\mathfrak{g}) = \infty$ for all $\theta \leq 1$ and $\mathcal{H}^{h_\theta}(B_\mathfrak{g}) = 0$ for all $\theta > 1$. Thus, using the terminology of the previous section, the function $\tilde{h}_\mathfrak{g} := \frac{x^{\frac{2}{\beta}}}{\log(\frac{1}{x})}$ belongs to $\mathcal{E}(B_\mathfrak{g})$, which means that $\tilde{h}_\mathfrak{g}$ is an expected dimension function for $B_\mathfrak{g}$. Note that, in the chain $\mathcal{C}$, we were able to find an appropriate correction factor to locate the precise dimension function.

We actually can go further, improving the precision on the logarithmic gap up to any arbitrary order. Let us define, for any $N \in \mathbb{N}$, the function L_N and the chain $\mathcal{C}_N$ as

$$L_N(x) = \log \circ \log \circ \cdots \circ \log \left(\frac{1}{x} \right) \qquad (N \text{ times}),$$

and

$$\mathcal{C}_N := \left\{ h_\theta(x) := \frac{x^{\frac{2}{\beta}}}{L_N(x)^\theta \prod_{j=1}^{N-1} L_j(x)}; \theta \in \mathbb{R} \right\}.$$

An easy induction argument implies that $\mathcal{H}^{h_\theta}(B_{\mathfrak{g}}) = \infty$ for all $\theta \leq 1$ and $\mathcal{H}^{h_\theta}(B_{\mathfrak{g}}) = 0$ for all $\theta > 1$.

5.2   A Jarník-type theorem

In this section we present a proof of a dimension estimate for the set of well approximable numbers. We remark here that the proofs of the following propositions are rather elementary and do not require any assumption on the dimension function h (cf. Theorem 5.1.1). In addition, we will need in Chapter 6 a construction from the proofs below.

For some particular choice of $\mathfrak{g}$, we will show that $\frac{1}{\left(\mathfrak{g}^{-1}(\frac{1}{x}) \right)^2}$ belongs to $\mathcal{E}(B_{\mathfrak{g}})$ and provide sharp estimates on the dimensional gaps. We present the lower bounds an the upper bounds separately.

5.2.1   Upper bounds for the Hausdorff dimension of $B_{\mathfrak{g}}$

For this part of the proof we do not need to impose any conditions on the function $\mathfrak{g}$.

Proposition 5.2.1. *Let $\mathfrak{g}$ be any positive increasing function and let $\mathfrak{h} = \frac{1}{\left(\mathfrak{g}^{-1}(\frac{1}{x}) \right)^2}$. If $h \in \mathbb{H}_d$, $h \succ \mathfrak{h}$ and $\Delta(\mathfrak{h}, h)(x) = \frac{h(x)}{\mathfrak{h}(x)} = h(x) \left(\mathfrak{g}^{-1}(\frac{1}{x}) \right)^2$ satisfies*

$$\sum_q \frac{1}{q} \Delta(\mathfrak{h}, h) \left(\frac{1}{\mathfrak{g}(q)} \right) < \infty, \tag{5.10}$$

then $\mathcal{H}^h(B_{\mathfrak{g}}) = 0$.

Proof. Define

$$G_q := \{ x \in [0,1] \setminus \mathbb{Q} : \|xq\| \leq \frac{q}{\mathfrak{g}(q)} \}. \tag{5.11}$$

For each $q \in \mathbb{N}$, G_q is the union of $q-1$ intervals of length $2\mathfrak{g}(q)^{-1}$ and two more intervals of length $\mathfrak{g}(q)^{-1}$ at the endpoints of $[0,1]$:

$$G_q = \left(0, \frac{1}{\mathfrak{g}(q)} \right) \cup \left(\frac{1}{\mathfrak{g}(q)}, 1 \right) \cup \bigcup_{1 \leq r \leq q-1} \left(\frac{r}{q} - \frac{1}{\mathfrak{g}(q)}, \frac{r}{q} + \frac{1}{\mathfrak{g}(q)} \right)$$

In addition, for all $k \in \mathbb{N}$, we have

$$B_{\mathfrak{g}} \subset \bigcup_{q \geq k} G_q.$$

Therefore, for $\delta > 0$, if $\frac{2}{\mathfrak{g}(q)} \leq \delta$,

$$\mathcal{H}_{\delta}^h(B_{\mathfrak{g}}) \leq \sum_{q \geq k}(q+1)h(2\mathfrak{g}(q)^{-1}) \lesssim \sum_{q \geq k}(q+1)h(\mathfrak{g}(q)^{-1})$$

since h is doubling. Therefore,

$$\mathcal{H}_{\delta}^h(B_{\mathfrak{g}}) \lesssim \sum_{q \geq k} q\Delta(\mathfrak{h}, h)\left(\frac{1}{\mathfrak{g}(q)}\right)\frac{1}{(\mathfrak{g}^{-1}(\mathfrak{g}(q)))^2} = \sum_{q \geq k}\frac{1}{q}\Delta(\mathfrak{h}, h)\left(\frac{1}{\mathfrak{g}(q)}\right).$$

Finally, the convergence of (5.10) implies $\mathcal{H}^h(B_{\mathfrak{g}}) = 0$. $\qquad\square$

Note that this is exactly the same condition for the upper bound in Theorem 5.1.1. We illustrate with the following zero dimensional set of extremely well approximable numbers.

Example 5.2.2. Define $\mathfrak{g}(x) = e^{x^2}$ and consider the set $B_{\mathfrak{g}}$. The expected dimension function $\mathfrak{h}$ of Proposition 5.2.1 is $\mathfrak{h}(x) = \frac{1}{\log(\frac{1}{x})}$. Consider the family $h_{\theta}(x) = \frac{1}{\log^{\theta}(\frac{1}{x})}$ $(\theta > 1)$. We have that $h_{\theta} \succ \mathfrak{h}$ and condition (5.10) becomes

$$\sum_q \frac{1}{q}\Delta(\mathfrak{h}, h_{\theta})\left(\frac{1}{\mathfrak{g}(q)}\right) = \sum_q \frac{1}{q}\frac{1}{\log^{\theta-1}(\mathfrak{g}(q))} \leq \sum_q \frac{1}{q^{1+2(\theta-1)}} < \infty,$$

which implies that $\mathcal{H}^{h_{\theta}}(B_{\mathfrak{g}}) = 0$ for all $\theta > 1$.

5.2.2 Lower bounds for $B_{\mathfrak{g}}$

For the lower bound, we will consider a defining function $\mathfrak{g}$ which satisfies (5.7) and also

$$\mathfrak{g}^{-1}(ab) \lesssim \mathfrak{g}^{-1}(a) + \mathfrak{g}^{-1}(b) \quad \text{for all } a, b \geq 1. \tag{5.12}$$

Take for example $\mathfrak{g}(x) = e^{x^p}$ for $p > 0$. Then $\mathfrak{g}^{-1}(x) = \log^{\frac{1}{p}}(x)$, which satisfies (5.12).

We define the set $B_{\mathfrak{g}}$ and consider again $\mathfrak{h}(x) = \frac{1}{\left(\mathfrak{g}^{-1}(\frac{1}{x})\right)^2}$. In this section we find conditions on $\Delta(h, \mathfrak{h}) = \frac{\mathfrak{h}}{h}$ with $h \prec \mathfrak{h}$ to ensure that $\mathcal{H}^h(B_{\mathfrak{g}}) > 0$. We have the following proposition:

Proposition 5.2.3. *Let $\mathfrak{g}$ be a positive, increasing function satisfying (5.12). Let $h \in \mathbb{H}_d$ such that $h \prec \mathfrak{h}(x) = \frac{1}{\left(\mathfrak{g}^{-1}(\frac{1}{x})\right)^2}$. Consider a sequence $\{n_k\}$ that satisfies:*

(A) $n_k \geq 3\mathfrak{g}(2n_{k-1})$.

(B) $\log(n_k) \leq \mathfrak{g}(n_{k-1})$.

If $\Delta(h, \mathfrak{h})(x) = \frac{\mathfrak{h}(x)}{h(x)} = \frac{1}{h(x)\mathfrak{g}^{-1}(\frac{1}{x})^2}$ *satisfies*

$$\varliminf_k \frac{1}{6^k \mathfrak{g}^2(n_{k-2})\Delta(h, \mathfrak{h})\left(\frac{1}{\log(n_k)\mathfrak{g}(n_{k-1})}\right)} > 0, \tag{5.13}$$

then $\mathcal{H}^h(B_\mathfrak{g}) > 0$.

Observe that *(A) and (B) imply a certain control on the growth of the sequence* $\{n_k\}$.

Proof. Consider the same sets G_q as in Proposition 5.2.1 and define

$$G'_q := G_q \cap \left(\frac{1}{\mathfrak{g}(q)}, 1 - \frac{1}{\mathfrak{g}(q)}\right).$$

Now, for each $n \in \mathbb{N}$ consider two prime numbers p_1, p_2 such that $n \leq p_1 < p_2 < 2n$. We will prove that G'_{p_1} and G'_{p_2} are disjoint and well separated. Note that if $\frac{r_1}{p_1}$ and $\frac{r_2}{p_2}$ are centers of two of the intervals belonging to G'_{p_1} and G'_{p_2}, we have

$$\left|\frac{r_1}{p_1} - \frac{r_2}{p_2}\right| = \frac{1}{p_1 p_2}|r_1 p_2 - r_2 p_1| \geq \frac{1}{4n^2}$$

since $r_1 p_2 - r_2 p_1 \neq 0$. Therefore, taking into account this separation between the centers and the length of the intervals, we conclude that for $x \in G'_{p_1}$ and $y \in G'_{p_2}$,

$$|x - y| \geq \frac{1}{4n^2} - \frac{2}{\mathfrak{g}(n)} \geq \frac{1}{8n^2} \qquad (\text{ since } \mathfrak{g}(n) \gg n^2).$$

Let $\mathcal{P}^n_m$ be the set of all the prime numbers between m and n and define

$$H_n := \bigcup_{p \in \mathcal{P}^{2n}_n} G'_p.$$

Then H_n is the union of intervals of length at least $\frac{2}{\mathfrak{g}(2n)}$ that are separated by a distance of at least $\frac{1}{8n^2}$.

Now we observe the following: If I is an interval with $|I| > \frac{3}{n}$, then at least $\frac{p|I|}{3}$ of the intervals of G'_p are completely contained on I. To verify this last statement, cut I into three consecutive and congruent subintervals. Then, in the middle interval there are at least $\frac{p|I|}{3}$ points of the form $\frac{m}{p}$. All the intervals of G'_p centered at these points are completely contained in I, since the length of each interval of G'_p is $\frac{2}{\mathfrak{g}(p)} < \frac{|I|}{3}$.

In addition, by the Prime Number Theorem, we know that $\#(\mathcal{P}^n_1) \sim \frac{n}{\log(n)}$, so we can find n_0 such that

$$\#(\mathcal{P}^{2n}_n) \geq \frac{n}{2\log(n)} \text{ for } n \geq n_0.$$

Hence, if I is an interval with $|I| > \frac{3}{n}$, then there are at least

$$\frac{p|I|}{3}\frac{n}{2\log(n)} > \frac{n^2|I|}{6\log(n)}$$

intervals of H_n contained on I. Now we will construct a Cantor type subset E of $B_\mathfrak{g}$ and apply Lemma 2.5.3.

Consider the sequence $\{n_k\}$ of the hypothesis of the proposition and let $E_0 = [0,1]$. Define E_k as the union of all the intervals of H_{n_k} contained in E_{k-1}. Then E_k is built up of intervals of length at least $\frac{1}{\mathfrak{g}(2n_k)}$ and separated by at least $\varepsilon_k = \frac{1}{8n_k^2}$. Moreover, since $\frac{1}{\mathfrak{g}(2n_{k-1})} \geq \frac{3}{n_k}$, each interval of E_{k-1} contains at least

$$m_k := \frac{n_k^2}{6\log(n_k)\mathfrak{g}(2n_{k-1})}$$

intervals of E_k.

Now we can apply Lemma 2.5.3. Consider $h \prec \mathfrak{h}$. Then

$$
\begin{aligned}
D_k^h &= m_1 \cdot m_2 \cdots m_{k-1}h(\varepsilon_k m_k) \\
&= \frac{6^{-(k-2)}n_2^2 \cdots n_{k-1}^2}{\log(n_2)\cdots\log(n_{k-1})\mathfrak{g}(n_1)\cdots\mathfrak{g}(n_{k-2})}h\left(\frac{6}{\log(n_k)\mathfrak{g}(n_{k-1})}\right)
\end{aligned}
$$

Now we note that $n_k \geq \log(n_k)$ and, by hypothesis (A), we also have that $n_k \geq \mathfrak{g}(2n_{k-1}) \geq \mathfrak{g}(n_{k-1})$. In addition, h is doubling, therefore it follows that we can bound the first factor to obtain that

$$
\begin{aligned}
D_k^h &\gtrsim \frac{6^{-k}n_{k-1}^2}{\log(n_{k-1})\mathfrak{g}(n_{k-2})}\frac{1}{\Delta(h,\mathfrak{h})\left(\frac{1}{\log(n_k)\mathfrak{g}(n_{k-1})}\right)\left(\mathfrak{g}^{-1}\left(\log(n_k)\mathfrak{g}(n_{k-1})\right)\right)^2} \\
&\gtrsim \frac{6^{-k}n_{k-1}^2}{\mathfrak{g}^2(n_{k-2})}\frac{1}{\Delta(h,\mathfrak{h})\left(\frac{1}{\log(n_k)\mathfrak{g}(n_{k-1})}\right)\left(\mathfrak{g}^{-1}(\log(n_k)) + n_{k-1}\right)^2}
\end{aligned}
$$

since, by hypothesis (B), n_k satisfies $\log(n_{k-1}) \leq \mathfrak{g}(n_{k-2})$ and $\mathfrak{g}$ satisfies (5.12). Now, again by hypothesis (B),

$$D_k^h \geq \frac{1}{6^k\mathfrak{g}^2(n_{k-2})}\frac{1}{\Delta(h,\mathfrak{h})\left(\frac{1}{\log(n_k)\mathfrak{g}(n_{k-1})}\right)}.$$

Thus, if

$$\varliminf_k \frac{1}{6^k\mathfrak{g}^2(n_{k-2})}\frac{1}{\Delta(h,\mathfrak{h})\left(\frac{1}{\log(n_k)\mathfrak{g}(n_{k-1})}\right)} > 0,$$

then $\mathcal{H}^h(E) > 0$ and therefore $\mathcal{H}^h(B_\mathfrak{g}) > 0$. $\qquad\square$

We also have an example to illustrate this last result.

Example 5.2.4. Define $\mathfrak{g}_\eta(x) = e^{x^{\frac{2}{\eta}}}$ for $\eta > 0$ and consider the set $B_{\mathfrak{g}_\eta}$. Then $\mathfrak{h}_\eta(x) = \frac{1}{\log^\eta(\frac{1}{x})}$ will be an expected dimension function for the set $B_{\mathfrak{g}_\eta}$. Consider the family $h_\theta(x) = \frac{1}{\log^\theta(\frac{1}{x})}$ ($0 < \theta < \eta$), which satisfy $h_\theta \prec \mathfrak{h}_\eta$. In this context, $\Delta(h_\theta, \mathfrak{h}_\eta)(x) = \log^{\theta-\eta}(\frac{1}{x})$. Define the sequence n_k as follows:

$$n_k = e^{kn_{k-1}^{\frac{2}{\eta}}}.$$

Clearly the sequence is admissible, since

(A) $n_k \geq 3\mathfrak{g}(2n_{k-1})$, and

(B) $\log(n_k) \leq \mathfrak{g}(n_{k-1})$.

Inequality (5.13) now becomes

$$D_k^h \gtrsim \frac{(\log\log(n_k) + n_{k-1}^{\frac{2}{\eta}})^{\eta-\theta}}{6^k e^{2n_{k-2}^{\frac{2}{\eta}}}} \geq \frac{n_{k-1}^{2\frac{\eta-\theta}{\eta}}}{6^k e^{2n_{k-2}^{\frac{2}{\eta}}}}.$$

Finally, for any $\varepsilon > 0$ and $M > 0$, n_k satisfies, for large k,

$$\frac{n_{k-1}^\varepsilon}{6^k e^{Mn_{k-1}^{\frac{2}{\eta}}}} = \frac{e^{\varepsilon k n_{k-1}^{\frac{2}{\eta}}}}{6^k e^{Mn_{k-1}^{\frac{2}{\eta}}}} = \frac{e^{(\varepsilon k-M)n_{k-1}^{\frac{2}{\eta}}}}{6^k} \geq 1,$$

so we conclude that $\varliminf_k D_k^h > 0$.

Remark 5.2.5. We will use this example in Chapter 6 to construct a set E with positive $\mathfrak{h}$ measure with $\mathfrak{h}(x) = \frac{1}{\log(\frac{1}{x})}$. For any $\eta > 1$, the above construction provides a set $E = E_\eta$ such that $\mathcal{H}^\mathfrak{h}(E_\eta) > 0$.

5.3   An equivalent formulation of Jarník's Theorem

For our application to small Furstenberg sets in Chapter 6, we will need the following equivalent form of the theorem of Jarník.

Theorem 5.3.1. *Let $\mathfrak{n} = \{n_j\}_j$ be a increasing sequence with $n_{j+1} \geq n_j^j$ for all $j \in \mathbb{N}$. For $0 < \alpha \leq 1$, if $A_\alpha^\mathfrak{n}$ is defined as*

$$A_\alpha^\mathfrak{n} = \left\{ x \in [0,1] \setminus \mathbb{Q} : \forall j \; \exists \; p,q \; ; q \leq n_j^\alpha; |x - \frac{p}{q}| < \frac{1}{n_j^2} \right\},$$

then $\dim(A_\alpha^\mathfrak{n}) = \alpha$.

For the proofs of Theorem 5.3.1 and Theorem 5.1.3, we refer the reader to [Jar31], [Bes34], [Egg52], [Fal86], and [Fal03].

We want to relate the sets A_α and B_β. It is clear that for any $\alpha \in (0,1]$, we have the inclusion $A_\alpha \subset B_{\frac{2}{\alpha}}$. For $\alpha \in (0,1]$, if $x \in A_\alpha$ then for each $j \in \mathbb{N}$ there exists a rational $\frac{p_j}{q_j}$ with $q_j \leq n_j^\alpha$ such that

$$|x - \frac{p_j}{q_j}| < n_j^{-2},$$

which is equivalent to $|xq_j - p_j| < q_j n_j^{-2}$. Therefore $|xq_j - p_j| \leq q_j^{1-\frac{2}{\alpha}}$. Observe that if there were only finite values of q for a given x, then x has to be rational. For if $q_j = q_{j_0}$ for all $j \geq j_0$, then $|x - \frac{p_j}{q_{j_0}}| \to 0$ and this implies that $x \in \mathbb{Q}$. We conclude then that, for any $x \in A_\alpha$,

$$\|xq\| < \frac{1}{q^{\frac{2}{\alpha}-1}} \qquad \text{for infinite many } q$$

and therefore $x \in B_{\frac{2}{\alpha}}$. However, since the dimension of $A_\alpha^{\mathfrak{n}}$ coincides with the one of $B_{\frac{2}{\alpha}}$, one can expect that both sets have approximately comparable sizes.

We introduce the following definition, which is the extended version of the definition of the set $A_\alpha^{\mathfrak{n}}$ in Theorem 5.3.1.

Definition 5.3.2. Let $\mathfrak{n} = \{n_j\}_j$ be any increasing nonnegative sequence of integers. Let $\mathfrak{f}$ be a increasing function defined on $\mathbb{R}_+$. Define the set

$$A_{\mathfrak{f}}^{\mathfrak{n}} := \left\{ x \in [0,1] \setminus \mathbb{Q} : \forall j \; \exists\, p,q \; ; q \leq \mathfrak{f}(n_j); |x - \frac{p}{q}| < \frac{1}{n_j^2} \right\}.$$

The preceding observation about the inclusion $A_\alpha \subset B_\beta$ can be extended to this general setting. For a given $\mathfrak{g}$ as in the definition of $B_{\mathfrak{g}}$, define $\Gamma_{\mathfrak{g}}(x) = \mathfrak{g}^{-1}(x^2)$. Then the same calculations show that $A_{\Gamma_{\mathfrak{g}}}^{\mathfrak{n}} \subset B_{\mathfrak{g}}$.

We will need a converse relation between those sets, since we want to prove a lower bound for the sets $A_{\mathfrak{f}}^{\mathfrak{n}}$ from the estimates provided in Proposition 5.2.3.

Lemma 5.3.3. Let $\mathfrak{g}$ and $B_{\mathfrak{g}}$ be as in Proposition 5.2.3. Define $\Gamma_{\mathfrak{g}}(x) = \mathfrak{g}^{-1}(x^2)$. Then, if $\mathfrak{m} = \{m_k\}$ is the sequence defining the set E in the proof of Proposition 5.2.3, then the set E is contained in $A_{\Gamma_{\mathfrak{g}}}^{\mathfrak{n}}$, where $\mathfrak{n} = \{n_k\} = \{\mathfrak{g}(m_k)^{\frac{1}{2}}\}$.

Proof. Recall that in the proof of Proposition 5.2.3 we define the sets G_q' as a union of intervals of the form $I = \left(\frac{r}{q} - \frac{1}{\mathfrak{g}(q)}; \frac{r}{q} + \frac{1}{\mathfrak{g}(q)} \right)$. The sets H_n was defined as $H_n := \bigcup_{p \in \mathcal{P}_n^{2n}} G_p'$, where $\mathcal{P}_n^{2n}$ is the set of primes between n and $2n$. We can therefore write

$$H_n := \bigcup I_j^n.$$

Now, given a sequence $\mathfrak{m} = \{m_k\}$, for each k, the set E_k is defined as the union of all the intervals of H_{m_k} that belong to E_{k-1}, where $E_0 = [0,1]$. If

$E = \bigcap E_k$, any $x \in E$ is in E_k and therefore in some of the $I_j^{m_k}$. It follows that there exists integers r and q, $q \le 2m_k$ such that

$$\left| x - \frac{r}{q} \right| < \frac{1}{\mathfrak{g}(q)} < \frac{1}{\mathfrak{g}(m_k)} = \frac{1}{n_k^2}, \qquad q \le 2\mathfrak{g}^{-1}(n_k^2).$$

Therefore $E \subset A_{\Gamma_\mathfrak{g}}^\mathfrak{n}$. $\qquad\square$

We remark that the above inclusion implies that any lower estimate on the size of E would also be a lower estimate for $A_{\Gamma_\mathfrak{g}}^\mathfrak{n}$ and in this sense it will be used in the next chapter. More precisely, we will use the following example which follows from the previous lemma and Example 5.2.4.

Example 5.3.4. Let $\mathfrak{h}(x) = \frac{1}{\log(\frac{1}{x})}$. For $\eta > 1$, consider the function $\mathfrak{g}_\eta$, the sequence $\mathfrak{m} = \{m_k\}$ and the set E_η as in Example 5.2.4. Define $\mathfrak{f} = \Gamma_{\mathfrak{g}_\eta}$ and $A = A_\mathfrak{f}^\mathfrak{n}$ as in Lemma 5.3.3. It follows that $\mathfrak{f}(x) = \log(x^2)^{\frac{\eta}{2}}$ and

$$A_\mathfrak{f}^\mathfrak{n} := \left\{ x \in [0,1] \setminus \mathbb{Q} : \forall j \; \exists \; p, q \; ; q \le \mathfrak{f}(n_j); |x - \frac{p}{q}| < \frac{1}{n_j^2} \right\}.$$

From Lemma 5.3.3 and Example 5.2.4 we obtain that $E_\eta \subset A_\mathfrak{f}^\mathfrak{n}$, $\mathcal{H}^\mathfrak{h}(E_\eta) > 0$ and therefore the set $A_\mathfrak{f}^n$ has positive $\mathcal{H}^\mathfrak{h}$-measure.

Chapter 6

Upper bounds for Furstenberg sets

In this section we will focus our attention on the right hand side of (3.1). Precisely, recall that (3.1) states that

$$\max\{2\alpha; \frac{1}{2} + \alpha\} \leq \gamma(\alpha) \leq \frac{1}{2} + \frac{3}{2}\alpha, \qquad \alpha \in (0, 1]. \tag{6.1}$$

where $\gamma(\alpha)$ is the infimum of all the possible values for the Hausdorff dimension of F_α-sets. First, we will provide an estimate on the upper bounds for the size of the class F_α in terms of logarithmic gaps. In second place, we will study the extreme case of $\alpha = 0$ and consider different classes of zero dimensional sets for the fibers, including the finite class F^K (see below for the precise definitions). Our main result in this chapter is to provide a construction of a $F_\mathfrak{h}$-set of Hausdorff dimension not greater than $\frac{1}{2}$ for the choice of $\mathfrak{h}(x) = \frac{1}{\log(\frac{1}{x})}$. This result is a complement of Theorem 3.6.1 in Chapter 3. In that theorem, we prove that *any* set $E \in F_\mathfrak{h}$ for that $\mathfrak{h}$ satisfies $\dim_H(E) \geq \frac{1}{2}$. With the construction of the present chapter, it follows that $\frac{1}{2}$ is the exact value for the Hausdorff dimension of the class $F_\mathfrak{h}$.

We believe that for the reader it will be easier to follow the original construction sketched in [Wol99b]. We will keep careful control of the involved quantities, which will allow us to adapt the construction to obtain the desired set developed in Section 6.2.

6.1 Upper Bounds for Classical Furstenberg Sets

Recall that we have defined the set $A_\alpha^\mathfrak{n}$ as follows (Theorem 5.3.1). Let $\mathfrak{n} = \{n_j\}_j$ be a increasing sequence with $n_{j+1} \geq n_j^j$ for all $j \in \mathbb{N}$. For $0 < \alpha \leq 1$, $A_\alpha^\mathfrak{n}$ is defined as

$$A_\alpha^\mathfrak{n} = \left\{ x \in [0, 1] \setminus \mathbb{Q} : \forall j \ \exists \ p, q \ ; q \leq n_j^\alpha; |x - \frac{p}{q}| < \frac{1}{n_j^2} \right\},$$

And also recall that $\dim_H(A_\alpha^{\mathfrak{n}}) = \alpha$. Moreover, $\mathcal{H}^\alpha(A_\alpha^{\mathfrak{n}}) = +\infty$.

We will show how to construct, for any $\alpha \in (0, 1)$, an F_α-set E with $\dim_H(E) \leq \frac{1}{2} + \frac{3}{2}\alpha$. We begin with some preliminary lemmas. Our first lemma is about a very well distributed (mod 1) sequence.

Lemma 6.1.1. *For $n \in \mathbb{N}$ and any real number $x \in [0, 1]$, there is a pair $0 \leq j, k \leq n - 1$ such that*

$$\left| x - \left(\sqrt{2}\frac{k}{n} - \frac{j}{n} \right) \right| \leq \frac{\log(n)}{n^2}.$$

This lemma is a consequence of Theorem 3.4 of [KN74], p125, regarding the problem of estimating the discrepancy of the fractional part of the sequence $\{n\alpha\}_{n\in\mathbb{N}}$ where α is a irrational of a certain type.

Let us now introduce the notion of G-set, a common ingredient in the construction of Kakeya and Furstenberg sets.

Definition 6.1.2. A G-set is a compact set $E \subseteq \mathbb{R}^2$ which is contained in the strip $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$ such that for any $m \in [0, 1]$ there is a line segment contained in E connecting $x = 0$ with $x = 1$ of slope m, i.e.

$$\forall m \in [0, 1] \; \exists \, b \in \mathbb{R} : mx + b \in E, \; \forall \, x \in [0, 1].$$

We also will need some notation for a thickened line.

Definition 6.1.3. Given a line segment $\ell(x) = mx + b$, we define the δ-tube associated to ℓ as

$$S_\ell^\delta := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1; |y - (mx + b)| \leq \delta\}.$$

Now we are ready to prove the following proposition.

Proposition 6.1.4. *For any $\alpha \in (0, 1]$, there exists a set $E \in F_\alpha$ with $\dim_H(E) \leq \frac{1}{2} + \frac{3}{2}\alpha$.*

Proof. Fix $n \in \mathbb{N}$. For $0 \leq j, k \leq n - 1$, define the line segments

$$\ell_{jk}(x) := (1 - x)\frac{j}{n} + x\sqrt{2}\frac{k}{n} \text{ for } x \in [0, 1],$$

and their δ_n-tubes $S_{\ell_{jk}}^{\delta_n}$ with $\delta_n = \frac{\log(n)}{n^2}$. From now on we will use the notation S_{jk}^n instead of $S_{\ell_{jk}}^{\delta_n}$. Also define

$$G_n := \bigcup_{jk} S_{jk}^n. \tag{6.2}$$

Note that, by Lemma 6.1.1, all the G_n are G-sets. Let now n_j be a sequence as in Theorem 5.3.1, i.e., $n_{j+1} > n_j^j$, and let

$$T = \left\{ x \in \left[\frac{1}{4}, \frac{3}{4} \right] : \forall j \; \exists \, p, q \; ; q \leq n_j^\alpha; |x - \frac{p}{q}| < \frac{1}{n_j^2} \right\}$$

By Theorem 5.3.1, $\dim_H(T) = \alpha$. If $\varphi(t) = \frac{1-t}{t\sqrt{2}}$ and $D = \varphi^{-1}\left(\left[\frac{1}{4}, \frac{3}{4}\right]\right)$, we have that $\varphi : D \to [\frac{1}{4}, \frac{3}{4}]$ is bi-Lipschitz. Therefore the set

$$T' = \left\{ t \in \mathbb{R} : \frac{1-t}{t\sqrt{2}} \in T \right\} = \varphi^{-1}(T)$$

also has Hausdorff dimension α.

The main idea is to construct a set for which we have, essentially, a copy of T' in each direction and simultaneously keep some optimal covering property.

Define, for each $n \in \mathbb{N}$,

$$\Gamma_n := \left\{ \frac{p}{q} \in \left[\frac{1}{4}, \frac{3}{4} \right], q \leq n^\alpha \right\}$$

and

$$Q_n = \left\{ t : \frac{1-t}{\sqrt{2}t} = \frac{p}{q} \in \Gamma_n \right\} = \varphi^{-1}(\Gamma_n).$$

To count the elements of Γ_n (and Q_n), we take into account that

$$\sum_{j=1}^{\lfloor n^\alpha \rfloor} j \leq \frac{1}{2} \lfloor n^\alpha \rfloor (\lfloor n^\alpha \rfloor + 1) \lesssim \lfloor n^\alpha \rfloor^2 \leq n^{2\alpha}.$$

Therefore, $\#(Q_n) \lesssim n^{2\alpha}$. For each $t \in Q_n$, define $S(t) := \{\ell_{jk}(t)\}_{j,k=1}^n$. Clearly, $\#(S(t)) \leq n^2$. But if we note that, if $t \in Q_n$, then

$$0 \leq \frac{\ell_{jk}(t)}{t\sqrt{2}} = \frac{1-t}{t\sqrt{2}} \frac{j}{n} + \frac{k}{n} = \frac{p}{q} \frac{j}{n} + \frac{k}{n} = \frac{pj + kq}{nq} < 2,$$

we can bound $\#(S(t))$ by the number of non-negative rationals smaller than 2 of denominator qn. Since $q \leq n^\alpha$, we have $\#(S(t)) \leq n^{1+\alpha}$. Considering *all* the elements of Q_n, we obtain $\#\left(\bigcup_{t \in Q_n} S(t) \right) \lesssim n^{1+3\alpha}$. Let us define

$$\Lambda_n := \left\{ (x, y) \in G_n : |x - t| \leq \frac{C}{n^2} \text{ for some } t \in Q_n \right\}.$$

Claim 6.1.5. For each n, take $\delta_n = \frac{\log(n)}{n^2}$. Then Λ_n can be covered by L_n balls of radio δ_n with $L_n \lesssim n^{1+3\alpha}$.

To see this, it suffices to set a parallelogram on each point of $S(t)$ for each t in Q_n. The lengths of the sides of the parallelogram are of order n^{-2} and $\frac{\log(n)}{n^2}$, so their diameter is bounded by a constant times $\frac{\log(n)}{n^2}$, which proves the claim.

We now begin with the recursive construction that leads to the desired set. Let F_0 be a G-set written as

$$F_0 = \bigcup_{i=1}^{M_0} S_{\ell_i^0}^{\delta^0},$$

(the union of M_0 δ^0-thickened line segments $\ell_i^0 = m_i^0 + b_i^0$ with appropriate orientation). Each F_j to be constructed will be a G-set of the form

$$F_j := \bigcup_{i=1}^{M_j} S_{\ell_i^j}^{\delta_j}, \qquad \text{with } \ell_i^j = m_i^j + b_i^j.$$

Having constructed F_j, consider the M_j affine mappings

$$A_i^j : [0,1] \times [-1,1] \to S_{\ell_i^j}^{\delta_j} \qquad 1 \le i \le M_j,$$

defined by

$$A_i^j \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ m_i^j & \delta^j \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ b_i^j \end{pmatrix}.$$

We choose n_{j+1} large enough to satisfy

$$\log(n_{j+1}) > M_j \tag{6.3}$$

and apply A_i^j to the sets $G_{n_{j+1}}$ defined in (6.2) to obtain

$$F_{j+1} = \bigcup_{i=1}^{M_j} A_i^j(G_{n_{j+1}}).$$

Since $G_{n_{j+1}}$ is a union of thickened line segments, we have that

$$F_{j+1} = \bigcup_{i=1}^{M_{j+1}} S_{\ell_i^{j+1}}^{\delta^{j+1}},$$

for an appropriate choice of M_{j+1}, δ_{j+1} and M_{j+1} line segments ℓ_i^{j+1}. From the definition of the mappings A_i^j and since the set $G_{n_{j+1}}$ is a G-set, we conclude that F_{j+1} is also a G-set. Define

$$E_j := \{(x,y) \in F_j : x \in T'\}.$$

To cover E_j, we note that if $(x,y) \in E_j$, then $x \in T'$, and therefore there exists a rational $\frac{p}{q} \in \Gamma_{n_j}$ with

$$\frac{1}{n_j^2} > \left| \frac{1-x}{x\sqrt{2}} - \frac{p}{q} \right| = |\varphi(x) - \varphi(r)| \ge \frac{|x-r|}{\sqrt{2}}, \qquad \text{for some } r \in Q_{n_j}.$$

Therefore $(x,y) \in \bigcup_{i=1}^{M_{j-1}} A_i^{j-1}(\Lambda_{n_j})$, so we conclude that E_j can be covered by $M_{j-1} n_j^{1+3\alpha}$ balls of diameter at most $\frac{\log(n_j)}{n_j^2}$. Since we chose n_j such that $\log(n_j) > M_{j-1}$, we obtain that E_j admits a covering by $\log(n_j) n_j^{1+3\alpha}$ balls of the same diameter. Therefore, if we set $F = \bigcap_j F_j$ and $E := \{(x,y) \in F : x \in T'\}$ we obtain that

$$\dim_H(E) \le \underline{\dim}_B(E) \le \varliminf_j \frac{\log\left(\log(n_j) n_j^{1+3\alpha}\right)}{\log\left(n_j^2 \log^{-1}(n_j)\right)} = \frac{1+3\alpha}{2},$$

where $\underline{\dim}_B$ stands for the lower box dimension. In addition, F is a G-set, so it must contain a line segment in each direction $m \in [0,1]$. If ℓ is such a line segment, then

$$\dim_H(\ell \cap E) = \dim_H(T') \geq \alpha.$$

The final set of the proposition is obtained by taking eight copies of E, rotated to achieve *all* the directions in $\mathbb{S}$.   □

The next proposition provides an estimate on the size of these optimal constructions of F_α sets in terms of Hausdorff measures. Note that since the the Hausdorff dimension of the set E constructed above is less or equal than $\frac{1+3\alpha}{2}$, this implies directly that $\mathcal{H}^s(E) = 0$ for any $s > \frac{1+3\alpha}{2}$. But it is not true that we also have that $\mathcal{H}^h(E) = 0$ for any $h \succ x^{\frac{1+3\alpha}{2}}$, again by the result of Rogers. We can therefore estimate the size of the logarithmic gap that guarantees that the measure drop to zero.

Proposition 6.1.6. *For $\alpha \in (0,1]$ and $\theta > 0$, define $h_\theta(x) = x^{\frac{1+3\alpha}{2}} \log^{-\theta}(\frac{1}{x})$. Then, for any $\theta > \frac{1+3\alpha}{2}$, there exists a set $E \in F_\alpha$ with $\mathcal{H}^{h_\theta}(E) = 0$.*

Proof. We follow the exact same construction of the preceding proposition and note that we can choose the defining sequence n_j such that $\log\log(n_{j+1}) > M_j$ instead of (6.3)). For each n, take $\delta_n = \frac{\log(n)}{n^2}$. As before, the set Λ_n can be covered with L_n balls of radio δ_n with $L_n \lesssim n^{1+3\alpha}$. Once again, define F_j, F, E_j and E as before. Now the sets F_j can be covered by $\log\log(n_j)n_j^{\frac{1+3\alpha}{2}}$ balls of diameter at most $\frac{\log(n_j)}{n_j^2}$. Then,

$$
\begin{aligned}
\mathcal{H}^{h_\theta}_{\delta_j}(E) &\lesssim& n_j^{1+3\alpha} \log\log(n_j) h_\theta\left(\frac{\log(n_j)}{n_j^2}\right) \\
&\lesssim& n_j^{1+3\alpha} \log\log(n_j) \left(\frac{\log(n_j)}{n_j^2}\right)^{\frac{1+3\alpha}{2}} \log^{-\theta}\left(\frac{n_j^2}{\log(n_j)}\right) \\
&\lesssim& \log\log(n_j) \log(n_j)^{\frac{1+3\alpha}{2}-\theta} \lesssim \log^{\frac{1+3\alpha}{2}+\varepsilon-\theta}(n_j)
\end{aligned}
$$

for $x \geq K = K(\varepsilon)$. Therefore, for any $\theta > \frac{1+3\alpha}{2}$, the last expression goes to zero.

In words, this corollary is sharpening the known bound $\gamma(\alpha) \leq \frac{1+3\alpha}{2}$ from (6.1), since there is an estimate on the logarithmic gap..   □

6.2   Upper Bounds for Furstenberg-type Sets

In this section we will focus on the class F_α at the endpoint $\alpha = 0$. This class consists of those sets which have a zero dimensional set in every direction. In this total generality nothing new can be said, therefore we will consider some particular subclasses of F_0 associated to specific choices of zero dimensional

dimension functions. Note that, roughly speaking, the inequalities in (3.1) would imply that $\gamma(0) = \frac{1}{2}$. We will show that the value $\frac{1}{2}$ is indeed sharp in some cases. We first consider the finite case, which can be understood as an extreme zero dimensional case. Next we address the main purpose of this section, which is to construct a small Furstenberg set of the class $F_\mathfrak{h}$ for $\mathfrak{h}(x) = \frac{1}{\log(\frac{1}{x})}$. Recall that we proved in Chapter 3 that any set $E \in F_\mathfrak{h}$ must satisfy $\dim(E) \geq \frac{1}{2}$. We will provide a construction of an optimal set, in the sense of having the smallest dimension possible. In other words, we will prove that $\frac{1}{2}$ is the exact dimension of the class $F_\mathfrak{h}$.

6.2.1 The case $\alpha = 0$, K points

For $K \in \mathbb{N}$, $K \geq 2$, a set will be a F^K-set or a Furstenberg set of type K if for any direction $e \in \mathbb{S}$, there are at least K points contained in E lined up in the direction of e. In this section we sketch how to easily adapt Proposition 6.1.4 to obtain a set $E \in F^K$ with $\dim_H(E) \leq \frac{1}{2}$ for any $K \in \mathbb{N}$.

Proposition 6.2.1. *For any $K \in \mathbb{N}$, there exists a set E in the class F^K with $\dim_H(E) \leq \frac{1}{2}$.*

Proof. Define the line segments ℓ_{jk} and the sets G_n in the same way as in Proposition 6.1.4. Consider the set corresponding sets Γ and Q:

$$\Gamma := \left\{ \frac{K+j}{4K} : 1 \leq j \leq K \right\},$$

$$Q = \left\{ t : \frac{1-t}{\sqrt{2}t} \in \Gamma \right\} = \varphi^{-1}(\Gamma) = \{t_1, \ldots, t_K\}.$$

Clearly, for any $t \in Q$, $\#(S(t)) \leq Kn$ and therefore

$$\#\left(\bigcup_{t \in Q} S(t) \right) \leq K^2 n.$$

Now, we define

$$\Lambda_n := \{(x,y) \in G_n : x = t_i \text{ for some } t_i \in Q\}.$$

As before, for each n, set $\delta_n = \frac{\log(n)}{n^2}$. Then the set Λ_n admits a covering by L_n balls of radius δ_n with $L_n \lesssim n$. We can repeat the recursive construction and obtain the sets F_j, which can be covered by $\log(n_j)n_j$ balls of diameter at most $\frac{\log(n_j)}{n_j^2}$. Define F in the same way as before. The sets E_j and E are

$$E_j := \{(x,y) \in F_j : x \in Q\}, \qquad E := \{(x,y) \in F : x \in Q\}.$$

It follows that $\dim_H(E) \leq \frac{1}{2}$ and for any slope $m \in [0,1]$, we have a line segment ℓ in that direction contained in F, so we have that

$$\#(\ell \cap E) = \#(Q) = K.$$

$\square$

6.2.2 The case $\alpha = 0$, continuum many points

To go beyond the finite case, it will be necessary to put in each direction some set with many points but with certain structure. This is the reason why we need Jarník's theorem.

Theorem 6.2.2. *Let* $\mathfrak{h} = \frac{1}{\log(\frac{1}{x})}$. *There is a set* $E \in F_{\mathfrak{h}}$ *such that* $\dim_H(E) \leq \frac{1}{2}$.

Proof. The heart of the matter is essentially Proposition 6.1.4, but we must replace the set T by a generalized version of it. More precisely, consider the set $T = A_{\mathfrak{f}}^{\mathfrak{n}}$ from Example 5.3.4:

$$T = \{x \in [0,1] \setminus \mathbb{Q} : \forall j \; \exists \; p, q \; ; q \leq \mathfrak{f}(n_j); |x - \frac{p}{q}| < \frac{1}{n_j^2}\}.$$

Here $\eta > 1$, $\mathfrak{f}(x) = \log(x^2)^{\frac{\eta}{2}}$ and $\mathfrak{n} = \{n_j\}$ is defined by $n_k = \mathfrak{g}(m_k)^{\frac{1}{2}}$. It follows that $n_k = e^{\frac{1}{2} n_{k-1}^{4k}}$. We will use essentially a copy of T in each direction in the construction of the desired set to fulfill the conditions required to be an $F_{\mathfrak{h}}$-set. For any $\eta > 1$, we have from Example 5.3.4 that $\mathcal{H}^{\mathfrak{h}}(T) > 0$. Now, if we define

$$T' = \left\{t \in \mathbb{R} : \frac{1-t}{t\sqrt{2}} \in T\right\} = \varphi^{-1}(T),$$

then T' has also positive $\mathcal{H}^{\mathfrak{h}}$-measure. Let us define the corresponding sets of Proposition 6.1.4 for this generalized case.

$$\Gamma_n := \left\{\frac{p}{q} \in \left[\frac{1}{4}, \frac{3}{4}\right], q \leq \mathfrak{f}(n)\right\},$$

$$Q_n = \left\{t : \frac{1-t}{\sqrt{2}t} = \frac{p}{q} \in \Gamma_n\right\} = \varphi^{-1}(\Gamma_n).$$

Now the estimate is $\#(Q_n) \lesssim \mathfrak{f}^2(n) = \log^\eta(n^2) \sim \log^\eta(n)$, since

$$\sum_{j=1}^{\lfloor \mathfrak{f}(n) \rfloor} j \leq \frac{1}{2} \lfloor \mathfrak{f}(n) \rfloor (\lfloor \mathfrak{f}(n) \rfloor + 1) \lesssim \lfloor \mathfrak{f}(n) \rfloor^2.$$

For each $t \in Q_n$, define $S(t) := \{\ell_{jk}(t)\}_{j,k=1}^n$. If $t \in Q_n$, following the previous ideas, we obtain that

$$\#(S(t)) \lesssim n \log^\eta(n),$$

and therefore

$$\# \left(\bigcup_{t \in Q_n} S(t)\right) \lesssim n \log(n)^{\frac{3\eta}{2}}.$$

Now we estimate the size of a covering of

$$\Lambda_n := \left\{(x, y) \in G_n : |x - t| \leq \frac{C}{n^2} \text{ for some } t \in Q_n\right\}.$$

For each n, take $\delta_n = \frac{\log(n)}{n^2}$. As before, the set Λ_n can be covered with L_n balls of radio δ_n with $L_n \lesssim n \log(n)^{\frac{3\eta}{2}}$.

Once again, define F_j, F, E_j and E as before. Now the sets F_j can be covered by less than $M_{j-1} n_j \log(n_j)^{\frac{3\eta}{2}}$ balls of diameter at most $\frac{\log(n_j)}{n_j^2}$. Now we can verify that, since each G_n consist of n^2 tubes, we have that $M_j = M_0 n_1^2 \cdots n_j^2$. We can also verify that the sequence $\{n_j\}$ satisfies the relation $\log n_{j+1} \geq M_j = M_0 n_1^2 \cdots n_j^2$, and therefore we have the bound

$$\dim_H(E) \leq \underline{\dim}_B(E) \leq \varliminf_j \frac{\log\left(\log(n_j) n_j \log(n_j)^{\frac{3\eta}{2}}\right)}{\log\left(n_j^2 \log^{-1}(n_j)\right)} = \frac{1}{2},$$

Finally, for any $m \in [0,1]$ we have a line segment ℓ with slope m contained in F. It follows that $\mathcal{H}^{\mathfrak{h}}(\ell \cap E) = \mathcal{H}^{\mathfrak{h}}(T') > 0$.   □

6.2.3   Very small Furstenberg sets

In this section we discuss some examples of very small Furstenberg sets. Precisely, we show three examples:

Example 6.2.3: There is a F^2-set with zero Hausdorff dimension.

Example 6.2.4: For any $h \in \mathbb{H}$, there exists G in F^2 such that $\mathcal{H}^h(G) = 0$.

Example 6.2.5: There is an F^2 set E such that $\dim_H(E) = \frac{1}{2} = \dim_P(E)$.

The corollary here is that there exists really small F^K-sets for the case of $K = 2$. It is clear that the set G in Example 6.2.4 will depend on the choice of h. In the case of being h a zero dimensional function, this construction extends the result in Example 6.2.3. The third example illustrates the relation between Hausdorff and Packing dimension for Furstenberg sets. Recall that we prove in Chapter 3, (3.22) that for any $K \geq 2$, any Furstenberg set E of the class F^K must satisfy

$$1 \leq \dim_H(E) + \dim_P(E) \leq 2\dim_P(E).$$

The present construction can be understood as optimal in the sense of obtaining the smallest possible dimensions, both Hausdorff and Packing.

Example 6.2.3. This example of an F^2-set G of dimension zero will be constructed using the result from Example 1.6.3. In that example, we constructed two sets $E, F \subseteq [0,1]$ with $\dim_H(E) = \dim_H(F) = 0$ and such that $[0,1] \subseteq E + F$.

Consider now $G = E \times \{1\} \cup -F \times \{0\}$. This set G has clearly dimension 0, and contains two points in every direction $\theta \in [0; \frac{\pi}{4}]$. For, if $\theta \in [0; \frac{\pi}{4}]$, let $c = \tan(\theta)$, so $c \in [0,1]$. By the choice of E and F, we can find $x \in E$ and $y \in F$ with $c = x + y$. The points $(-y, 0)$ and $(x, 1)$ belong to G and determine a segment in the direction θ (Figure 6.1).
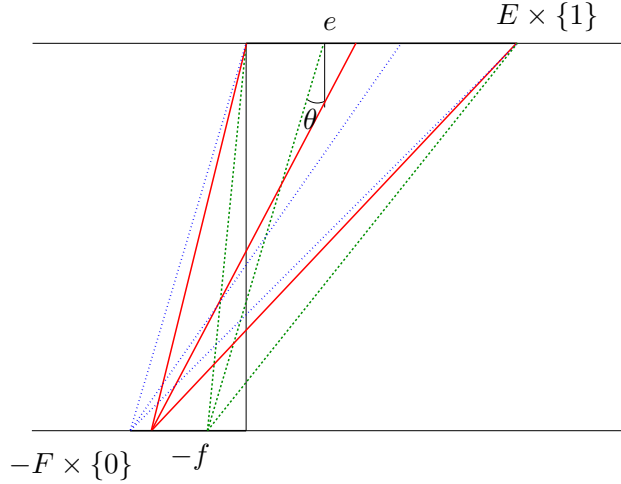
Figure 6.1: An F^2-set of zero Hausdorff dimension

Example 6.2.4. For this second example, since the size of the set is the size of the union of E and F defined as in the previous example, the goal will be to choose the defining sequence $m = \{m_k\}$ in an appropriate way. Recall that the sets E and F was defined in Example 1.6.3 as

$$E := \{x \in [0,1] : r_j = 0 \text{ if } m_k + 1 \leq j \leq m_{k+1}; k \text{ even}\}$$

$$F := \{x \in [0,1] : r_j = 0 \text{ if } m_k + 1 \leq j \leq m_{k+1}; k \text{ odd }\}$$

where $\{m_k; m_0 = 0\}_k$ is an increasing sequence such that $m_k \to +\infty$. Now we estimate the size of the set E. Given $k \in \mathbb{N}$, k even, define $\ell_k = m_k - m_{k-1} + \cdots + m_2 - m_1$. It is clear that E can be covered by 2^{l_k} intervals of length $2^{-m_{k+1}}$. Therefore, if the sequence m_k increases fast enough, then

$$\dim_H(E) \leq \underline{\dim}_B(E) \leq \varliminf_k \frac{\log(2^{\ell_k})}{\log(2^{m_{k+1}})} \lesssim \varliminf_k \frac{2^{\ell_k}}{2^{m_{k+1}}} = 0.$$

Since the same argument shows that $\dim_H(F) = 0$, this estimate proves that the set G in the previous example has Hausdorff dimension equal to zero. Now, for the finer estimate on the $\mathcal{H}^h$-measure of the set, we must impose a more restrictive condition on the sequence $\{m_k\}$.

Recall that the covering property implies that, for a given $h \in \mathbb{H}$, we have that

$$\mathcal{H}^h(E) \leq 2^{l_k} h(2^{-m_{k+1}}).$$

Therefore we need to choose a sequence $\{m_j\}$, depending on h, such that the above quantity goes to zero with k. Since $\ell_k \leq m_k$, we can define recursively the sequence $\{m_k\}$ to satisfy the relation

$$2^{m_k} h(2^{-m_{k+1}}) = \frac{1}{k}.$$

This last condition is equivalent to $m_{k+1} = \log\left(\frac{1}{h^{-1}\left(\frac{1}{k2^{m_k}}\right)}\right)$. As an concrete example, take $h(x) = \frac{1}{\log(\frac{1}{x})}$. In this case we obtain that the sequence $\{m_k\}$ can be defined as

$$m_{k+1} = k2^{m_k}.$$

Example 6.2.5. The construction is essentially the same, but we use two different sets to obtain all directions. Let A be the set of all the numbers whose expansion in base 4 uses only the digits 0 and 1. On the other hand, let B the set of those numbers which only uses the digits 0 and 2. Both sets have Packing and Hausdorff dimension equal to $\frac{1}{2}$ and $[0,1] \subseteq A + B$. The construction follows the same pattern as in the previous examples.

Remark 6.2.6. We remark here that our construction of the F^2-set seems not be extendable to $K \geq 3$. However, [Shm], Tamás Keleti and András Máthé claim that they can construct, for any $K \in \mathbb{N}$, a zero dimensional Furstenberg set in the class F^K. Moreover, they are able to locate the exact cut off function in the following sense: for any $h \in \mathbb{H}$ that is dimensionally smaller than $\mathfrak{h}(x) = \frac{1}{\log\log(\frac{1}{x})}$, a zero dimensional Furstenberg set $E \in F_h$ can be constructed. On the other hand, if h is dimensionally bigger than $\mathfrak{h}$, then any Furstenberg set of the class F_h must have Hausdorff dimension at least $\frac{1}{2}$. These results would completely solve the problem about the size of the Furstenberg sets with *zero dimensional fibers*.

Chapter 7

The Restriction Problem

7.1 Introduction

In the present chapter we will focus our attention to the so called "Restriction Problem" for the Fourier transform. This is indeed a whole family of related problems concerning restriction properties of the Fourier transform to small sets.

Given a compact hypersurface $S \subset \mathbb{R}^n$ endowed with a surface measure μ (the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ with spherical measure is the typical example), the question would be if, for a given $f \in L^p(\mathbb{R}^n)$, there is a chance of having a meaningful restriction of $\widehat{f}$ to S. Here $\widehat{f}$ is the Fourier Transform of f (see Chapter 1 for precise definitions and properties). This formulation of the restriction problem does not make any sense since it is known that the Fourier transform maps L^p to $L^{p'}$ with p' the dual exponent of p. The correct formulation must be made therefore in terms of operator bounds. If we achieve a bound like

$$\|\widehat{f}\|_{(L^q(S),d\mu)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \tag{7.1}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$, then by standard density arguments $\widehat{f}|_S$ can be defined. In that case, we say that there is a restriction theorem for S, denoted by $R_S(p \to q)$. It is easy to see, by testing the above inequality for appropriate indicator functions, that there are no non-trivial restriction theorems for flat surfaces. More precisely, if S is a hyperplane, the only restriction estimate that can hold is the trivial $R_S(1 \to \infty)$, since we always have

$$\|\widehat{f}\|_\infty \leq \|f\|_1. \tag{7.2}$$

In that direction, if we consider a surface $S \subseteq \mathbb{R}^n$ such that it is locally parameterized by a function φ, $\varphi(0) = \nabla\varphi(0) = 0$ and satisfies a condition like $\varphi(x) = O(|x|^k)$, then there is a necessary condition for the exponents: If there is a restriction theorem $R_S(p \to q)$ for S, then

$$\frac{1}{q} \geq \frac{n+k-1}{n-1}\left(1 - \frac{1}{p}\right). \tag{7.3}$$

The geometric interpretation of k is about the order of contact with the tangent space. For the sphere, we now that $k = 2$, and therefore one of the classic necessary condition for a restriction theorem is that

$$\frac{1}{q} \geq \frac{n+1}{n-1}\left(1 - \frac{1}{p}\right). \tag{7.4}$$

The other necessary condition for the case of the sphere is obtained by looking at the restriction problem in a dual formulation. The inequality (7.1) is equivalent to the estimate

$$\|\widehat{g d\sigma}\|_{L^{p'}(\mathbb{R}^n)} \leq C\|g\|_{L^{q'}(\mathbb{S}^{n-1})}, \tag{7.5}$$

and in this form it is known as an "Extension Estimate". Testing this inequality for $g \equiv 1$, we need that

$$\|\widehat{d\sigma}\|_{L^{p'}(\mathbb{R}^n)} \leq C.$$

Now we use the key ingredient that reflects the curvature of the sphere. This curvature implies an estimate on the decay at infinity for the Fourier transform of μ, which can be deduced from classical estimates for oscillatory integrals (see [Ste93]). The known estimate is

$$|\widehat{d\sigma}(\xi)| = O(|\xi|^{-\frac{n-1}{2}}), \tag{7.6}$$

so we conclude that

$$\frac{1}{p} > \frac{n+1}{2n}. \tag{7.7}$$

The famous unsolved Restriction Conjecture asserts that conditions (7.4) and (7.7) are also sufficient.
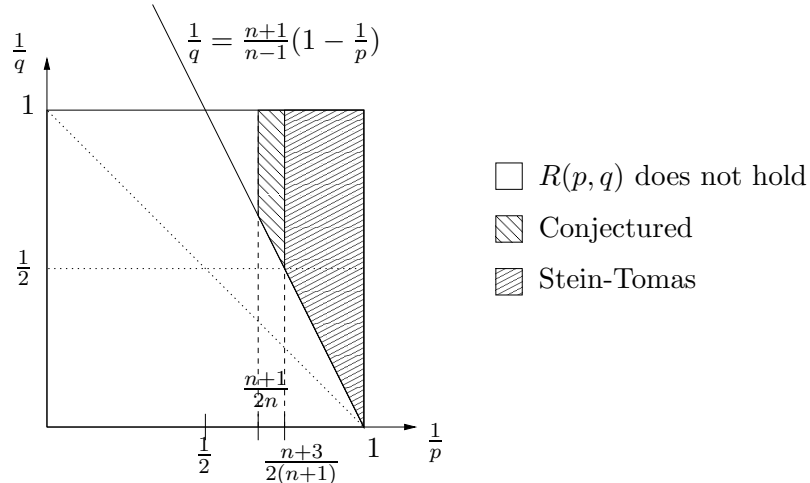
Figure 7.1: Known bounds for $\mathbb{S}^{n-1}$, pictured for $n = 3$.

A classical known result is the Stein-Tomas restriction theorem in [Tom75]. The theorem is that there is a restriction theorem $R_{\mathbb{S}^{n-1}}(p \to 2)$ for the sphere for any p such that $\frac{1}{p} > \frac{n+3}{2(n+1)}$. These conditions are pictured in Figure 7.1.

7.1.1 Connection with Kakeya and PDE

The connection between Kakeya and Restriction can be made by looking for norm estimates for a superposition of Knapp examples (indicator functions of neighborhoods of spherical caps). This is because the Fourier transform of a spherical cap around the point e is like a tube in the direction of e. Therefore, if we can control the norm of a superposition of Knapp examples, we also have some control about the size of a union of tubes of fixed eccentricity, and this last quantity is clearly related to the Kakeya set conjecture. It has also been shown that there is a partial converse to this implication. A positive answer to the Kakeya set conjecture would imply an advance on the Restriction Conjecture (see [Wol99b]).

For the relation with PDE, consider the Helmholtz equation:

$$\Delta u + 4\pi^2 u = 0.$$

Given a function $g \in L^p(\mathbb{S}^{n-1})$, we have that $\widehat{gd\sigma}$ is a distributional solution of the above equation. To see this, take a test function $f \in \mathcal{S}(\mathbb{R}^n)$ and compute

$$
\begin{aligned}
(\Delta\widehat{gd\sigma} + 4\pi^2\widehat{gd\sigma})(f) &= \widehat{gd\sigma}(\Delta f) + 4\pi^2\widehat{gd\sigma}(f) \\
&= gd\sigma(\widehat{\Delta f} + 4\pi^2\widehat{f}) \\
&= 4\pi^2 gd\sigma((1 - |\xi|^2)\widehat{f}) = 0.
\end{aligned}
$$

The last equality follows from the fact that $gd\sigma$ is supported on the sphere. Any estimate like (7.5) will provide information about the size of the solutions of the Helmholtz equation.

The same kind of arguments allows to verify that $\widehat{gd\mu}$, with $d\mu$ the measure of the paraboloid $\{(\tau, \xi) : \tau = 2\pi|\xi|^2\}$ is a solution of the Schrödinger equation:

$$i\partial_t u = \Delta_x u.$$

7.2 A restriction theorem for Hausdorff measures

The contribution of this chapter is to approach this problem in the dimension functions setting. Our work was inspired by the work of Mockenhaupt in [Moc00]. There the author notes that the curvature of the surface can be substituted by an appropriate decay estimate on the Fourier transform of the surface measure, such as

$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\frac{\beta}{2}}. \tag{7.8}$$

Here the convention is to write the exponent as $\frac{\beta}{2}$ since we want to keep the consistency with the definition of Fourier dimension (see Chapter 1, Definition 1.5.7). The other key ingredient is the dimensionality of that measure, i.e., an estimate like

$$\mu(B(x, r)) \lesssim r^\alpha, \tag{7.9}$$

for all balls $B(x, r)$ centered on the surface and for all radii r. In that case a restriction theorem can be proved for the support of μ when the exponent of the underlying spaces will clearly depend on the parameters α and β.

The natural extension to the dimension functions setting would be to study the existence of restriction theorems for a given set $E \subset \mathbb{R}^n$ supporting a measure μ such that

$$\mu(B(x, r)) \lesssim h(r),$$

for any ball $B(x, r)$ for some $h \in \mathbb{H}$ and

$$|\widehat{\mu}(\xi)| \lesssim g(|\xi|),$$

for some bounded positive function g such that $g(\xi) \to 0$ when $|\xi| \to \infty$. Then, the theorem of Stein-Tomas will give sufficient conditions on the decay of g and the growth of h to obtain a restriction theorem. In this direction, it is interesting to study the possible constrains that control these orders. In other words, we ask if there is an inequality analogous to $\dim_F(E) \leq \dim_H(E)$ (Corollary 1.5.6). Since this inequality follows from energy and potential estimates, we study the *h-Energy* $I_h(\mu)$ of a measure μ defined, for $h \in \mathbb{H}$, as

$$I_h(\mu) = \iint \frac{1}{h(|x - y|)} \, d\mu(x) d\mu(y). \tag{7.10}$$

We provide a characterization of the size condition (7.16) in terms of finiteness of this energy for $g \prec h$.

7.2.1   Notation, lemmas, techniques and further remarks

We introduce some notation. For a measurable function f on $\mathbb{R}^n$, $x \in \mathbb{R}^n$, and $a > o$ we define the *translation, dilation,* and *reflection* of f by

$$\begin{array}{rcl}
\tau^y(f)(x) & = & f(x - y) \\
\delta^a(f)(x) & = & f(ax) \\
\widetilde{f}(x) & = & f(-x).
\end{array} \tag{7.11}$$

For future reference, we list some fundamental properties of the Fourier transform. The proofs can be found, for example, in [Wol03], [Duo01] and [Gra04].

Proposition 7.2.1. *Consider $f, g \in \mathcal{S}(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, $b \in \mathbb{R}$, α multiindex and $a > 0$.*

1. *$\|\widehat{f}\|_\infty \leq \|f\|_1$.*

2. *$\widehat{f + g} = \widehat{f} + \widehat{g}$, $\widehat{bf} = b\widehat{f}$, $\widehat{\widetilde{f}} = \widetilde{\widehat{f}}$, $\widehat{\overline{f}} = \overline{\widetilde{\widehat{f}}}$.*

3. *$\widehat{\tau^y(f)}(\xi) = e^{-2\pi i y\xi} \widehat{f}(\xi)$, $(e^{-2\pi i \cdot y} f(\cdot))\widehat{}(\xi) = \tau^y(\widehat{f})(\xi)$.*

4. *$\delta^a(f)\widehat{} = a^{-n}\delta^{a^{-1}}(\widehat{f})$.*

5. *$(\partial^\alpha f)\widehat{} = (2\pi i\xi)^\alpha \widehat{f}(\xi)$.*

6. $(\partial^\alpha \widehat{f})(\xi) = ((-2\pi i \cdot)^\alpha f(\cdot))\,\widehat{}\,(\xi).$

7. $\widehat{f} \in \mathcal{S}(\mathbb{R}^n).$

8. $\widehat{f * g} = \widehat{f}\widehat{g}$, where $(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x - y)dy.$

9. For T an invertible matrix, $\widehat{f \circ T}(\xi) = |\det T^{-1}|\widehat{f}(T^{-t}\xi).$

10. $\displaystyle\int_{\mathbb{R}^n} f(x)\widehat{g}(x)\ dx = \int_{\mathbb{R}^n} \widehat{f}(x)g(x)\ dx.$

11. Define $f^\vee(x) = \widehat{f}(-x)$ for $f \in \mathcal{S}(\mathbb{R}^n)$. Then $(\widehat{f})^\vee = f = \widehat{f^\vee}.$

12. $\displaystyle\int_{\mathbb{R}^n} f(x)\overline{h}(x)\ dx = \int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{h}(\xi)}\ d\xi.$

13. **Plancherel's identity**: $\|f\|_2 = \|\widehat{f}\|_2.$

14. $\displaystyle\int_{\mathbb{R}^n} f(x)g(x)\ dx = \int_{\mathbb{R}^n} \widehat{f}(\xi)g^\vee(\xi)\ d\xi.$

15. Hausdorff-Young: $(1 \leq p \leq 2)$ $\|\widehat{f}\|_{p'} \leq \|f\|_p$ $\quad \forall f \in L^p(\mathbb{R}^n).$

16. Young: If $\frac{1}{p} + \frac{1}{q} \geq 1$, $f \in L^p$, $g \in L^q$ y r with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ then .

$$\|f * g\|_r \leq \|f\|_p \|g\|_q. \tag{7.12}$$

Note that item 13 and the density of $\mathcal{S}(\mathbb{R}^n)$ in L^2 allows us to define the Fourier transform $\widehat{g}$ for all $g \in L^2(\mathbb{R}^n)$ making the Fourier operator $g \to \widehat{g}$ an isometry. Also note that for $L^1 \cap L^2$ we have two definitions of Fourier transform. We can check that they are consistent by splitting any $f \in L^1 \cap L^2$ into $f = f_1 + f_2$ with $f_i \in L^i$.

7.2.2 Endpoint cases

As we said before, we always have $R_S(1 \to \infty)$. In the opposite direction, there is no restriction theorem of the form $R_S(2 \to p)$ for any q. To see this consider a sequence of Schwartz functions $\{f_n\}$ whose Fourier transforms are such that $\widehat{f_n} \to \chi_S$, where χ_S is the indicator function of the surface S. Therefore, by Plancherel,

$$\|f_n\|_{L^2(\mathbb{R}^n)} = \|\widehat{f_n}\|_{L^2(\mathbb{R}^n)} \to 0,$$

and it follows, since $\|\widehat{f_n}\|_{L^q(S)} \gtrsim 1$, that there is no possible inequality of the form

$$\|f\|_{L^q(S)} \leq \|f\|_{L^2(\mathbb{R}^n)} \qquad \text{uniformly in } f \in L^2(\mathbb{R}^n).$$

7.2.3   Compact surfaces

We note here that since we are looking at *compact* surfaces and, in general, *compact* sets, any restriction theorem $R_S(p \to r)$ immediately implies a restriction theorem $R_S(p \to q)$ for any $q \leq r$. The reason is the inclusion property for the Lebesgue spaces for compact sets:

$$L^r(B) \subset L^q(B) \subset L^1(B), \quad 1 \leq q \leq r,$$

reflected by the following norm inequalities:

$$\|f\|_{L^q(B)} \leq C\|f\|_{L^r(B)}, \qquad \text{for } q \leq r,$$

since, by Hölder,

$$
\begin{aligned}
\|f\|_{L^q(B)}^q = \int_B |f|^q \, dx &\leq \left(\int_B |f|^r dx\right)^{\frac{q}{r}} \left(\int_B 1^{\frac{r}{r-q}} dx\right)^{\frac{r-q}{r}} \\
&= \|f\|_{L^r(B)}^q |B|^{\frac{r-q}{r}}.
\end{aligned}
$$

Therefore, if we have a restriction theorem $R_S(p \to r)$ with surface measure σ, then

$$\|\widehat{f}\|_{L^q(S)} \leq \sigma(S)^{\frac{r-q}{rq}} \|\widehat{f}\|_{L^r(S)} \leq \sigma(S)^{\frac{r-q}{rq}} C\|f\|_{L^p(\mathbb{R}^n)}. \tag{7.13}$$

7.2.4   Interpolation

We will use the next classic interpolation theorem.

Theorem 7.2.2 (M. Riesz-Thorin). *Let p_0, p_1, q_0, q_1 such that*

$$1 \leq p_0, p_1, q_0, q_1 \leq \infty \tag{7.14}$$

and, for $0 < \theta < 1$, let us define p and q as

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad , \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If T is a linear operator from $L^{p_0} + L^{p_1}$ to $L^{q_0} + L^{q_1}$ such that

$$\|Tf\|_{q_0} \leq M_0\|f\|_{p_0} \quad \text{for } f \in L^{p_0}$$

$$\|Tf\|_{q_1} \leq M_1\|f\|_{p_1} \quad \text{for } f \in L^{p_1}$$

then

$$\|Tf\|_q \leq M_0^{1-\theta} M_1^{\theta}\|f\|_p \quad \text{for } f \in L^p$$

This theorem says that if we can prove a pair of norm estimates with exponents $(p_0, q_0), (p_1, q_1)$, then we also have an estimate for any pair (p, q) in the line segment that joins $(\frac{1}{p_0}, \frac{1}{q_0})$ with $(\frac{1}{p_1}, \frac{1}{q_1})$.

As a consequence of (7.13), (7.2) and this last theorem, we conclude that any (r, s) restriction theorem directly implies the existence of restriction theorems for the shaded region in Figure 7.2.
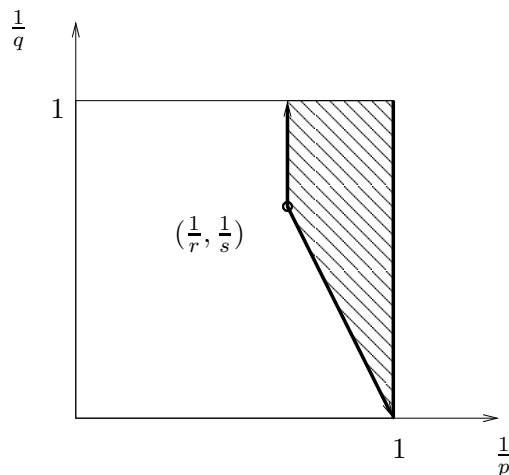
Figure 7.2: Riesz - Thorin interpolation

7.2.5 $R_S(p \to 2)$ restriction theorems

We will study restriction theorems of the form $R_S(p \to 2)$. The advantage arises form the use of the following lemma, whose proof is immediate.

Lemma 7.2.3 (T^*T-method)**.** *Let $T : L^p \to L^2$ be a linear operator, let $T^* : L^2 \to L^{p'}$ be the adjoint. Then T is bounded if and only if $T^*T : L^p \to L^{p'}$ is bounded.*

Let us see how this lemma reformulates the restriction problem. We consider a surface S endowed with a measure μ. We have the operator $R : \mathcal{S}(\mathbb{R}^n) \to L^2(S, \mu)$ that, applied to f, satisfies

$$R(f)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \xi} f(x) \ dx \qquad \xi \in S.$$

The adjoint R^* is

$$R^*(g)(x) = \int_S e^{-2\pi i x \xi} g(\xi) d\mu(\xi) = \widehat{g d\mu}(x) \qquad x \in \mathbb{R}^n,$$

since

$$
\begin{aligned}
R^*(g)(f) &= \overline{g(R(f))} \\
&= \int_S g(\xi) \overline{\int_{\mathbb{R}^n} e^{-2\pi i x \xi} f(x) \ dx} d\mu(\xi) \\
&= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} e^{2\pi i x \xi} g(\xi) \ d\mu(\xi) dx \\
&= \int_{\mathbb{R}^n} f(x) \widehat{g d\mu} \ dx,
\end{aligned}
$$

and therefore $R^*(g) = \widetilde{\widehat{gd\sigma}}$.

We can compute R^*R directly:

$$
\begin{aligned}
R^*R(f)(x) &= \int_{\mathbb{S}^{n-1}} e^{2\pi i x \xi} (Rf)(\xi) d\mu(\xi) \\
&= \int_{\mathbb{S}^{n-1}} e^{2\pi i x \xi} \left(\int_{\mathbb{R}^n} e^{-2\pi i y \xi} f(y) dy \right) d\mu(\xi) \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} e^{2\pi i \xi (x-y)} d\mu(\xi) \right) f(y) dy \\
&= (f * K)(x),
\end{aligned}
$$

where $K(x) = \widehat{d\mu}(-x)$.

The above reformulation allows us to prove a restriction theorem as a consequence of the following lemma. We need an $(L^p, L^{p'})$ estimate for the convolution operator $f \mapsto f * \widehat{d\mu}$.

Theorem 7.2.4 (Hardy-Littlewood-Sobolev)**.** *Let $0 < \alpha < n$, $1 < p, q < \infty$ and*

$$
\frac{1}{q} + 1 = \frac{1}{p} + \frac{\alpha}{n}.
$$

Then

$$
\left\| f * \frac{1}{|x|^\alpha} \right\|_q \le C \|f\|_p.
$$

We remark that this theorem can be understood as an endpoint Young inequality, since $1/|x|^\alpha$ is almost in $L^{\frac{n}{\alpha}}$.

We apply Theorem 7.2.4 to $f * \widehat{d\sigma}$. From (7.6) we know that

$$
|\widehat{d\sigma}| = O(|x|^{-\frac{n-1}{2}}).
$$

If we define $\frac{1}{p'} + 1 = \frac{1}{p} + \frac{n-1}{2n}$ (which is the same as $\frac{1}{p} = \frac{3n+1}{4n}$) we obtain

$$
\|f * \widehat{d\sigma}\|_{\frac{4n}{n-1}} \le C \|f\|_{\frac{4n}{3n+1}} \tag{7.15}
$$

It follows from Lemma 7.2.3 that a restriction theorem $R(\frac{4n}{3n+1} \to 2)$ is valid for the sphere $\mathbb{S}^{n-1}$.

7.2.6 A Stein-Tomas-like theorem for fractal measures

In this section we will prove a Stein-Tomas-like theorem in the general setting announced in Section 7.1. We recall that we will work with measures μ that obey the following estimates:

$$
\mu(B(x,r)) \lesssim h(r) \tag{7.16}
$$

for any ball $B(x,r)$ for some $h \in \mathbb{H}$ and

$$
|\widehat{\mu}(\xi)| \lesssim g(|\xi|) \tag{7.17}
$$

for some bounded, positive function g such that $g(|\xi|) \to 0$ at infinity.

The conditions above are somehow reflecting the Fourier and Hausdorff dimension of $\text{supp}(\mu)$ (in the case of h and g being power functions, this is indeed precise). The purpose of this approach is not only the extension to more general conditions on the measures, but also to consider the problem of finding restriction estimates for zero dimensional sets. In this zero dimensional context, we have to consider decay rates for the Fourier transform that are slower than any reciprocal of a power. For, if this is not the case, Corollary 1.5.6 would imply the positivity of the Hausdorff dimension of the support of the measure. Those measures μ whose Fourier transform vanishes at infinity are called *Rachman measures* (see [Lyo95]) and an example of such a measure supported on a zero dimensional set can be found in [Blu00]. There the author proves that the Liouville set $\mathbb{L}$ carries a positive non-trivial Rachman measure.

Now we prove the main theorem of this chapter.

Theorem 7.2.5 ($R_E(p \to 2)$). *Let E be a subset of $\mathbb{R}^n$ supporting a measure μ satisfying (7.16) for $h \in \mathbb{H}_d$ with doubling constant $C_h < 2^n$ and (7.17) for a given g. Let Γ be the sequence defined as $\Gamma_k = g(2^{k-1})^{\frac{1}{p} - \frac{1}{p'}} (2^{nk} h(2^{-k}))^{\frac{2}{p'}}$, where p and p' are conjugate exponents. If $\Gamma \in \ell^1$ for some p, then a restriction theorem $R_E(p \to 2)$ holds.*

Proof. We start by decomposing the operator $f * \widehat{d\mu}$ by introducing a dyadic partition. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a radial function such that $\phi(x) = 1$ for $|x| < 1$ and with support contained in a ball centered on the origin with radius 2. Define

$$\psi_k(x) = \phi(\frac{x}{2^k}) - \phi(\frac{x}{2^{k-1}}).$$

We have that

$$
\begin{aligned}
\psi_0(x) &= \phi(x) - \phi(2x), \\
\psi_1(x) &= \phi(\frac{x}{2}) - \phi(x), \\
\psi_2(x) &= \phi(\frac{x}{4}) - \phi(\frac{x}{2}),
\end{aligned}
$$

and also the scaling property

$$\psi_k(x) = \psi_0(\frac{x}{2^k})$$

The support of each ψ_k is contained in the annulus

$$C_k = \{x \in \mathbb{R}^n : 2^{k-1} < |x| < 2^{k+1}\},$$

since

$$|x| < 2^{k-1} < \frac{|x|}{2^k} < \frac{1}{2} \ y \ \frac{|x|}{2^{k-1}} < 1 < \psi_k(x) = \phi(\frac{x}{2^k}) - \phi(\frac{x}{2^{k-1}}) = 0$$

and

$$|x| < 2^{k+1} < \frac{|x|}{2^k} > 2 \ y \ \frac{|x|}{2^{k+1}} > 4 < \psi_k(x) = \phi(\frac{x}{2^k}) - \phi(\frac{x}{2^{k-1}}) = 0.$$

Now, for any $x \in \mathbb{R}^n$ we have the identity

$$\sum_{k>0} \psi_k(x) = -\phi(x) + \lim_{k\to\infty} \phi(\frac{x}{2^k}) = 1 - \phi(x).$$

Hence, we can decompose the operator as

$$(f * \widehat{d\sigma})(x) = (f * (\phi + \sum_{k>0} \psi_k)\widehat{d\sigma})(x) = (f * (\phi\widehat{d\sigma})) + \sum_{k>0} f * (\psi_k\widehat{d\sigma}).$$

Therefore, the aim would be to bound two terms:

$$\|f * \widehat{d\sigma}\|_{p'} \leq \left\|f * (\phi\widehat{d\sigma})\right\|_{p'} + \left\|\sum f * (\psi_k\widehat{d\sigma})\right\|_{p'}$$

The first term can be bounded using, for example, Young's inequality, since $\phi\widehat{d\sigma}$ is a C^∞ function of compact support, and therefore it is in any L^p. We apply Young (7.12) with

$$\frac{1}{p'} = \frac{1}{p} + \frac{2}{p'} - 1$$

and conclude that

$$\|f * (\phi\widehat{d\sigma})\|_{p'} \leq \|f\|_p \|\phi\widehat{d\sigma}\|_{\frac{p'}{2}}$$

Now we focus on the second term. We will obtain a (p, p') estimate by interpolation between $(1, \infty)$ and $(2, 2)$ bounds. The $(1, \infty)$ bound is not difficult, since

$$\|f * (\psi_k\widehat{d\sigma})\|_\infty \leq \|f\|_1 \|\psi_k\widehat{d\sigma}\|_\infty \leq \|f\|_1 g(2^{k-1}) \tag{7.18}$$

by the localization property for $\text{supp}(\psi_k) \subset C_k$ and the assumption (7.17).

For the $(2, 2)$ bound, we recall that for any convolution operator $T : L^2 \to L^2$, $Tf = f * K$ the operator norm is the L^∞ norm of the Fourier transform of the kernel. This is a direct consequence of Hölder and Plancherel. Then, $\|T\| = \|\widehat{K}\|_\infty$. In this case the kernel is $\psi_k\widehat{d\mu}$, and a direct calculation shows that we have to control

$$\|\widehat{\psi_k} * d\mu\|_\infty.$$

We begin with a pointwise estimate for $\widehat{\psi_k}$:

$$\widehat{\psi_k}(\xi) = 2^{nk}\widehat{\psi_0}(2^k\xi).$$

Now we use the fact that, since ψ_0 is in the Schwartz class, the same holds for $\widehat{\psi_0}$, and therefore for any $N \in \mathcal{N}_\delta$ there exists a constant $C_N > 0$ such that

$$|\widehat{\psi_0}(2^k\xi)| \leq C_N \frac{1}{(1 + 2^k|\xi|)^N}.$$

It follows that $|\widehat{\psi_k}(\xi)| \lesssim \frac{2^{nk}}{(1+2^k|\xi|)^N}$. Let ξ be fixed and define, for $j \geq 0$, the balls $B_j := B(\xi, 2^{j-k})$ centered on ξ. Then we can decompose the convolution integral as

$$
\begin{aligned}
|(\widehat{\psi_k} * d\mu)(\xi)| &\leq C_N 2^{nk} \int_{\mathbb{R}^n} (1 + 2^k|\xi - \omega|)^{-N} d\mu(\omega) \\
&\leq C_N 2^{nk} \int_{B_0} (1 + 2^k|\xi - \omega|)^{-N} d\mu(\omega) + \\
&\qquad + C_N 2^{nk} \sum_{j \geq 0} \int_{B_{j+1} \backslash B_j} (1 + 2^k|\xi - \omega|)^{-N} d\mu(\omega) \\
&\leq C_N 2^{nk} \mu(B_0) + C_N 2^{nk} \sum_{j \geq 0} 2^{-jN} \mu(B_{j+1} \backslash B_j) \\
&= C_N 2^{nk} \left(h(2^{-k}) + \sum_{j \geq 0} 2^{-jN} h(2^{j+1-k}) \right).
\end{aligned}
$$

Now, if we choose $N = n$ and use the doubling condition for h, we obtain

$$
\begin{aligned}
|(\widehat{\psi_k} * d\mu)(\xi)| &\leq C_n 2^{nk} \left(h(2^{-k}) + \sum_{j \geq 0} 2^{-jn} C_h^{j+1} h(2^{-k}) \right) \\
&\lesssim C_n 2^{nk} h(2^{-k}) \sum_{j \geq 0} \left(\frac{C_h}{2^n} \right)^j.
\end{aligned}
$$

If the doubling constant satisfies $C_h < 2^n$, then

$$
\|f * (\psi_k \widehat{d\sigma})\|_2 \leq 2^{nk} h(2^{-k}) \|f\|_2. \tag{7.19}
$$

Finally, with the two bounds from (7.19) and (7.18), we apply Riesz-Thorin to obtain a (p, p') estimate for $1 < p < 2$. Let $\theta \in (0,1)$ be defined by $\theta = \frac{2}{p} - 1$. Then

$$
\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2} \quad , \quad \frac{1}{p'} = \frac{\theta}{\infty} + \frac{1-\theta}{2},
$$

and therefore

$$
\left\| f * (\psi_k \widehat{d\sigma}) \right\|_{p'} \leq g(2^{k-1})^\theta (2^{nk} h(2^{-k}))^{1-\theta} \|f\|_p.
$$

The obtained operator bounds for each scale 2^{-k} is precisely the sequence Γ of the hypothesis. Substituting $\theta = \frac{2}{p} - 1 = \frac{1}{p} - \frac{1}{p'}$ and $1 - \theta = \frac{2}{p'}$, we obtain that

$$
\Gamma_k = g(2^{k-1})^{\frac{1}{p} - \frac{1}{p'}} (2^{nk} h(2^{-k}))^{\frac{2}{p'}}. \tag{7.20}
$$

The summability of Γ_k allows us to bound

$$
\sum_{k>0} \left\| f * (\psi_k \widehat{d\sigma}) \right\|_{p'} \lesssim \|f\|_p
$$

Since

$$f * \phi \widehat{d\sigma} + \sum_{k=0}^{N} f * (\psi_k \widehat{d\sigma}) \xrightarrow[N]{} f * \widehat{d\sigma}$$

pointwise, from Fatou's lemma it follows that

$$\left\| f * \widehat{d\sigma} \right\|_{p'} \leq \lim_{N \to \infty} \left\| f * \phi \widehat{d\sigma} \right\|_{p'} + \sum_{k=0}^{N} \left\| f * (\psi_k \widehat{d\sigma}) \right\|_{p'} \leq C \|f\|_p$$

for any p such that $\Gamma \in \ell^1$.

$\square$

Remark 7.2.6. We remark that in the case of power functions, $h_\alpha(x) = x^\alpha$, we recover the estimate of Mockenhaupt for the $(2,2)$ bound: $\|f * (\psi_k \widehat{d\sigma})\|_2 \leq 2^{k(n-\alpha)} \|f\|_2$. Also note that in those cases the doubling condition always satisfies the required property. Any admissible power must be $\alpha < n$, since otherwise the dimension function would lead to the null measure if $\alpha > n$ or to the trivial case if $\alpha = n$. Therefore, $h_\alpha(2x) = 2^\alpha x^\alpha < 2^n h_\alpha(x)$.

For the special case of E being a Salem set (see the discussion after Definition 1.5.7) or more generally, for any set supporting a measure μ satisfying (7.8) and (7.9), the admissible values for p are those such that $\frac{1}{p} > \frac{4(n-\alpha)+\beta}{4(n-\alpha)+2\beta}$. In particular, for the sphere $\mathbb{S}^{n-1}$, $\alpha = \beta = n - 1$, and therefore we obtain that $\frac{1}{p} > \frac{n+3}{2(n+1)}$ as announced in Section 7.1.

Remark 7.2.7. We also remark that the previous theorem allows us to consider those cases in which the measure satisfies a growth condition that is slightly weaker than a certain power. For example, suppose that we have a set E supporting a measure μ such that (7.8) holds for a given $\beta > 0$ but we can only control the growth of $\mu(B(x,r))$ by an expression of the order of $r^\alpha \log(\frac{1}{r})$. Even in this case, when we clearly can not bound $\mu(B(x,r))$ by r^α, the theorem holds directly in the exact same way and in the same range of exponents as it would be in the case of a measure μ satisfying (7.8) and (7.9).

7.3   Further questions and work in progress

In this section we want to address the restriction problem in the zero dimensional setting. More precisely, we ask if there is a meaningful version of a restriction theorem for a zero dimensional set $E \subseteq \mathbb{R}^n$. Clearly, this set E must support a Rachman measure with an appropriate decay rate to allow the sequence Γ to be summable. More precisely, we post the following question:

Question 7.3.1. Is it possible to find a zero dimensional set $E \subseteq \mathbb{R}^n$ endowed with a measure μ with properties (7.16) and (7.17) for which $R_E(p \to q)$ holds for some (p, q)?

It is natural to conjecture that there must be a limitation on the maximum possible decay for $\widehat{\mu}$ depending on size properties of E. As we said in the introduction of this chapter, in the classical setting we have that this restraint is reflected by the inequality

$$\dim_F(E) \leq \dim_H(E) \text{ for all } E \subset \mathbb{R}^n. \tag{7.21}$$

We therefore post another question:

Question 7.3.2. Suppose that a set E supports a measure μ satisfying (7.16) and (7.17). How are g and h related? Is there an analogue for the above inequality? It is true that something like $g^2(\frac{1}{x}) \prec h$ must hold?

In the direction of solving the previous questions, one could try to generalize the energy and potential methods that characterize the Hausdorff dimension of a set and, consequently, allow to derive inequality (7.21). We include in this section a more general version of Lemma 1.4.1. Recall that the h-Energy of a measure was defined in (7.10) as

$$I_h(\mu) = \iint \frac{1}{h(|x-y|)} \, d\mu(x)d\mu(y). \tag{7.22}$$

Lemma 7.3.3. *Consider a probability measure μ with compact support, $h \in \mathbb{H}_d$.*

1. *If μ satisfies $\mu(B(x,r)) \leq Ch(r)$, then $I_g(\mu) < \infty$ for all $g \prec h$ such that $\sum_{j\geq 1} \frac{h}{g}(2^{-j}) < +\infty$.*

2. *Conversely, if μ is a probability measure with compact support and with $I_h(\mu) < \infty$, then there is another probability measure ν such that $\nu(X) \leq 2\mu(X)$ for all sets X and such that ν satisfies $\nu(B(x,r)) \leq Ch(r)$.*

Proof. For the first item, since μ is compactly supported and

$$I_g(\mu) = \int V_\mu^g(y) \, d\mu(y),$$

the finiteness of I_g would follow from a uniform bound for the g-potential $V_\mu^g(y)$. Let $g \prec h$ and $y \in \text{supp}(\mu)$. Consider also the annulus

$$C_j := B(y, 2^{-j+1}) \setminus B(y, 2^{-j}).$$

If we decompose the integral, we obtain

$$V_\mu^g(y) = \int \frac{1}{g(|x-y|)} \, d\mu(x) = \sum_j \int_{C_j} \frac{1}{g(|x-y|)} \, d\mu(x).$$

Now we use the doubling property for h and note that for any $x \in C_j$ we have that $g(|x-y|) \geq g(2^{-j})$. Then,

$$V_\mu^g(y) \leq \sum_j \frac{1}{g(2^{-j})} \mu(C_j) \lesssim \sum_j \frac{h}{g}(2^{-j}) < \infty,$$

where the constants involved in the inequalities are independent of y.

We prove the second item, that can be understood as a converse statement. Define F as

$$F = \left\{ x \in \mathbb{R}^n : V_\mu^h \leq 2I_h(\mu) \right\}.$$

Then, by Chebyshev we obtain that

$$\mu \left(\left\{ x \in \mathbb{R}^n : V_\mu^h \leq 2I_h(\mu) \right\} \right) \leq \frac{1}{2I_h(\mu)} \int V_\mu^h \, d\mu(x) = \frac{1}{2},$$

and it follows that $\mu(F) \geq \frac{1}{2}$. The desired measure ν would be defined as

$$\nu(X) = \frac{\mu(X \cap F)}{\mu(F)}.$$

From this definition it is clear that $\nu(X) \leq 2\mu(X)$ for all $X \in \mathbb{R}^n$. Now, let $x \in \mathbb{R}^n$, $r > 0$ and let $B(x, r)$ be a ball. We consider two cases. First, take $x \in F$. We compute the h-potential of ν as follows,

$$V_\nu^h = \int_{\mathbb{R}^n} \frac{1}{h(|x - y|)} \, d\nu(y) \geq \int_{B(x,r)} \frac{1}{h(|x - y|)} \, d\nu(y) \geq \frac{1}{h(r)} \nu(B(x, r)).$$

Using that $x \in F$, we conclude

$$\frac{1}{h(r)} \nu(B(x, r)) \leq V_\nu^h \leq 2V_\mu^h \leq 4I_h(\mu),$$

which is the desired bound. On the other hand, if $x \notin F$, for those $r > 0$ such that $B(x, r) \cap F = \emptyset$, we have $\nu(B(x, r)) = 0$. For the other values of r, since $B(x, r) \cap F \neq \emptyset$, we can take $y \in B(x, r) \cap F$ and then

$$\nu(B(x, r)) \leq \nu(B(y, 2r)) \leq Ch(2r) \leq Ch(r).$$

$\square$

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